A Note on Euclidean Ramsey Theory

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Abstract. Suppose $n \geq 2$ and $X, Y \subseteq \mathbb{R}^n$ are such that X is infinite and Y is the set of vertices of an n -simplex. Then there is blue/red coloring of $\mathbb{R}^n$ such that no set similar to X is monochromatically blue and no set similar to Y is monochromatically red.

Consider some finite-dimensional Euclidean space $\mathbb{R}^n$. If $X_0, X_1, \dots, X_{k-1} \subseteq \mathbb{R}^n$, then we write

$$\mathbb{R}^n \longrightarrow (X_0, X_1, \dots, X_{k-1})$$

if for any function $\psi: \mathbb{R}^n \longrightarrow k$ there are $i < k$ and $H \subseteq \mathbb{R}^n$ such that H is similar to X_i and ψ is constantly i on H . If $\mathbb{R}^n \longrightarrow (X_0, X_1, \dots, X_{k-1})$ is false, then we write $\mathbb{R}^n \not\rightarrow (X_0, X_1, \dots, X_{k-1})$.

The following theorem, but with the stronger conclusion with “homothetic” replacing “similar”, is the Gallai–Witt Theorem, proved independently by Gallai (see [6]¹) and Witt [9].

Theorem A. *If $X_0, X_1, \dots, X_{k-1} \subseteq \mathbb{R}^n$ are finite, then $\mathbb{R}^n \longrightarrow (X_0, X_1, \dots, X_{k-1})$.*

For infinite sets, Theorem A fails as shown by the following easy consequence of the much stronger Theorem 19 of [4].

Theorem B. *If $X \subseteq \mathbb{R}^n$ is infinite, then $\mathbb{R}^n \not\rightarrow (X, X)$.*

¹ The result of Gallai (Grünwald) as quoted on page 123 of [6] is restricted to lattice points. However, on page 130 of [6] there is stated and proved a theorem that is slightly stronger than even the homothetic version of Theorem A.

Theorem B shows that the symmetric 2-color version of Theorem A fails for an infinite set. Similar arguments can be used to show the failure of the asymmetric version; specifically, if $X, Y \subseteq \mathbb{R}^n$ are infinite, then $\mathbb{R}^n \not\rightarrow (X, Y)$. Baumgartner [1] proved the following theorem showing the asymmetric version can fail even if only one of the sets is infinite.

Theorem C. *If $X, Y \subseteq \mathbb{R}^n$ are such that X is an infinite arithmetic progression and Y is a three-element arithmetic progression, then $\mathbb{R}^n \not\rightarrow (X, Y)$.*

An *affine subspace* of $\mathbb{R}^n$ is a translate of a subspace of $\mathbb{R}^n$. For example, a line is just a one-dimensional affine subspace. If $Y \subseteq \mathbb{R}^n$, then let $\text{aff}(Y)$ be the affine subspace generated by Y . In general, $|Y| \geq \dim(\text{aff}(Y)) + 1$, but if $|Y| = \dim(\text{aff}(Y)) + 1$, then Y is (*affinely*) *independent*, and if $|Y| > \dim(\text{aff}(Y)) + 1$, then Y is *dependent*. For example, every three-element arithmetic progression is dependent. If $Y \subseteq \mathbb{R}^n$ and $k < \omega$, then Y is (the set of vertices of) a k -simplex iff $|Y| = k + 1$ and Y is independent.

The following theorem is the main new result.

Theorem D. *If $n \geq 2$ and $X, Y \subseteq \mathbb{R}^n$ are such that X is infinite and Y is an n -simplex, then $\mathbb{R}^n \not\rightarrow (X, Y)$.*

Theorem D answers a question of Komjáth [5], who proved the special case in which $n = 2$. His proof was along the lines of Ceder's proof in [2] and made use of Theorem C. As pointed out by Komjáth [5], Theorem D is, in some sense, an optimal result as shown by the following proposition which is easily proved by induction.

Proposition E. *If $X, Y \subseteq \mathbb{R}^n$ are such that X is a circle, Y is independent, and $|Y| \leq n$, then $\mathbb{R}^n \rightarrow (X, Y)$.*

It would be interesting to have a characterization of those countable subsets $X, Y \subseteq \mathbb{R}^n$ for which $\mathbb{R}^n \rightarrow (X, Y)$.

1. Preliminaries

We use ω to denote the set of natural numbers, and we systematically write $n < \omega$ instead of $n \in \omega$. As usual, each natural number is the set of smaller natural numbers; that is, $n = \{0, 1, \dots, n-1\}$. Throughout this paper we consider a fixed n -dimensional Euclidean space $\mathbb{R}^n$, where $2 \leq n < \omega$. For $x \in \mathbb{R}^n$, we let $\|x\|$ be the usual Euclidean norm of x . If $X \subseteq \mathbb{R}$, then we write $X = \{x_0, x_1, x_2, \dots, x_{m-1}\}_{<}$ to indicate that $x_0 < x_1 < x_2 < \dots < x_{m-1}$. We use the notational convention that if we consider a point in $\mathbb{R}^m$, say $t \in \mathbb{R}^m$, then we let $t = (t_0, t_1, \dots, t_{m-1})$.

We always have in mind, for this paper, a fixed countable, real-closed subfield $\mathbb{F} \subseteq \mathbb{R}$ and also a transcendence basis $\mathcal{T}$ for $\mathbb{R}$ over $\mathbb{F}$. (A transcendence basis can easily be shown to exist since we are assuming the Axiom of Choice.) If $D \subseteq \mathbb{R}$, then we say that the k -ary relation $R \subseteq \mathbb{R}^k$ is *D-definable* if it is definable by a first-order formula in the ordered field $(\mathbb{R}, +, \times, 0, 1, <)$ using parameters just from $D \cup \mathbb{F}$. If R is $\emptyset$ -

definable, then we say simply that it is *definable*. If $B \subseteq \mathbb{R}^m$ and we write something like $f: B \rightarrow \mathbb{R}$, then we understand that f is a function whose domain $\text{dom}(f) = B$. Therefore, if f is D -definable, then its domain B must also be D -definable. Notice that R is semi-algebraic iff it is $\mathbb{R}$ -definable.

Definable analytic functions are crucial to our method. The properties that we need about these functions are basic properties that follow from the o-minimality of $\mathbb{R}$, which is a consequence of the Tarski–Seidenberg Theorem on the elimination of quantifiers. The book by van den Dries [3] is a suggested reference.

The following basic facts about definable analytic functions are crucial for our method. Proofs of the following results can be found in [8].

