Staircase Connected Sets

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Abstract. A compact set $S \subset \mathbb{R}^2$ is staircase connected if every two points $a, b \in S$ can be connected by a polygonal path with sides parallel to the coordinate axes, which is both x -monotone and y -monotone. $\xi(a, b)$ denotes the smallest number of edges of such a path. $\xi(\cdot, \cdot)$ is an integer-valued metric on S . We investigate this metric and introduce stars and kernels. Our main result is that the r th kernel is nonempty, compact and staircase connected provided $r \geq \frac{1}{2} \cdot \text{stdiam}(S) + 1$.

1. Staircase Sets

Call a polygonal path $P = \langle a_0, a_1, \dots, a_n \rangle$ in $\mathbb{R}^2$ **orthogonal** if every edge of P is parallel to one of the coordinate axes (either horizontal or vertical). An orthogonal path P is a **staircase** if all its horizontal edges point in the same direction (east or west) and all its vertical edges point in the same direction (south or north).

A set $S \subset \mathbb{R}^2$ is **orthogonally connected (staircase connected)** if $S \neq \emptyset$ and every two points $p, q \in S$ can be joined within S by an orthogonal path (staircase). Note that if the segment $[p, q]$ is horizontal or vertical, then $[p, q]$ is the unique staircase that connects p and q .

A set $S \subset \mathbb{R}^2$ is **horizontally convex (vertically convex)** if S includes every horizontal (vertical) segment with endpoints in S . If S is both horizontally and vertically convex, then we say that S is **orthogonally convex**.

Proposition 1.1. *A set $S \subset \mathbb{R}^2$ is staircase connected if and only if S is orthogonally connected and orthogonally convex.*

Fig. 1. Shortcut.

Proof. The “only if” part is obvious. To prove the converse, consider two points $a, b \in S$ and let $P = \langle a, x_1, x_2, \dots, x_{i-1}, b \rangle$ be an orthogonal path in S from a to b with the smallest possible number of edges. If P is not a staircase, then P has two consecutive horizontal (or vertical) edges $[x_{i-1}, x_i]$ and $[x_{i+1}, x_{i+2}]$ with opposite directions. Since S is orthogonally convex, P can be replaced by a shorter orthogonal path within S , contradicting our initial choice of P . (See Fig. 1.) $\square$

Definition. A set $S \subset \mathbb{R}^2$ is called an S_k -set if every two points $a, b \in S$ can be joined within S by a staircase with k or fewer edges. We define the **staircase diameter** of S ($\text{stdiam}(S)$) to be the smallest k such that S is an S_k -set.

Proposition 1.2. Suppose $S \subset \mathbb{R}^2$ is compact and staircase connected. Then S is an S_k -set for some finite k .

Proof. Let L_1, L_2 be the two vertical supporting lines of S (L_1 from the left, L_2 from the right). If $L_1 = L_2$ then S is a line segment (or a single point) and there is nothing to prove. So we may assume L_1 and L_2 are parallel. Take two points $c \in L_1 \cap S$ and $d \in L_2 \cap S$. Join c and d by a staircase $P = \langle c, x_1, x_2, \dots, x_{k-1}, d \rangle$ within S . Given any two points $a, b \in S$ we can join a to a point $a' \in P$ and b to a point $b' \in P$ by vertical segments within S (S is vertically convex), and obtain an orthogonal path $\langle a, a', \dots, b', b \rangle$ with at most $k + 2$ edges joining a to b within S . By Proposition 1.1 there is a staircase with $k + 2$ or fewer edges that joins a and b within S . $\square$

Remarks. 1. Compactness of S is necessary in Proposition 1.2 (see the example in Fig. 2):

$$S = \{(\gamma, \eta) \in \mathbb{R}^2: |\gamma| < 1, |\eta| < 1, (\gamma + 1)^2 + (\eta - 1)^2 \leq 4, (\gamma - 1)^2 + (\eta + 1)^2 \leq 4\}.$$

2. The analogue of Proposition 1.2 for polygonally connected sets, and even for orthogonally connected sets, is false, as Fig. 3 shows.

2. The Staircase Distance

Definition. Let $S \subset \mathbb{R}^2$ be a compact and staircase connected set, and let a, b be two points in S . We denote by $\mu(a, b)$ a staircase path with the minimal number of edges

Fig. 2. A noncompact staircase connected set with infinite diameter.

connecting a and b within S . The number of edges of any polygonal path P will be denoted by $e(P)$. Finally define $\xi(a, b) = e(\mu(a, b))$.