Proposition 1.1. *Suppose $f: U \rightarrow \mathbb{R}$ is a definable analytic function, where $U \subseteq \mathbb{R}^m$ is an open connected set. Let $t_0, t_1, t_2, \dots, t_{m-1} \in \mathcal{T}$ be distinct such that $t \in U$ and $f(t) = 0$. Then f is identically 0 on U .*

Corollary 1.2. *Suppose $f: U \rightarrow \mathbb{R}$ is a definable analytic function, where $U \subseteq \mathbb{R}^m$ is an open convex set. Let $t_0, t_1, t_2, \dots, t_{m-1} \in \mathcal{T}$ (not necessarily distinct) be such that $t \in U$ and $f(t) = 0$. Let $u \in U$ be such that whenever $i < j < m$ and $t_i = t_j$, then $u_i = u_j$. Then $f(u) = 0$.*

Proposition 1.3. *Suppose $f: U \rightarrow \mathbb{R}$ is a definable function, where $U \subseteq \mathbb{R}^m$ is open, and $t_0, t_1, t_2, \dots, t_{m-1} \in \mathcal{T}$ are distinct such that $t \in U$. Then there is an open neighborhood V of t such that $f|_V$ is an analytic function.*

If $X \subseteq \mathbb{R}^n$, then a function $\alpha: X \rightarrow \mathbb{R}$ is a *similarity* if there is a positive $c \in \mathbb{R}$ such that whenever $x, y \in X$, then $\|\alpha(x) - \alpha(y)\| = c\|x - y\|$. The number c (which is unique as long as $|X| \geq 2$) is the *scaling factor* of α .

A function on $\mathbb{R}^n$ will sometimes be referred to as a coloring with the concomitant terminology. For example, subsets of $\mathbb{R}^n$ on which the coloring is constant are referred to as *monochromatic*. With this terminology, $\mathbb{R}^n \rightarrow (X, Y)$ means that whenever ψ is a 2-coloring of $\mathbb{R}^n$, then either there is a subset $X' \subseteq \mathbb{R}^n$ similar to X on which χ is monochromatically 0 or there is a subset $Y' \subseteq \mathbb{R}^n$ similar to Y on which ψ is monochromatically 1. A 2-coloring ψ *confirms* that $\mathbb{R}^n \not\rightarrow (X, Y)$ if, whenever $X', Y' \subseteq \mathbb{R}^n$ are similar to X, Y respectively, then ψ is neither monochromatically 0 on X' nor monochromatically 1 on Y' .

2. Acceptable Colorings

In this section we continue the assumptions of the previous section that $2 \leq n < \omega$, $\mathbb{F} \subseteq \mathbb{R}$ is a countable, real-closed field, and $\mathcal{T} \subseteq \mathbb{R}$ is a transcendence basis for $\mathbb{R}$ over $\mathbb{F}$.

It is known [7] that there is a countable coloring of $\mathbb{R}^n$ for which there are no monochromatic equilateral triangles. (For $n = 2$, this had been previously proved by Ceder [2].) The proof is easily adapted to a proof of the stronger result that for any given countable set of triangles, there is a countable coloring of $\mathbb{R}^n$ that has no monochromatic trian-

gle similar to any in the countable set. The result in [7] was subsequently improved in [8] to isosceles triangles: there is a countable coloring of $\mathbb{R}^n$ for which there are no monochromatic isosceles triangles. A somewhat stronger result is Lemma 2.1.

If $R \subseteq \mathbb{R}^k$ is $\mathbb{R}$ -definable, then there is a unique, minimal subset $T \subseteq \mathcal{T}$ such that R is T -definable; we call this set T the *support* of R and denote it by $\text{supp}(R)$. If $a \in \mathbb{R}^k$, then we usually write $\text{supp}(a)$ instead of $\text{supp}(\{a\})$. Notice that if $a = (a_0, a_1, \dots, a_{k-1}) \in \mathbb{R}^k$, then $\text{supp}(a) = \text{supp}(a_0) \cup \text{supp}(a_1) \cup \dots \cup \text{supp}(a_{k-1})$.

If $a \in \mathbb{R}^k$ and $\text{supp}(a) = \{t_0, t_1, \dots, t_{m-1}\}_{<}$, then there is a definable open neighborhood $U \subseteq \mathbb{R}^m$ of $(t_0, t_1, \dots, t_{m-1})$ and a definable analytic function $f: U \rightarrow \mathbb{R}^k$ such that $f(t_0, t_1, \dots, t_{m-1}) = a$.

We wish to place a restriction on the domains of the analytic functions. If $I, J \subseteq \mathbb{R}$ are intervals, then we write $I < J$ if $x < y$ whenever $x \in I$ and $y \in J$. If $k < \omega$, then we say that the interval $I \subseteq \mathbb{R}$ is a *dyadic* interval with *norm* k if there is an integer i such that I is the bounded, open interval $(i/2^k, (i+1)/2^k)$. If $m < \omega$ and $B \subseteq \mathbb{R}^m$, then B is an *m-box* with *norm* k if there are dyadic intervals $I_0 < I_1 < \dots < I_{m-1}$, each having norm k , such that $B = I_0 \times I_1 \times \dots \times I_{m-1}$. If B is an *m-box* for some $m < \omega$, then B is a *box*. We let $\mathcal{C}$ be the set of all definable analytic functions $f: B \rightarrow \mathbb{R}^n$, where B is a box. Clearly, $\mathcal{C}$ is countable. The *norm* of a function in $\mathcal{C}$ is the norm of its domain.

There is a small technical point that needs clarification. If $m = 0$, then $B = \{\emptyset\}$ is the unique *m-box*. For our purposes, the only definable analytic functions $f: B \rightarrow \mathbb{R}^n$, where B is the 0-box, are those for which $f: B \rightarrow \mathbb{F}^n$. Since the norm of this box B is not well defined, we conventionally let $\nu(B) = -\infty$.

If X is either a dyadic interval, a box, or a function in $\mathcal{C}$, then we let $\nu(X)$ be its norm.

If $a \in \mathbb{R}^n$, then f is an *acceptable color for a* if $f \in \mathcal{C}$, $\text{supp}(a) = \{t_0, t_1, \dots, t_{m-1}\}_{<}$, $(t_0, t_1, \dots, t_{m-1}) \in \text{dom}(f)$, and $f(t_0, t_1, \dots, t_{m-1}) = a$. A function $\chi: \mathbb{R}^n \rightarrow \mathcal{C}$ is an *acceptable coloring* if, whenever $a \in \mathbb{R}^n$, then $\chi(a)$ is an acceptable color for a .

For each $a \in \mathbb{R}^n$, it is always possible to find an acceptable color for a . Moreover, there are infinitely many acceptable colors for a . For, if $f \in \mathcal{C}$ is an acceptable color for a , where $\nu(f) = k$, then for each $\ell \geq k$, there is an acceptable color $g \in \mathcal{C}$ such that $\nu(g) = \ell$. (In fact, for each $\ell \geq k$ there is exactly one g that is acceptable for a such that $\nu(g) = \ell$, and it must be that $g \subseteq f$.) Therefore, there are many acceptable colorings.

Lemma 2.1. *Let $\chi: \mathbb{R}^n \rightarrow \mathcal{C}$ be an acceptable coloring. Suppose $a, b, c \in \mathbb{R}^n$ are distinct points and $r \in \mathbb{F}$ is such that $\|a - b\| = r\|b - c\|$. Then $\chi(a) \neq \chi(b)$ or $\chi(b) \neq \chi(c)$.*

Proof. Suppose $\chi(a) = \chi(b) = \chi(c) = f$, where $\text{dom}(f) = B \subseteq \mathbb{R}^m$. Let $F: B^3 \rightarrow \mathbb{R}$ be the definable analytic function

$$F(x, y, z) = \|f(x) - f(y)\|^2 - r^2 \|f(y) - f(z)\|^2.$$

Thus, $F(x, y, z) = 0$ iff $\|f(x) - f(y)\| = r\|f(y) - f(z)\|$. Let

$$\text{supp}(a) = \{t_0, t_1, \dots, t_{m-1}\}_{<},$$

$$\text{supp}(b) = \{u_0, u_1, \dots, u_{m-1}\}_{<},$$

$$\text{supp}(c) = \{v_0, v_1, \dots, v_{m-1}\}_{<}.$$

Hence, $f(t) = a$, $f(u) = b$, and $f(v) = c$. Then $F(t, u, v) = 0$. Since $a \neq c$, there is $j < m$ such that $t_j \neq v_j$ and, therefore, $t_j \neq u_j$ or $u_j \neq v_j$. Without loss of generality, we assume that $t_j \neq u_j$. Let $w \in \mathbb{R}^m$ agree with t except that $w_j = u_j$. (This means $w_i = t_i$ if $j \neq i < m$ and $w_i = u_i$ if $i = j$.) Then $w \in B$, so it follows from Corollary 1.2 that $F(t, w, w) = 0$. Thus, $\|a - f(w)\| = r\|f(w) - f(w)\| = 0$. Therefore, $a = f(w)$ and $t_j \notin \{w_0, w_1, \dots, w_{m-1}\} \subseteq T$, contradicting that $t_j \in \text{supp}(a)$. $\square$

We say that a subset $\mathcal{C}_0 \subseteq \mathcal{C}$ is *thin* if, for each $\ell < \omega$, $\{f \in \mathcal{C}_0 : v(f) = \ell\}$ is finite.

Lemma 2.2. *Let $\chi: \mathbb{R}^n \rightarrow \mathcal{C}$ be an acceptable coloring. Let $f_0, f_1, \dots, f_{k-1} \in \mathcal{C}$, and let $F: (\mathbb{R}^n)^k \rightarrow \mathbb{R}^n$ be a definable analytic function. Then there is a thin $\mathcal{C}_0 \subseteq \mathcal{C}$ such that whenever $a_0, a_1, \dots, a_{k-1} \in \mathbb{R}^n$ are such that $\chi(a_i) = f_i$ for each $i < k$, then $\chi(F(a_0, a_1, \dots, a_{k-1})) \in \mathcal{C}_0$.*

Proof. For each $i < k$, let $\text{dom}(f_i) = B_i = I_{i0} \times I_{i1} \times \dots \times I_{i_{m_i-1}} \subseteq \mathbb{R}^{m_i}$. Let $m = m_0 + m_1 + \dots + m_{k-1}$, and then let $B = B_0 \times B_1 \times \dots \times B_{k-1} \subseteq \mathbb{R}^m$. Let $G: B \rightarrow \mathbb{R}^n$ be the definable analytic function such that

$$G(x_0, x_1, \dots, x_{k-1}) = F(f_0(x_0), f_1(x_1), \dots, f_{k-1}(x_{k-1}))$$

whenever $(x_0, x_1, \dots, x_{k-1}) \in B$.

Let $\mathcal{G}$ be the smallest set of functions $H: J_0 \times J_1 \times \dots \times J_{\ell-1} \rightarrow \mathbb{R}$, with each J_i being a dyadic interval, such that $G \in \mathcal{G}$ and $\mathcal{G}$ has the closure properties that $H' \in \mathcal{G}$ whenever $H \in \mathcal{G}$ and one of the following holds:

- (1) H' is obtained from H by permuting the coordinates in the domain.
- (2) $\text{dom}(H) = J_0 \times J_1 \times \dots \times J_\ell$, J' is one of the halves of J_ℓ , $\text{dom}(H') = J_0 \times J_1 \times \dots \times J_{\ell-1} \times J'$, and $H' \subseteq H$.
- (3) $\text{dom}(H) = J_0 \times J_1 \times \dots \times J_\ell$, $\text{dom}(H') = J_0 \times J_1 \times \dots \times J_{\ell-1}$, q is the midpoint of J_ℓ , and

$$H'(x_0, x_1, \dots, x_{\ell-1}) = H(x_0, x_1, \dots, x_{\ell-1}, q).$$

- (4) $\text{dom}(H) = J_0 \times J_1 \times \dots \times J_\ell$, $\text{dom}(H') = J_0 \times J_1 \times \dots \times J_{\ell-1}$, $J_\ell = J_{\ell-1}$ and $H'(x_0, x_1, \dots, x_{\ell-1}) = H(x_0, x_1, \dots, x_{\ell-2}, x_{\ell-1}, x_{\ell-1})$.

Let $\mathcal{C}_0 = \mathcal{G} \cap \mathcal{C}$. It is easily checked that $\mathcal{C}_0$ is thin.

It can be verified that whenever $a_0, a_1, \dots, a_{k-1} \in \mathbb{R}^n$ are such that $\chi(a_i) = f_i$ for each $i < k$, then $\chi(F(a_0, a_1, \dots, a_{k-1})) \in \mathcal{C}_0$. Modulo some details, this completes the proof. $\square$

Two special cases of this lemma are worth noting, each requiring a lemma. (The second of these special cases will be used in the proof of Theorem D in the last section. Although the first will not be used in the proof of Theorem D, its lemma will be.)

Lemma 2.3. *Let $Y = \{y_0, y_1, \dots, y_n\} \subseteq \mathbb{F}^n$ be an n -simplex, and let $a \in \mathbb{F}^n$. Then there is a definable analytic function $F: (\mathbb{R}^n)^{n+1} \rightarrow \mathbb{R}^n$ such that whenever $\alpha: \mathbb{R}^n \rightarrow$*

$\mathbb{R}^n$ is a similarity, then

$$\alpha(a) = F(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_n)).$$

Proof. There are uniquely determined $r_0, r_1, \dots, r_n \in \mathbb{R}$ such that $r_0 + r_1 + \dots + r_n = 1$ and $a = r_0 y_0 + r_1 y_1 + \dots + r_n y_n$. Let $F: (\mathbb{R}^n)^{n+1} \rightarrow \mathbb{R}^n$ be such that whenever $x_0, x_1, \dots, x_n \in \mathbb{R}^n$, then

$$F(x_0, x_1, \dots, x_n) = r_0 x_0 + r_1 x_1 + \dots + r_n x_n.$$

Then F is definable since each $r_i \in \mathbb{F}$. Clearly, F is as required. $\square$

Corollary 2.4. *Let $\chi: \mathbb{R}^n \rightarrow \mathcal{C}$ be an acceptable coloring. Let $\{y_0, y_1, \dots, y_n\} \subseteq \mathbb{F}^n$ be an n -simplex, let $a \in \mathbb{F}^n$, and let $f_0, f_1, \dots, f_n \in \mathcal{C}$. Then there is a thin $\mathcal{C}_0 \subseteq \mathcal{C}$ such that whenever $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity such that $\chi(\alpha(y_i)) = f_i$ for $i \leq n$, then $\chi(\alpha(a)) \in \mathcal{C}_0$.*