In this section we prove that if $S \subset \mathbb{R}^2$ is a compact staircase-connected set, then $\xi(\cdot, \cdot)$ is an integer-valued metric. We also show some properties of this metric and prove the equivalent of the monotonicity lemma of [MP1].

Proposition 2.1. *Let S be a staircase-connected set in $\mathbb{R}^2$. Then $\xi(\cdot, \cdot)$ is an integer-valued metric.*

Proof. It is clear that for every two points $a, b \in S$, $\xi(a, b) = \xi(b, a)$, $\xi(a, b) \geq 0$ and $\xi(a, b) = 0$ if and only if $a = b$. It just remains to show the triangle inequality, that is, for every three points a, b, c in S , $\xi(a, b) \leq \xi(a, c) + \xi(c, b)$.

Let $\mu(a, c)$ be a staircase with $\xi(a, c)$ edges that connects a to c within S . Let $\mu(c, b)$ be a staircase with $\xi(c, b)$ edges that connects c to b within S . Let P be the concatenation of $\mu(a, c)$ and $\mu(c, b)$. P is an orthogonal path with at most $\xi(a, c) + \xi(c, b)$ edges that connects a to b within S . Let $\mu(a, b)$ be a shortest orthogonal path from a to b within S . Clearly $e(\mu(a, b)) \leq e(P)$ and $\mu(a, b)$ is a staircase by Proposition 1.1. $\square$

Proposition 2.2. *Let S be a staircase-connected set in $\mathbb{R}^2$. Suppose $a, c \in S$ and let b be a point in S that belongs to the rectangle determined by a and c (with sides parallel to the coordinate axes). Then $\xi(a, b) \leq \xi(a, c)$ and $\xi(b, c) \leq \xi(a, c)$.*

Fig. 3. A compact orthogonally connected set with infinite diameter.

Proof. A minimal staircase $\mu(a, c)$ lies entirely within the rectangle determined by a and c . Assume, without loss of generality, that a is northwest of c .

If $b \in \mu(a, c)$ then clearly $\xi(a, b) \leq \xi(a, c)$.

If $b \notin \mu(a, c)$ then b lies either below or above $\mu(a, c)$. If b lies below $\mu(a, c)$, go from b up vertically until you hit $\mu(a, c)$ at a point b' , then continue from b' to a on $\mu(a, c)$. This yields a staircase P' from b to a within S , and P' has at most $\xi(a, c)$ edges, since it misses entirely the last edge of $\mu(a, c)$, adjacent to c . (No point of the rectangle determined by a and c lies below that edge.) If b lies above $\mu(a, c)$, go from b horizontally to the left until you hit $\mu(a, c)$ at a point b' , then continue to a on $\mu(a, c)$. This again yields a staircase P' from b to a that misses the last edge of $\mu(a, c)$, and therefore has at most $\xi(a, c)$ edges. $\square$

Corollary 2.3. *Under the conditions of Proposition 2.2, $\xi(a, b) + \xi(b, c) - 2 \leq \xi(a, c) \leq \xi(a, b) + \xi(b, c)$.*

Proof. The upper bound is trivial, since $\xi(\cdot, \cdot)$ is a metric. To prove the lower bound, assume once more that a is northwest of c .

Let $\mu(a, c)$ be a minimal staircase connecting a to c within S . If b is a vertex of $\mu(a, c)$, then $\xi(a, c) = \xi(a, b) + \xi(b, c)$. If b is an interior point of an edge of $\mu(a, c)$, then $\xi(a, c) \leq \xi(a, b) + \xi(b, c) \leq \xi(a, c) + 1$.

If $b \notin \mu(a, c)$, then b lies either below or above $\mu(a, c)$. Assume, for example, that b lies below $\mu(a, c)$. The vertical line through b ($= L_b^v$) hits $\mu(a, c)$ at a point b' above b . The horizontal line through b ($= L_b^h$) hits $\mu(a, c)$ at a point b'' to the right of b . Since $\mu(a, c)$ is a descending path, b' precedes b'' on $\mu(a, c)$.

Going from a to b' on $\mu(a, c)$, from b' to b'' through b (first down, then right), and from b'' to c on $\mu(a, c)$, we obtain a staircase P' from a to c via b within S . This staircase consists of edges and parts of edges of $\mu(a, c)$, plus two edges ($[b', b]$ and $[b, b'']$) outside $\mu(a, c)$. Thus $e(P') \leq 2 + e(\mu(a, c)) = 2 + \xi(a, c)$.