Lemma 2.5. *Let $\{y_0, y_1, \dots, y_n\} \subseteq \mathbb{F}^n$ be an n -simplex. Then there are definable analytic functions $G, G': (\mathbb{R}^n)^n \rightarrow \mathbb{R}^n$ such that whenever $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity, then*

$$\alpha(y_n) = G(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_{n-1}))$$

or

$$\alpha(y_n) = G'(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_{n-1})).$$

Proof. Let $N: (\mathbb{R}^n)^{n-1} \rightarrow \mathbb{R}^n$ be the n -dimensional analogue of the cross-product.² Then N is a definable analytic function such that if $z_1, z_2, \dots, z_{n-1} \in \mathbb{R}^n$ are linearly independent vectors, then $N(z_1, z_2, \dots, z_{n-1})$ is a nonzero vector that is orthogonal to each of $z_1, z_2, \dots, z_{n-1}$, and if $z_1, z_2, \dots, z_{n-1}$ are linearly dependent, then $N(z_1, z_2, \dots, z_{n-1})$ is the zero vector. Let $r_1, r_2, \dots, r_n \in \mathbb{R}$ be such that $y_n - y_0 = r_1(y_1 - y_0) + r_2(y_2 - y_0) + \dots + r_{n-1}(y_{n-1} - y_0) + r_n N(y_1 - y_0, y_2 - y_0, \dots, y_{n-1} - y_0)$. Clearly, each $r_i \in \mathbb{F}$. Now we define G, G' so that

$$G(x_0, x_1, \dots, x_{n-1}) = r_1 z_1 + r_2 z_2 + \dots + r_{n-1} z_{n-1} + r_n N(z_1, z_2, \dots, z_{n-1}) + x_0,$$

and

$$G'(x_0, x_1, \dots, x_{n-1}) = r_1 z_1 + r_2 z_2 + \dots + r_{n-1} z_{n-1} - r_n N(z_1, z_2, \dots, z_{n-1}) + x_0,$$

where $z_i = x_i - x_0$.

It can be checked that G, G' are as required. $\square$

² To be definitive, suppose $z_1, z_2, \dots, z_{n-1} \in \mathbb{R}^n$. Let M be the $(n-1) \times n$ matrix whose i th row is z_i . For $1 \leq j \leq n$, let M_j be the $(n-1) \times (n-1)$ submatrix of M obtained by deleting its j th column. Then let $N(z_1, z_2, \dots, z_{n-1})$ be the vector in $\mathbb{R}^n$ whose j th coordinate is $(-1)^{j+1} \det(M_j)$.

Corollary 2.6. *Let $\chi: \mathbb{R}^n \rightarrow \mathcal{C}$ be an acceptable coloring. Let $\{y_0, y_1, \dots, y_n\} \subseteq \mathbb{F}^n$ be an n -simplex, and let $f_0, f_1, \dots, f_{n-1} \in \mathcal{C}$. Then there is a thin $\mathcal{C}_1 \subseteq \mathcal{C}$ such that whenever $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity such that $\chi(\alpha(y_i)) = f_i$ for $i < n$, then $\chi(\alpha(y_n)) \in \mathcal{C}_1$.*

3. Preferred Colorings

In this section we continue with the assumptions of the previous sections that $\mathbb{F}$ is a countable real-closed subfield of $\mathbb{R}$ and $\mathcal{T}$ is a transcendence basis for $\mathbb{R}$ over $\mathbb{F}$. Some definitions are needed. A subset $S \subseteq \mathbb{R}^n$ is an $(n-1)$ -sphere with radius r and center c if $0 < r \in \mathbb{R}$, $c \in \mathbb{R}^n$ and $S = \{x \in \mathbb{R}^n: \|x - c\| = r\}$. More generally, if $k < n$, then S is a k -sphere with radius r and center c if $c \in \text{aff}(S)$ and there is an $(n-1)$ -sphere S' with radius r and center c such that $S = S' \cap \text{aff}(S)$ and $\dim(\text{aff}(S)) = k+1$. By convention, we let $\mathbb{R}^n$ be the (unique) n -sphere; we arbitrarily let the origin be its center and 1 its radius.

If S is a k -sphere for some $k \leq n$, then S is a *sphere*. A *circle* is a 1-sphere. If S is a sphere, then its *axis* is $\{a \in \mathbb{R}^n: \forall x, y \in S, \|x - a\| = \|y - a\|\}$. If $k < n$ and S is a sphere, then S is a k -sphere iff its axis is an $(n-k-1)$ -dimensional affine subspace.

Now consider some $a \in \mathbb{R}^n$. Let $\text{supp}(a) = T = \{t_0, t_1, \dots, t_{m-1}\}_<$, and then, for each $i < m$, let $T_i = T \setminus \{t_i\}$. Let S_i be the intersection of all T_i -definable spheres S such that $a \in S$. Since a is not T_i -definable, there is d such that $1 \leq d \leq n$ and S_i is a T_i -definable d -sphere. We refer to S_i as the i th *definable sphere* of a . Notice that it is possible that $d = n$.

Clearly (or by the Hilbert Basis Theorem), the i th definable sphere is T_i -definable. Hence, it is T -definable. There is a uniform way to define it, which we now describe. Fix some acceptable color $f: B \rightarrow \mathbb{R}^n$, where $B \subseteq \mathbb{R}^m$, and also fix some $i < m$. Let $\theta_{f,i}(u, x, v)$ be an $(m+n+1)$ -ary formula which asserts: $u \in B$, $v > 0$ and x is in the smallest sphere containing $\{f(u_0, u_1, \dots, u_{i-1}, u, u_{i+1}, \dots, u_{m-1}): \|u - u_i\| < v\}$. Then let $\varphi_{f,i}(u, x)$ be an $(m+n)$ -ary formula asserting that $u \in B$ and x is such that, for every $v > 0$, x is in the sphere defined by $\theta_{f,i}(u, x, v)$.

It is clear that if $t \in B$, then $\varphi_{f,i}(t, x)$ defines a sphere. Furthermore, if $a \in \mathbb{R}^n$, $f \in \mathcal{C}$ is an acceptable color for a , and $\text{supp}(a) = \{t_0, t_1, \dots, t_{m-1}\}$, then $\varphi_{f,i}(t, x)$ defines the i th definable sphere of a . To see this last statement, let S_i be the i th definable sphere of a , and let S'_i be the sphere defined by $\varphi_{f,i}(t, x)$. Clearly, $S_i \subseteq S'_i$. Now let $F: B \rightarrow \mathbb{R}$ be the function where $F(u) = \inf\{\|f(u) - x\|^2: x \in S_i\}$. Then F is a T_i -definable analytic function and $F(t) = 0$. It then follows from Corollary 1.2 that $F(t') = 0$ whenever t' agrees with t except that perhaps $t'_i \neq t_i$. This implies that $S'_i \subseteq S_i$. Therefore, we have proved that $S_i = S'_i$.