On the other hand, P' is a union of two subpaths: P'_1 (from a to b) and P'_2 (from b to c). Hence,

$$e(P') = e(P'_1) + e(P'_2) \geq \xi(a, b) + \xi(b, c),$$

ergo

$$\xi(a, b) + \xi(b, c) \leq 2 + \xi(a, c).$$

If b lies above $\mu(a, c)$, we repeat the same procedure with b' the first point of $\mu(a, c)$ going from b horizontally to the left and b'' the first point of $\mu(a, c)$ going from b vertically down. $\square$

The staircase version of the monotonicity lemma is a straightforward corollary of Proposition 2.2.

Corollary 2.4. *Let S be a staircase-connected set in $\mathbb{R}^2$. Let $x, y \in S$ and $b, c \in \mu(x, y)$ be such that b precedes c on $\mu(x, y)$. Then $\xi(x, b) \leq \xi(x, c)$.*

Fig. 4. Diameter 3 attained by both an ascending and a descending pair.

3. Ascending and Descending Sets

Definition. A pair a, b of points in $\mathbb{R}^2$ ($a = (a_1, a_2)$; $b = (b_1, b_2)$) is called **ascending** if $(a_1 - b_1)(a_2 - b_2) > 0$. The pair a, b is **descending** if $(a_1 - b_1)(a_2 - b_2) < 0$.

Example 1. The set depicted in Fig. 4 is compact and staircase connected. The pair a, c is descending with $\xi(a, c) = 3$, and the pair b, d is ascending with $\xi(b, d) = 3$.

Definition. For points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$ we write $a \leq b$ if $a_1 \leq b_1$ and $a_2 \leq b_2$. (Also, $a < b$ if $a_1 < b_1$ and $a_2 < b_2$.)

Proposition 3.1. Let $S \subset \mathbb{R}^2$ be a staircase-connected set. Let $e = (e_1, e_2)$ and $f = (f_1, f_2)$ be a descending pair of points of S . If there is a point $a \in S$ such that $a \leq e$, $a \leq f$ (or $e \leq a$, $f \leq a$), then $\xi(e, f) \leq 2$. The same holds when e, f is an ascending pair, and the point a satisfies $a_1 \geq \max(e_1, f_1)$, $a_2 \leq \min(e_2, f_2)$ (or $a_1 \leq \min(e_1, f_1)$, $a_2 \geq \max(e_2, f_2)$).

Proof. Assume without loss of generality that $a \leq e$, $a \leq f$, $e_1 < f_1$, and $e_2 > f_2$. Let P be a staircase from a to f . Let V be a vertical segment going down from e to a point $e' \in P$. Let H be a horizontal segment going from f to a point $v \in V$. Then $\langle f, v, e \rangle$ is a 2-staircase that connects f and e within S . (Clearly, $\xi(e, f) > 1$ since $e_1 \neq f_1, e_2 \neq f_2$). $\square$

Theorem 3.2. Let $S \subset \mathbb{R}^2$ be a staircase-connected set. If there is an ascending (descending) pair a, b of points in S with $\xi(a, b) \geq 4$, then for every descending (ascending) pair e, f we have $\xi(e, f) = 2$.

Proof. Assume without loss of generality that a, b is an ascending pair of points in S ($a_1 < b_1$ and $a_2 < b_2$).

Consider the cross $C = \{(x, y) \in \mathbb{R}^2: a_1 \leq x \leq b_1 \vee a_2 \leq y \leq b_2\}$. The complement of C in $\mathbb{R}^2$ is the union of four open quadrants, say **NE**, **NW**, **SE**, **SW**. If S meets **NW**,

then the vertex $c = (a_1, b_2)$ of C is in S , since S is staircase connected and orthogonally convex, and therefore $\xi(a, b) = 2$, contrary to our assumption. Thus $\mathbf{NW} \cap S = \emptyset$. By the same argument, $\mathbf{SE} \cap S = \emptyset$.

Let P be a staircase with $\xi(a, b)$ edges from a to b in S . The path P divides the horizontal strip of the cross ($\{(x, y) \in \mathbb{R}^2: a_2 \leq y \leq b_2\}$) into two closed half-strips: $\mathbf{E}$ (right of P) and $\mathbf{W}$ (left of P), with $\mathbf{E} \cap \mathbf{W} = P$. It also divides the vertical strip ($\{(x, y) \in \mathbb{R}^2: a_1 \leq x \leq b_1\}$) into two closed half-strips: $\mathbf{S}$ (below P) and $\mathbf{N}$ (above P), with $\mathbf{S} \cap \mathbf{N} = P$.

Let e, f be a descending pair of points in S ($e_1 < f_1, e_2 > f_2$). If $e \leq b$ and $f \leq b$ (or, equivalently $e, f \in \mathbf{W} \cup \mathbf{SW} \cup \mathbf{S}$), then $\xi(e, f) = 2$ by Proposition 3.1. The same holds when $a \leq e, a \leq f$ (i.e., $e, f \in \mathbf{E} \cup \mathbf{NE} \cup \mathbf{N}$).

This covers all cases when one of the points e, f does not belong to the cross C , or when both e and f belong to the same half-strip, or to adjacent half-strips ($\mathbf{E}$ and $\mathbf{N}$, or $\mathbf{W}$ and $\mathbf{S}$).

There remains the case where e and f belong to opposite half-strips ($\mathbf{E}$ and $\mathbf{W}$, or $\mathbf{N}$ and $\mathbf{S}$).

Assume, without loss of generality, that $e \in \mathbf{W}$ and $f \in \mathbf{E}$. Since they do not belong to two adjacent half-strips, none of them lies within the central rectangle $[a, d, b, c]$. Draw a horizontal segment H from e to P , and another horizontal segment H' from f to P . H meets $[a, c]$ at the point $(a_1, e_2) \in S$, and H' meets $[d, b]$ at $(b_1, f_2) \in S$. Since $f_2 < e_2 \leq b_2$, (b_1, e_2) is also in S (vertical convexity). Thus $\langle a, (a_1, e_2), (b_1, e_2), b \rangle$ is a 3-staircase connecting a to b within S , which contradicts our assumption that $\xi(a, b) \geq 4$. $\square$

In view of Theorem 3.2, we divide the staircase-connected sets satisfying $\text{stdiam}(S) = k \geq 4$ into two disjoint classes: **ascending sets** and **descending sets**. S is ascending (descending) if it includes an ascending (descending) pair a, b with $\xi(a, b) \geq 3$.

Reflection in the x -axis (or y -axis) transforms ascending sets into descending sets and vice versa.

4. Staircase Stars and Kernels

Let $S \subset \mathbb{R}^2$ be a staircase-connected set. We define the **staircase r -star** of a point z in S ($\text{sst}_r(z, S)$) to be the set of all points $y \in S$ which can be joined to z within S by an r -staircase, i.e., by a staircase of at most r edges. The intersection of all staircase r -stars is called the **staircase r -kernel** of S and denoted by $\text{Ker}_r(S)$.

Assume $\text{stdiam}(S) = k$. If $r < k/2$ then clearly $\text{Ker}_r(S) = \emptyset$. If $r = k/2$ then $\text{Ker}_r(S)$ may or may not be empty, as shown in the examples in Fig. 5.

In the first example, $k = 4, r = 2$ and $\text{Ker}_2(S)$ consists of two points, a and b . In the second example, $k = 4, r = 2$ and $\text{Ker}_2(S)$ is empty.

For $S \subset \mathbb{R}^2$ compact and staircase connected, we will show that if $r > k/2$ then $\text{Ker}_r(S) \neq \emptyset$. We will also show that if $r \geq k/2 + 1$ then $\text{Ker}_r(S)$ is again compact and staircase connected.

When $r = k/2 + 1/2$, $\text{Ker}_r(S)$ need not be staircase connected, but it is connected, in fact, connected by monotone paths (see [MP2]).

Example 2. Take a rectangle R with sides parallel to the coordinate axes, whose length

Fig. 5. Two sets of diameter 4. The second kernel is nonempty on the left, empty on the right.

is 2κ times larger than its height (κ a positive integer). Tilt R by 45° . This yields a compact staircase-connected set S with $\text{stdiam}(S) = 2\kappa + 1$. The staircase $(\kappa + 1)$ -kernel of S is just the segment that joins the midpoints of the long sides of S . Thus, $\text{Ker}_{\kappa+1}(S)$ is not staircase connected (though it is connected).

Lemma 4.1. *Let S be a staircase-connected set in $\mathbb{R}^2$, let r be any positive integer and let x be any point of S . Then $\text{sst}_r(x, S)$ is orthogonally convex.*

Proof. Suppose $[y, z]$ is a horizontal or vertical segment with endpoints in $\text{sst}_r(x, S)$. We must show that $[y, z] \subset \text{sst}_r(x, S)$.

Assume, without loss of generality, that $[y, z]$ is horizontal and that $y_1 < z_1$, $y_2 = z_2 = h$. Take a point $u = (u_1, h) \in [y, z]$. We must show that $\xi(u, x) \leq r$. If $x_1 \leq u_1$, then $\xi(x, u) \leq \xi(x, z) \leq r$ by Proposition 2.2. If $x_1 \geq u_1$, then $\xi(x, u) \leq \xi(x, y) \leq r$, again by Proposition 2.2. $\square$

A similar result, valid for simply connected orthogonal polygons, has been proved by Topala in [T]. These sets have also been investigated by Breen in [B1] and [B2].