Suppose that $f: B \rightarrow \mathbb{R}^n$ is an acceptable color, where $B \subseteq \mathbb{R}^m$. Making use of the formulas $\varphi_{f,i}(u, x)$, we define the functions $d_{f,i}: B \rightarrow \omega$, and $r_{f,i}: B \rightarrow \mathbb{R}$ such that whenever $t \in B$ and S_i is the sphere defined by $\varphi_{f,i}(t, x)$, then S_i is a $d_{f,i}(t)$ -sphere of radius $r_{f,i}(t)$. All these functions are definable. We say that f is *preferred* if, for each $i < m$, the functions $d_{f,i}$ and $r_{f,i}$ are analytic and whenever $u, v \in B$, then $r_{f,i}(u) \leq 2r_{f,i}(v)$. Of course, if $d_{f,i}$ is analytic, then it is constant. A coloring $\chi: \mathbb{R}^n \rightarrow \mathcal{C}$ is *preferred* if $\chi(x)$ is preferred for each $x \in \mathbb{R}^n$. Preferred colorings exist by Proposition 1.3.

We make a further refinement of this definition. If $\varepsilon > 0$ and $f: B \rightarrow \mathbb{R}^n$ is a preferred color, where $B \subseteq \mathbb{R}^m$, then f is ε -preferred if whenever $u, v \in B$ and $i < m$, then

$$\|f(u) - f(v)\| < \varepsilon r_{f,i}(v).$$

The coloring χ is ε -preferred if $\chi(x)$ is ε -preferred for each $x \in \mathbb{R}^n$. Clearly, ε -preferred colorings exist.

The following is the main result of this section.

Theorem 3.1. *Suppose $n \geq 3$, and suppose $Y = \{y_0, y_1, \dots, y_n\} \subseteq \mathbb{F}^n$ is an n -simplex. There is $\varepsilon > 0$ such that whenever $\mu: \mathbb{R}^n \rightarrow \mathcal{C}$ is ε -preferred and $f_0, f_1, \dots, f_{n-2} \in \mathcal{C}$, then there is a thin $\mathcal{C}_2 \subseteq \mathcal{C}$ such that whenever $g \in \mathcal{C}$ and similarity $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are such that $f_0, f_1, \dots, f_{n-2}$ are acceptable colors for $\alpha(y_0), \alpha(y_1), \dots, \alpha(y_{n-2})$, respectively, and $\mu(\alpha(y_{n-1})) = \mu(\alpha(y_n)) = g$, then $g \in \mathcal{C}_2$.*

Proof. Let C_{n-1} be the set of $x \in \mathbb{R}^n$ such that $(y_0, y_1, \dots, y_{n-2}, y_{n-1}) \mapsto (y_0, y_1, \dots, y_{n-2}, x)$ is a similarity, and let C_n be the set of $x \in \mathbb{R}^n$ such that $(y_0, y_1, \dots, y_{n-2}, y_n) \mapsto (y_0, y_1, \dots, y_{n-2}, x)$ is a similarity. Each of C_{n-1} and C_n is a sphere with axis $H = \text{aff}(\{y_0, y_1, \dots, y_{n-2}\})$. Since Y is an n -simplex, $\dim(H) = n - 2$, so both C_{n-1}, C_n are circles. Let r_{n-1}, r_n be their respective radii. Let

$$\varepsilon = \|y_n - y_{n-1}\|/2 \max(r_n, r_{n-1}).$$

We will prove that this choice of ε works.

For the remainder of this proof, fix an ε -preferred coloring μ and colors $f_0, f_1, \dots, f_{n-2} \in \mathcal{C}$. Let $\text{dom}(f_i) = B_i = I_{i0} \times I_{i1} \times \dots \times I_{im_i-1}$ for $i \leq n - 2$. We say that $(a_0, a_1, \dots, a_n) \in (\mathbb{R}^n)^{n+1}$ is nice if $(y_0, y_1, \dots, y_n) \mapsto (a_0, a_1, \dots, a_n)$ is a similarity, $f_0, f_1, \dots, f_{n-2}$ are acceptable colors for $a_0, a_1, \dots, a_{n-2}$, respectively, and $\mu(a_{n-1}) = \mu(a_n)$.

Lemma 3.1.1. *If $(a_0, a_1, \dots, a_n)$ is nice, then $\text{supp}(a_{n-1}) \cup \text{supp}(a_n) \subseteq \text{supp}(a_0) \cup \text{supp}(a_1) \cup \dots \cup \text{supp}(a_{n-2})$.*

Proof. It suffices to prove that $\text{supp}(a_n) \subseteq \text{supp}(a_0) \cup \text{supp}(a_1) \cup \dots \cup \text{supp}(a_{n-2})$, for then by a similar proof (or by symmetry), we can also conclude that $\text{supp}(a_{n-1}) \subseteq \text{supp}(a_0) \cup \text{supp}(a_1) \cup \dots \cup \text{supp}(a_{n-2})$.

Let $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a similarity such that $\alpha(y_i) = a_i$ for $i \leq n$, and let c be its scaling factor. For each $i \leq n$, let $\text{supp}(a_i) = \{t_{i0}, t_{i1}, \dots, t_{im_i-1}\}_<$. For a contradiction, suppose $j < m_n$ is such that $t_{nj} \notin \text{supp}(a_i)$ for every $i \leq n - 2$. Let $g = \mu(a_{n-1}) = \mu(a_n)$, where $g: B \rightarrow \mathbb{R}^n$. For $1 \leq i \leq n - 2$, define $F_i: B_0 \times B_i \times B \rightarrow \mathbb{R}$ by

$$F_i(u, v, w) = \|y_0 - y_i\|^2 \|f_0(u) - g(w)\|^2 - \|y_0 - y_n\|^2 \|f_0(u) - f_i(v)\|^2,$$

so that F_i is a definable analytic function. Clearly, $F_i(t_0, t_i, t_n) = 0$. Thus, by Corollary 1.2, if we let t'_n be obtained from t_n by keeping all coordinates fixed except that t_{nj} is replaced by a number sufficiently close to it, then we get that $F_i(t_0, t_i, t'_n) = 0$. It follows from this, by Corollary 1.2 and Proposition 1.3, that the j th definable sphere of a_n is the

circle $C = \alpha(C_n)$ having radius $r = cr_n$. Since μ is preferred, the j th definable sphere C' of a_{n-1} is also a circle. Let r' be the radius of C' , and, since μ is preferred, $r' \leq 2r$. Since μ is ε -preferred, we get that

$$\begin{aligned} \varepsilon &> \|a_n - a_{n-1}\| / \max(r, r') \\ &\geq \|a_n - a_{n-1}\| / 2r \\ &= \|y_n - y_{n-1}\| / 2r_n \geq \varepsilon, \end{aligned}$$

which is a contradiction. $\square$

Lemma 3.1.2. *If $a_0, a_1, \dots, a_{n-2} \in \mathbb{R}^n$, then there are only finitely many pairs (a_{n-1}, a_n) such that $(a_0, a_1, \dots, a_n)$ is nice.*

Proof. We can assume that there is at least one pair (a_{n-1}, a_n) such that $(a_0, a_1, \dots, a_n)$ is nice. Hence, we can let $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a similarity such that $\alpha(y_i) = a_i$ for each $i \leq n-2$. Then $c = \|a_0 - a_1\| / \|y_0 - y_1\|$ is its scaling factor.

Let $T = \text{supp}(a_0) \cup \text{supp}(a_1) \cup \dots \cup \text{supp}(a_{n-2})$, which is finite. Suppose, for a contradiction, that there are infinitely many pairs (a_{n-1}, a_n) such that $(a_0, a_1, \dots, a_n)$ is nice.

By Lemma 3.1.1, for each such pair, $\text{supp}(a_{n-1}) \cup \text{supp}(a_n) \subseteq T$ and, moreover, $|\text{supp}(a_{n-1})| = |\text{supp}(a_n)|$ and $\text{supp}(a_{n-1}) \neq \text{supp}(a_n)$. Thus, there are $m < k$,

$$\begin{aligned} T_n &= \{t_0, t_1, \dots, t_{m-1}\} \subseteq T, \\ T_{n-1} &= \{t'_0, t'_1, \dots, t'_{m-1}\} \subseteq T, \end{aligned}$$

and three distinct such pairs $(a_{n-1}, a_n), (a'_{n-1}, a'_n), (a''_{n-1}, a''_n)$ such that $\text{supp}(a_n) = \text{supp}(a'_n) = \text{supp}(a''_n) = T_n$ and $\text{supp}(a_{n-1}) = \text{supp}(a'_{n-1}) = \text{supp}(a''_{n-1}) = T_{n-1}$. Notice that not only are these three pairs distinct, but also $a_n \neq a'_n \neq a''_n \neq a_n$ and $a_{n-1} \neq a'_{n-1} \neq a''_{n-1} \neq a_{n-1}$.

Clearly, $T_n \neq T_{n-1}$ and $|T_n| = |T_{n-1}|$, so $m \geq 1$. Let $g = \mu(a_n) = \mu(a_{n-1})$, $g' = \mu(a'_n) = \mu(a'_{n-1})$, and $g'' = \mu(a''_n) = \mu(a''_{n-1})$. Let $B = \text{dom}(g) \cap \text{dom}(g') \cap \text{dom}(g'') \subseteq \mathbb{R}^m$. Clearly, $t, t' \in B$.

Let C be the unique circle for which $a_n, a'_n, a''_n \in C$, and let C' be the unique circle for which $a_{n-1}, a'_{n-1}, a''_{n-1} \in C'$. Clearly, $C = \alpha(C_n)$ and $C' = \alpha(C_{n-1})$. The radii of C and C' are cr_n and cr_{n-1} respectively, and both of these circles have the same axis, namely $\alpha(H) = \text{aff}(\{a_0, a_1, \dots, a_{n-2}\})$. Since cr_n is the radius of C , it is T_n -definable, so that c is T_n -definable. Similarly, c is T_{n-1} -definable, so c is $(T_n \cap T_{n-1})$ -definable.

Let $T_n \setminus T_{n-1} = \{s_0, s_1, \dots, s_k\}$, and let $T_{n-1} \setminus T_n = \{s'_0, s'_1, \dots, s'_k\}$. Let $J_0, J_1, \dots, J_k$ be those factors of B such that $s, s' \in J_0 \times J_1 \times \dots \times J_k$. Let $B' = J_0 \times J_1 \times \dots \times J_k$. Let $h: B' \rightarrow \mathbb{R}^n$ be the $(T_n \cap T_{n-1})$ -definable function which is a “slice” of $g: B \rightarrow \mathbb{R}^n$ defined as follows: if $(u_0, u_1, \dots, u_k) \in B'$, then $h(u_0, u_1, \dots, u_k) = g(x_0, x_1, \dots, x_{m-1})$, where $x_i = t_i$ if $t_i \in T_n \cap T_{n-1}$ and $x_i = u_j$ if $x_i \in J_j$.

Now let $F: (B')^2 \rightarrow \mathbb{R}$ be such that

$$F(u, v) = \|h(u) - h(v)\|^2 - c^2 \|y_n - y_{n-1}\|^2.$$

Then F is a $(T_n \cap T_{n-1})$ -definable analytic function, and $F(s, s') = 0$. By Corollary 1.2, F is constantly 0 on $(B')^2$. In particular, $0 = F(s, s) = -c^2 \|y_n - y_{n-1}\|^2$ implying that $y_n = y_{n-1}$, which is a contradiction. $\square$

We wish to define the profile of a nice $(a_0, a_1, \dots, a_n)$. Recall that the boxes B_i and intervals I_{ij} , where $i \leq n-2$ and $j < m_i$, have already been defined (in the second paragraph of the proof of Theorem 3.1). Let $B_n = \text{dom}(\mu(a_n)) \subseteq \mathbb{R}^{m_n}$. Then also $B_n = \text{dom}(\mu(a_{n-1}))$, so let $m_{n-1} = m_n$. Then let $\text{supp}(a_i) = \{t_{i0}, t_{i1}, \dots, t_{im_i-1}\}_{<}$. Let $P = \{\langle i, j \rangle : i \leq n \text{ and } j < m_i\}$, and let E be the equivalence relation on P , where $\langle i, j \rangle, \langle i', j' \rangle$ are equivalent iff $t_{ij} = t_{i'j'}$. Then the *profile* of $(a_0, a_1, \dots, a_n)$ is the pair $\langle E, B_n \rangle$.

Clearly, for any box B , there are at most finitely many possible profiles $\langle E, B \rangle$.

Lemma 3.1.3. *Suppose $(a_0, a_1, \dots, a_n), (b_0, b_1, \dots, b_n)$ are nice and have the same profiles. Then there are a'_{n-1}, a'_n such that $(a_0, a_1, \dots, a_{n-2}, a'_{n-1}, a'_n)$ is nice and $\mu(a'_n) = \mu(b_n)$.*

Proof. Let $g = \mu(a_{n-1}) = \mu(a_n)$ and $h = \mu(b_{n-1}) = \mu(b_n)$, and then let $B = \text{dom}(g) = \text{dom}(h)$. For $i \leq n$, let

$$\begin{aligned} \text{supp}(a_i) &= \{t_{i0}, t_{i1}, \dots, t_{im_i-1}\}_{<}, \\ \text{supp}(b_i) &= \{u_{i0}, u_{i1}, \dots, u_{im_i-1}\}_{<}. \end{aligned}$$

For $i \leq n-2$ and $e \in \{n-1, n\}$, let $F_{ei}: B_0 \times B_1 \times \dots \times B_{n-2} \times B^2 \longrightarrow \mathbb{R}$ be such that if $x = (x_0, x_1, \dots, x_n) \in B_0 \times B_1 \times \dots \times B_{n-2} \times B^2$, then

$$F_{ei}(x) = \|y_0 - y_1\|^2 \|h(x_e) - f_i(x_i)\|^2 - \|y_e - y_i\|^2 \|f_0(x_0) - f_1(x_1)\|^2.$$