Corollary 4.2. *Let S be a staircase-connected set in $\mathbb{R}^2$, let r be any positive integer and let x be any point of S . Then $\text{sst}_r(x, S)$ is staircase connected.*

Proof. This corollary is an easy consequence of Proposition 1.1 and Lemma 4.1. $\square$

Lemma 4.3. *Let S be a compact staircase-connected set in $\mathbb{R}^2$, let r be any positive integer and let x be any point of S . Then $\text{sst}_r(x, S)$ is compact.*

Proof. Since S is compact, it suffices to show $\text{sst}_r(x, S)$ is closed. Consider an infinite sequence of points $\{q_i\}_{i=1}^\infty$ in $\text{sst}_r(x, S)$ that converges to a point q in S . We need to show that $q \in \text{sst}_r(x, S)$, i.e., there exists a staircase path in S of at most r edges joining x and q .

Denote by L_x^h and L_x^v the horizontal and vertical lines through x , respectively. If q belongs to any of these two lines, then $[x, q] \subset S$, since S is orthogonally convex, and the claim follows. If not, then q lies in the interior of one of the four quadrants determined by L_x^h and L_x^v . Assume without loss of generality that q and $\{q_i\}_{i=1}^\infty$ are interior points of the northeast quadrant, and also that $\xi(x, q_i) = r$ for every i . If so, for every q_i there exists an

r -staircase $\mu(x, q_i) = \langle x = p_0^i, p_1^i, \dots, p_r^i = q_i \rangle$ joining x to q_i via S . Without loss of generality, assume that all the segments $[x, p_1^i]$ are horizontal, that is, the vectors induced by these segments have direction west to east. It follows that every segment $[p_j^i, p_{j+1}^i]$ with j even is horizontal, while every segment $[p_j^i, p_{j+1}^i]$ with j odd is pointing upward. By passing to a subsequence if necessary, we may assume that for every j ($0 \leq j \leq r$), the sequence $\{p_j^i\}_{i=1}^\infty$ converges to a point p_j in S . Then every sequence of horizontal (or vertical) segments $[p_j^i, p_{j+1}^i]$ will converge to the horizontal (or vertical) segment $[p_j, p_{j+1}]$ (which may degenerate to a point). The path $\langle x = p_0, \dots, p_r = q \rangle$ is a staircase path in S with at most r edges joining x and q , and our claim is proved. $\square$

Corollary 4.4. *Let S be a compact staircase-connected set in $\mathbb{R}^2$, then for every positive integer r , $\text{Ker}_r(S)$ is compact.*

We define the **staircase radius** of S ($\text{strad}(S)$) to be the smallest integer r such that $\text{Ker}_r(S) \neq \emptyset$.

Theorem 4.5. *If $S \subset \mathbb{R}^2$ is a compact staircase-connected set, and $\text{stdiam}(S) = k \geq 1$, then*

$$\text{strad}(S) = \begin{cases} k/2 \text{ or } k/2 + 1, & k \text{ even,} \\ (k+1)/2, & k \text{ odd.} \end{cases}$$

Proof. If $k = 1$ then S is a line segment (horizontal or vertical). If $k = 2$ then $\text{strad}(S)$ may be either 1 (only when S is a union of a horizontal segment and a vertical segment, with a common point) or 2.

Assume, from now on, that $k \geq 3$. Let $[a_1, c_1]$ and $[b_2, d_2]$ be the projections of S onto the first and second coordinates, respectively. (Clearly $-\infty < a_1 < c_1 < \infty$, $-\infty < b_2 < d_2 < \infty$.) $R(S) = [a_1, c_1] \times [b_2, d_2]$ is the smallest rectangle with sides parallel to the coordinate axes that includes S .

Define

$$I_1 = [(a_1, a'_2), (a_1, a_2)] = [a', a],$$

$$I_2 = [(b'_1, b_2), (b_1, b_2)] = [b', b],$$

$$I_3 = [(c_1, c_2), (c_1, c'_2)] = [c, c'],$$

$$I_4 = [(d_1, d_2), (d'_1, d_2)] = [d, d']$$

to be the intersections of S with the left edge, bottom edge, right edge and top edge of $R(S)$, respectively ($a'_2 \leq a_2, b'_1 \leq b_1, c_2 \leq c'_2, d_1 \leq d'_1$).