Also, define $F: B_0 \times B_1 \times \dots \times B_{n-2} \times B^2 \longrightarrow \mathbb{R}$ by

$$F(x) = \|y_0 - y_1\|^2 \|h(x_{n-1}) - h(x_n)\|^2 - \|y_{n-1} - y_n\|^2 \|f_0(x_0) - f_1(x_1)\|^2.$$

Then F and each F_{ei} are definable analytic functions and

$$F(u_0, u_1, \dots, u_n) = F_{ei}(u_0, u_1, \dots, u_n) = 0.$$

Since $(a_0, a_1, \dots, a_n), (b_0, b_1, \dots, b_n)$ have the same profiles, Corollary 1.2 can be invoked to show that

$$F(t_0, t_1, \dots, t_n) = F_{ei}(t_0, t_1, \dots, t_n) = 0.$$

Letting $a'_{n-1} = h(t_{n-1})$ and $a'_n = h(t_n)$, we then easily see that $(a_0, a_1, \dots, a_{n-2}, a'_{n-1}, a'_n)$ is nice and $\mu(a'_n) = h$. $\square$

We now complete the proof of Theorem 3.1. Let $\mathcal{C}_2$ be the set of $g \in \mathcal{C}$ for which there is a nice $(a_0, a_1, \dots, a_n)$ such that $g = \mu(a_n)$. If $\mathcal{C}_2$ is not thin, then let $k < \omega$ and $g_0, g_1, g_2, \dots$ be infinitely many distinct colors in $\mathcal{C}_2$ such that $v(g_i) = k$ for every $i < \omega$.

For each $i < \omega$, let $(a_{i0}, a_{i1}, \dots, a_{in})$ be nice such that $\mu(a_{in}) = g_i$. It follows from Lemma 3.1.1 that there are only finitely many boxes such that each $\text{dom}(g_i)$ is one of them. Thus, without loss, we can assume that there is a single B such that $\text{dom}(g_i) = B$ for each $i < \omega$. Since there are only finitely many profiles $\langle E, B \rangle$, we can further assume that each $(a_{i0}, a_{i1}, \dots, a_{in})$ has the same profile. Then, by Lemma 3.1.3, for each $i < \omega$, there are a'_{in-1}, a'_{in} such that $(a_{00}, a_{01}, \dots, a_{0n-2}, a'_{in-1}, a'_{in})$ is nice and $\mu(a'_{in}) = g_i$. Since the g_i 's are distinct, so must the a'_{in} 's be distinct. However, this contradicts Lemma 3.1.2, thereby proving the theorem. $\square$

4. Proof of Theorem D

This section contains the proof of Theorem D. Since, as previously mentioned, Komjáth [5] has already proved Theorem D in case $n = 2$, we assume throughout this section that $n \geq 3$.

To begin the proof of Theorem D, let $X, Y \subseteq \mathbb{R}^n$ be as in that theorem; that is, X is infinite and Y is an n -simplex. By taking a subset of X if needed, assume that X is countable, and then let $\mathbb{F} \subseteq \mathbb{R}$ be a countable, real-closed subfield such that $X, Y \subseteq \mathbb{F}^n$, and then the assumptions from the previous sections can be made.

Lemma 4.1. *Let $\chi: \mathbb{R}^n \rightarrow \mathcal{C}$ be an acceptable coloring. If $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity and $f \in \mathcal{C}$, then $\{a \in X: \chi(\alpha(a)) = f\}$ is finite.*

Proof. Let f be m -ary, and let $k = |\text{supp}(\alpha(Y))|$. If $a \in X$, then by Lemma 2.3, $\text{supp}(\alpha(a)) \subseteq \text{supp}(Y)$, so that $m \leq k$. Therefore, it follows that $|\{a \in X: \chi(\alpha(a)) = f\}| \leq \binom{k}{m}$. $\square$

For the rest of this section, we fix $\varepsilon > 0$ that will work in Theorem 3.1 no matter how we write Y as $\{y_0, y_1, \dots, y_n\}$. However, now fix $Y = \{y_0, y_1, \dots, y_n\}$. We let μ be the ε -preferred coloring that is *minimal* in the following sense: whenever μ' is an ε -preferred coloring and $a \in \mathbb{R}^n$, then $v(\mu(a)) \leq v(\mu'(a))$.

Of course, there are minimal ε -preferred colorings of any $\mathbb{R}^k$, where $1 \leq k < \omega$, not just for $\mathbb{R}^n$. If $1 \leq k, m < \omega$, then there is a canonical bijection from $\mathbb{R}^{km}$ to $(\mathbb{R}^k)^m$, giving us a canonical way to extend $\mu: \mathbb{R}^k \rightarrow \mathcal{C}_k$ to $\mu: (\mathbb{R}^k)^m \rightarrow \mathcal{C}_{km}$.

We refer to elements of $\mathcal{C}_{n(n+1)}$ as *types*. We say that the similarity $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ has *type* τ (in symbols: $\text{type}(\alpha) = \tau$) if τ is a type and $\mu(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_n)) = \tau$.

Lemma 4.2. *If $a \in X$ and $\text{type}(\alpha) = \text{type}(\beta)$, then $\mu(\alpha(a)) = \mu(\beta(a))$.*

Proof. Suppose the lemma is false; thus, $\text{type}(\alpha) = \text{type}(\beta) = \tau$ and $\mu(\alpha(a)) \neq \mu(\beta(a))$. By Lemma 2.3, let $F: (\mathbb{R}^n)^{n+1} \rightarrow \mathbb{R}^n$ be a definable analytic function such that $a = F(y_0, y_1, \dots, y_n)$, $\alpha(a) = F(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_n))$ and $\beta(a) = F(\beta(y_0), \beta(y_1), \dots, \beta(y_n))$.

Let $\text{dom}(\tau) = B = I_0 \times I_1 \times \dots \times I_{m-1} \subseteq \mathbb{R}^m$, and let

$$\text{supp}(\alpha(y_0), \alpha(y_1), \dots, \alpha(y_n)) = \{s_0, s_1, \dots, s_{m-1}\}_<$$

and

$$\text{supp}(\beta(y_0), \beta(y_1), \dots, \beta(y_n)) = \{t_0, t_1, \dots, t_{m-1}\}_{<}.$$

Let $\tau = (\tau_0, \tau_1, \dots, \tau_n)$, where each $\tau_i: B \rightarrow \mathbb{R}^n$, and then let $G: B \rightarrow \mathbb{R}^n$ be the definable analytic function such that

$$G(x_0, x_1, \dots, x_{m-1}) = F(\tau_0(x), \tau_1(x), \dots, \tau_n(x)).$$

In particular, $\alpha(a) = G(s_0, s_1, \dots, s_{m-1})$ and $\beta(a) = G(t_0, t_1, \dots, t_{m-1})$.