If $[a'_2, a_2] \cap [c_2, c'_2] \neq \emptyset$ then S has a horizontal transversal $T = [(a_1, y), (c_1, y)]$ ($y \in [a'_2, a_2] \cap [c_2, c'_2]$). In this case every point of S can be joined to T by a vertical segment in S , and therefore every point of S can be joined to any point of T by an orthogonal path with at most two edges in S . It follows that $\text{stdiam}(S) \leq 3$, $\text{strad}(S) \leq 2$. Since we assume that $k > 2$ we obtain $\text{stdiam}(S) = 3$, $\text{strad}(S) = 2$.

Assume, from now on, that $[a'_2, a_2] \cap [c_2, c'_2] = \emptyset$. Similarly, assume that $[b'_1, b_1] \cap [d_1, d'_1] = \emptyset$. Thus $a_2 < c_2$ or $a'_2 > c'_2$. Assume without loss of generality that $a_2 < c_2$

Fig. 6. Case I.

(otherwise reflect S in the y -axis). As for the horizontal sides, we have either $b_1 < d_1$ or $b'_1 > d'_1$.

Case I: $b'_1 > d'_1$. (See Fig. 6.) We show that in this case, $stdiam(S) \leq 3$ and $strad(S) \leq 2$.

Consider the points a', b' . They are a descending pair and satisfy $a' < c, b' < c$. By Proposition 3.1, they are connected in S by a 2-staircase with midvertex (b'_1, a'_2) . By the same token, we find a 2-staircase in S that connects d' to c' with midvertex (d'_1, c'_2) ; a 2-staircase in S that connects b to c with midvertex (b_1, c_2) ; and, finally, a 2-staircase in S that connects a to d with midvertex (d_1, a_2) .

These four 2-paths, together with the four (possibly degenerate) segments I_1, I_2, I_3, I_4 , bound a rectangle $D = [d_1, b_1] \times [a'_2, c'_2]$ with four rectangular protrusions, as shown in Fig. 6. This central figure, call it E , is certainly part of S .

$R(S) \setminus E$ consists of four pairwise disjoint rectangles, numbered I, II, III and IV (see Fig. 6).

We urge the reader to verify the following claims:

1. Each two points of E can be joined by a staircase path with at most three edges. Three edges are actually needed only to connect points in opposite protrusions.
2. Every point of S in one of the rectangles I, II, III, IV can be joined to any point of E by a staircase path with at most three edges.
3. Every two points in the same rectangle, say I, can be joined by a staircase path with at most two edges in S .
4. Every two points of S in adjacent rectangles (say I and II) can be joined by a staircase path with at most three edges in S .
5. Every two points of S in opposite rectangles (say I and III) can be joined by a staircase path with at most three edges in S .
6. Every point of S can be joined to any point of D by a staircase path with at most two edges in S .

It follows that $strad(S) = 2$, $stdiam(S) = 3$.

Case II: $b_1 < d_1$. As in Case I, consider the points b and c . S includes an ascending staircase path P from b to c with $e(P) = \xi(b, c)$. This path must start with a vertical edge and end with a horizontal edge. Therefore, it must have an even number of edges, say 2κ . Denote by w the middle vertex of P . We claim that every point of S (say $z = (z_1, z_2)$) can be joined to w by a staircase path of at most $\kappa + 1$ edges in S .

If $z_2 \leq c_2$, move from z horizontally until you hit P . Then continue on P towards w . This yields an orthogonal path of at most $\kappa + 1$ edges from z to w within S . If this path is not monotone, i.e., not a staircase path, then it can be shortened by suitable shortcuts until it becomes a staircase path (within S) from z to w (see Proposition 1.1).

If $z_1 \geq b_1$, we proceed in the same way, starting with a vertical move from z to P .

There remains the case $z_1 < b_1, z_2 > c_2$. The case where some point $z \in S$ satisfies this condition will be treated below.

If there are no points $z \in S$ with $z_1 < b_1, z_2 > c_2$, then $w \in \text{Ker}_{\kappa+1}(S)$, hence $\text{strad}(S) \leq \kappa + 1$. Clearly, $\text{strad}(S) \geq \frac{1}{2} \cdot \text{stdiam}(S) \geq \frac{1}{2} \cdot \xi(b, c) = \kappa$. Therefore, the possible values for radius and diameter are:

$$\begin{array}{ll} \text{strad}(S) = \kappa, & \text{stdiam}(S) = 2\kappa, \\ \text{strad}(S) = \kappa + 1, & \text{stdiam}(S) = 2\kappa, \\ \text{strad}(S) = \kappa + 1, & \text{stdiam}(S) = 2\kappa + 1, \\ \text{strad}(S) = \kappa + 1, & \text{stdiam}(S) = 2\kappa + 2, \end{array}$$

and our claim holds.