Let $J = \{i < m: s_i \in \text{supp}(\alpha(a))\}$ and $K = \{i < m: t_i \in \text{supp}(\beta(a))\}$. We will show that $J = K$.

Define $s', t' \in B$ as follows: let $s'_i = s_i$ and $t'_i = t_i$ if $i \in J$, and let s'_i, t'_i be the midpoint of I_i if $i \notin J$. Then $G(s'_0, s'_1, \dots, s'_{m-1}) = G(s_0, s_1, \dots, s_{m-1})$, so by Corollary 1.2, $G(t'_0, t'_1, \dots, t'_{m-1}) = G(t_0, t_1, \dots, t_{m-1})$. Thus, $K \subseteq J$. Then, by symmetry, $J = K$.

Let $J = \{j_0, j_1, \dots, j_{k-1}\}_{<}$, and let $B' = I_{j_0} \times I_{j_1} \times \dots \times I_{j_{k-1}} \subseteq \mathbb{R}^k$, and let $g: B' \rightarrow \mathbb{R}^n$ be such that $g(x_0, x_1, \dots, x_{k-1}) = G(y_0, y_1, \dots, y_{m-1})$, where $y_j = x_i$ if $j = j_i$ and $y_j = s'_j$ if $j \notin J$. Then, g is a definable analytic function and $\alpha(a) = g(s_0, s_1, \dots, s_{m-1})$ and $\beta(a) = g(t_0, t_1, \dots, t_{m-1})$. Thus, $g \subseteq \mu(\alpha(a))$ and $g \subseteq \mu(\beta(a))$. Since μ is a preferred coloring, then $\mu(\alpha(a)) = \mu(\beta(a))$. $\square$

We say that a function $\rho: D \rightarrow 2$ is *good* if $D \subseteq \mathcal{C}$ is finite and whenever $\alpha: Y \rightarrow \mathbb{R}^n$ is a similarity and $\mu(\alpha(Y)) \subseteq D$, then $\rho(\mu(\alpha(y))) = 0$ for some $y \in Y$.

Lemma 4.3. *Suppose that ρ is good and that τ is a type. Then there is a good $\rho' \supseteq \rho$ and $a \in X$ such that whenever $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity and $\text{type}(\alpha) = \tau$, then $\rho'(\mu(\alpha(a))) = 1$.*

Proof. Let $D_0 = \text{dom}(\rho)$, which is finite since ρ is good.

By Corollary 2.6, let $D_1 \subseteq \mathcal{C}$ be a thin set such that whenever $j \leq n$ and $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity for which $\mu(\alpha(y_i)) \in D_0$ if $j \neq i \leq n$, then $\mu(\alpha(y_j)) \in D_1$.

By Theorem 3.1, let $D_2 \subseteq \mathcal{C}$ be a thin set such that whenever $j < k \leq n$ and $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a similarity for which $\mu(\alpha(y_i)) \in D_0$ if $i \leq n$ and $i \notin \{j, k\}$, and $\mu(\alpha(y_j)) = \mu(\alpha(y_k))$, then $\mu(\alpha(y_j)) \in D_2$.

It follows from Lemmas 4.1 and 4.2 that we can let $a \in X$ and $f \notin D_0 \cup D_1 \cup D_2$ be such that $\mu(\alpha(a)) = f$ whenever $\text{type}(\alpha) = \tau$. Now let $\rho' \supseteq \rho$ be such that $\text{dom}(\rho') = D_0 \cup \{f\}$ and $\rho'(f) = 1$.

We claim that ρ' is good. If it were not, then there would be a similarity α such that $\alpha(Y) \subseteq \text{dom}(\rho')$ and $\rho'(\alpha(y)) = 1$ for each $y \in Y$. Let $k = |\{y \in Y: \mu(y) = f\}|$. We consider several cases.

$k = 0$. Then $\rho(\alpha(y)) = 1$ for all $y \in Y$, contradicting that ρ is good.

$k = 1$. Then $f \in D_1$, which is a contradiction.

$k = 2$. Then $f \in D_2$, which is a contradiction.

$k \geq 3$. This contradicts Lemma 2.1.

Thus, ρ' is good.

That ρ' has the required property is clear since whenever $\text{type}(\alpha) = \tau$, then $\rho'(\mu(\alpha(a))) = \rho'(f) = 1$. $\square$

Our goal now is to construct a function $\rho: \mathcal{C} \longrightarrow 2$ and then let $\psi = \rho \circ \mu$. If this is done properly, then ψ will be a 2-coloring of $\mathbb{R}^n$ that confirms Theorem D. The function ρ will be constructed as the union of an increasing sequence $\rho_0 \subseteq \rho_1 \subseteq \dots$ of good functions. The construction will proceed by stages: at stage i , where $i < \omega$, we will construct ρ_i .

We need a list $\langle f_j: j < \omega \rangle$ of all the colors in $\mathcal{C}$, and we also need a list $\langle \tau_j: j < \omega \rangle$ of all types.

To start this construction, we let ρ_0 be the empty function, which is good vacuously. The construction now splits into even and odd stages.

The odd stages. Suppose $i = 2j$. Then let $\rho_{i+1} \supseteq \rho_i$ be such that $\text{dom}(\rho_{i+1}) = \text{dom}(\rho_i) \cup \{f_j\}$ and $\rho_{i+1}(f_j) = 0$. It is easily checked that ρ_{i+1} is good.

The even stages. Suppose $i = 2j + 1$. Use Lemma 4.3 to get a good $\rho_{i+1} \supseteq \rho_i$. That is, whenever $\alpha: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is such that $\text{type}(\alpha) = \tau_j$, then there is $a \in X$ such that $\mu(\alpha(a)) \in \text{dom}(\rho_{i+1})$ and $\rho_{i+1}(\mu(\alpha(a))) = 1$.

We have completed the construction of the sequence $\langle \rho_i: i < \omega \rangle$. Let ρ be its union, and let $\psi = \rho \circ \mu$.

The odd stages assure that $\text{dom}(\rho) = \mathcal{C}$ and, therefore, that $\text{dom}(\psi) = \mathbb{R}^n$. We now show that the function $\psi: \mathbb{R}^n \longrightarrow 2$ confirms Theorem D. Let $\alpha: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a similarity, and suppose $\text{type}(\alpha) = \tau_j$. Let $X' = \alpha(X)$ and $Y' = \alpha(Y)$. We wish to show that there are $a' \in X'$ and $y' \in Y'$ such that $\psi(a') = 1$ and $\psi(y') = 0$.

Since Y' is finite, the odd stages assure that there is $i < \omega$ such that $i \geq 2j + 1$ and $\mu(Y') \subseteq \text{dom}(\rho_i)$. Since ρ_i is good, there is $y' \in Y'$ such that $\rho_i(\mu(y')) = 0$. Thus, $\psi(y') = 0$. The even stages assure that there is $a' \in X'$ such that $\mu(a') \in \text{dom}(\rho_i)$ and $\rho(\mu(a')) = 1$. Thus, $\psi(a') = 1$.

This completes the proof of Theorem D.

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