If there is a point $z \in S$ with $z_1 < b_1, z_2 > c_2$, then, by Proposition 3.1, $\xi(b, c) = 2\kappa = 2$, i.e., $\kappa = 1$. Therefore, $w = (b_1, c_2)$.

Applying the same proposition again to the descending pair z, w (with $z \leq d, w \leq d$), we find that $\xi(z, w) = 2$. Thus $w \in \text{Ker}_2(S)$, hence $\text{strad}(S) = 2$. In this case, $\text{stdiam}(S)$ may be 3 or 4, as shown in Fig. 7. ($\text{stdiam}(S) > 2$ since $\xi(a, c) = 3$.) $\square$

If we drop the compactness of S in the assumptions of Theorem 4.5, then $\text{Ker}_{(k+1)/2}(S)$ may be empty, as shown in the following example:

Fig. 7

Example 3. Consider the tilted rectangle R from Example 2. Erase from its boundary both (closed) long sides, and one of the short sides. This yields a set $S \subset \mathbb{R}^2$ with $\text{stdiam}(S) = 2\kappa + 1$ and whose staircase $(\kappa + 1)$ -kernel is empty.

The correct version of Theorem 4.5 without compactness is as follows:

Theorem 4.6. If $S \subset \mathbb{R}^2$ is a staircase-connected set, $\text{stdiam}(S) = k \geq 1$, then

$$\text{strad}(S) = \begin{cases} k/2 \text{ or } k/2 + 1, & k \text{ even,} \\ (k+1)/2 \text{ or } (k+3)/2, & k \text{ odd.} \end{cases}$$

Proof. The inequalities $\text{strad}(S) \leq \text{stdiam}(S) \leq 2 \cdot \text{strad}(S)$ hold without assuming compactness. Thus all we have to show is that

$$\text{strad}(S) \leq \frac{1}{2} \cdot \text{stdiam}(S) + \frac{3}{2}. \quad (*)$$

This is trivial for $\text{stdiam}(S) = k \leq 3$. Assume therefore $k \geq 4$ and suppose, without loss of generality, that S is ascending.

Define, for any point $y \in S$,

$$\alpha(y) = \max\{\xi(x, y) : x \in S, x \leq y\},$$

$$\beta(y) = \max\{\xi(y, z) : z \in S, y \leq z\}.$$

These functions have the following properties, whose proofs are straightforward:

- (A) α is nondecreasing; β is nonincreasing. That is, if $y, y' \in S$, $y \leq y'$, then $\alpha(y) \leq \alpha(y')$ and $\beta(y') \leq \beta(y)$.
- (B) If $y, y' \in S$, $y \leq y'$ and the segment $[y, y']$ is either horizontal or vertical, then $\alpha(y') \leq \max\{2, 1 + \alpha(y)\}$. Thus $\alpha(y') \leq 1 + \alpha(y)$ provided $\alpha(y) \geq 1$. Similarly, $\beta(y) \leq \max\{2, 1 + \beta(y')\}$.
- (C) For all $y \in S$, $\alpha(y) + \beta(y) \leq 2 + \text{stdiam}(S)$, by Corollary 2.3.
- (D) If $\text{stdiam}(S) = k \geq 4$ and S is ascending, then $\text{strad}(S) \leq r$ iff $\alpha(y) \leq r$ and $\beta(y) \leq r$ for some $y \in S$.

Now we turn to the proof of (*). Choose two points $a, b \in S$ such that $\xi(a, b) = \text{stdiam}(S) = k$. Since S is ascending, we may assume $a \leq b$. Let $\langle a = x_0, \dots, x_k = b \rangle$ be a k -staircase in S from a to b . Note that $\alpha(x_0) \leq 2$, $\beta(x_0) = k \geq 4$, $\alpha(x_k) = k \geq 4$, $\beta(x_k) \leq 2$. Let i be the last index for which $\alpha(x_i) \leq \beta(x_i)$. Thus $\alpha(x_{i+1}) > \beta(x_{i+1})$.

If (*) is false, then, by (D), $\beta(x_i) \geq \frac{1}{2}k + 2$ and therefore, by (C),

$$\alpha(x_i) \leq \frac{1}{2}k. \quad (**)$$

By the same token, if (*) is false then

$$\alpha(x_{i+1}) \geq \frac{1}{2}k + 2 \geq 4. \quad (***)$$

Inequalities (**) and (***) together contradict (B). $\square$

Theorem 4.7. *Let $S \subset \mathbb{R}^2$ be orthogonally convex, and let $\text{stdiam}(S) = k$. Assume $k/2 + 1 \leq r \leq k$, and suppose $a, c \in \text{Ker}_r(S)$. Then:*

1. *If $\xi(a, c) \neq 2$, then every point b of S that lies in the rectangle determined by a and c also belongs to $\text{Ker}_r(S)$. (In particular, every staircase from a to c within S lies in $\text{Ker}_r(S)$.)*
2. *If $\xi(a, c) = 2$ then there exists a 2-staircase $v(a, c)$ from a to c in $\text{Ker}_r(S)$.*

Note that if $r = k$, then $\text{Ker}_r(S) = S$ and the theorem does not say anything new. We may therefore assume that $k/2 + 1 \leq r < k$. This condition implies $k \geq 4$ and $r \geq 3$.

Proof. Since $k \geq 4$, we may assume without loss of generality that the set S is ascending (Theorem 3.2).

(1) If $\xi(a, c) = 1$ then $b \in \text{Ker}_r(S)$ by Lemma 4.1. So we may assume $\xi(a, c) \geq 3$, and therefore a, c is an ascending pair. If $x \in S$ is southeast or northwest of b , then $\xi(b, x) \leq 2$, since S is ascending. If x is northeast of b , then $\xi(b, x) \leq \xi(a, x) \leq r$, by the rectangle property (Proposition 2.2). If x is southwest of b , then $\xi(b, x) \leq \xi(c, x) \leq r$, again by the rectangle property.

(2) Let $\langle a, q, c \rangle$ be a 2-staircase path in S from a to c . If the pair a, c is ascending, then $q \in \text{Ker}_r(S)$, and therefore $\langle a, q, c \rangle \subset \text{Ker}_r(S)$. This is proved exactly as in (1).

Assume then that the pair a, c is descending (with c southeast of a), and assume without loss of generality that q is the intersection point of the lines L_a^h (horizontal through a) and L_c^v (vertical through c). If $q \in \text{Ker}_r(S)$, we are done. If $q \notin \text{Ker}_r(S)$, then there exists a point q' in S such that $\xi(q, q') > r$. If q' lies above L_a^h (including L_a^h) then $\xi(q', q) \leq \xi(q', c) \leq r$. If q' lies to the right of L_c^v (including L_c^v) then $\xi(q', q) \leq \xi(q', a) \leq r$. If q' lies between L_a^h and L_c^v (including both lines) then $\xi(q', q) \leq 2 \leq r$. The same holds when q' lies between L_a^v and L_c^h (including both lines).

We conclude that q' is strictly southwest of p (the intersection point of L_a^v and L_c^h). This implies that $p \in S$ as well (since S is staircase connected) and therefore the whole rectangle $[a, p, c, q]$ is in S .

If $p \in \text{Ker}_r(S)$, we are done. Otherwise there is a point $p' \in S$ with $\xi(p, p') > r$. Repeating the previous argument, we find that p' is strictly northeast of q .

Here $\xi(q, q') > r \geq \xi(q', a)$. Since a and q are adjacent, we find that $\xi(q, q') = r + 1$, $\xi(q', a) = r$. Similarly $\xi(q', c) = r$, $\xi(p, p') = r + 1$, $\xi(p', a) = r$ and $\xi(p', c) = r$.

Now consider the distance $\xi(p', q')$. Since a and c lie between p' and q' (i.e., a and c belong to the rectangle determined by p' and q'), we find that

$$k \geq \xi(p', q') \geq \xi(p', a) + \xi(a, q') - 2 = 2r - 2 \quad (\otimes)$$

by Corollary 2.3.

In the theorem we assume that $k/2 + 1 \leq r$, hence $k \leq 2r - 2$. Thus $k = 2r - 2$ and the inequalities in $(\otimes)$ hold as equalities.

Let us draw a k -staircase Q from q' to p' . Q starts with a horizontal edge and ends with a vertical edge, or vice versa. Both cases are essentially the same. In Fig. 8, the first edge is horizontal and r is odd (actually, $r = 7$). If r is even then the points z and w in Fig. 8 exchange roles, and some of the wording below must be adjusted accordingly.

Denote the vertices of Q by $(q' =) q'_0, q'_1, \dots, q'_{r-2}, z, p'_{r-2}, \dots, p'_0 (= p')$. If a lies on the line L_z^v or to its left, then $\xi(q', a) \leq r - 1$. If a lies on, or below the line L_w^h , then $\xi(q', a) \leq r - 2$. If a lies on, or above the line L_z^h , then $\xi(p', a) \leq r - 1$. If a lies on the line L_v^v or to its right, then $\xi(p', a) \leq r - 2$. The same holds for c instead of a .

The following theorem is an immediate consequence of Theorem 4.7:

Acknowledgements

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