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Covering Polygonal Annuli by Strips*

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Abstract. In 2000 Bezdek asked which plane convex bodies have the property that whenever an annulus, consisting of the body less a sufficiently small scaled copy of itself, is covered by strips, the sum of the widths of the strips must still be at least the minimal width of the body. We characterise the polygons for which this is so.

Introduction

In this note we give a complete answer for convex polygons to the following question of Bezdek.

Question [2], [3]. Given a plane convex body C, is there an $\varepsilon > 0$ such that whenever an annulus resulting from the removal of an ε-scaled copy of C from the interior of C is covered by finitely many strips, the sum of the widths of those strips must be at least the minimal width of C?

Here and throughout, strips are closed, and minimal width means the shortest distance between two different parallel supporting lines. It is an elementary fact that there is at least one chord of C joining, and perpendicular to, these support lines. In 1951 Bang [1] solved the Tarski plank problem, showing that to cover the whole of a convex region one needs strips of total width at least its minimal width. Thus the question is asking whether making a small hole in the set leaves this unchanged.

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It is tempting to believe that an equivalent question can be formulated by asking about the removal of small discs rather than scaled copies of C; this is, however, not quite true, since a C-shaped hole may be cut so close to the boundary that the disc circumscribing it protrudes outside C. Also note that if the question were asked not about small diameter but small area, then the answer would clearly be no: we can always cut a hole of arbitrarily small area that does reduce the total width of strips needed, simply by making it long and thin, perpendicular to a minimal-width chord, and with both ends very near to the boundary.

Bezdek himself answered the question in the affirmative for a special class of polygons.

Theorem 1 [2]. *Let P be a convex polygon whose inradius is exactly half of its minimal width. Then there is an $\varepsilon > 0$ such that whenever any annulus resulting from the removal of an ε-scaled copy of P from the interior of P is covered by finitely many strips, the sum of the widths of those strips must be at least the minimal width of P.*

So in particular, the answer to the question is yes for regular polygons with an even number of sides. Zhang and Ding have announced a positive result for parallelograms, giving an explicit bound for ε in terms of the geometry of the parallelogram [5]. Our theorem, which characterises those polygons having the property, is the following.

Theorem 2. *Let P be a convex polygon.*

(i) *If there is a minimal-width chord of P that meets a vertex of P and divides the angle at that vertex into two acute angles, then for every $\varepsilon > 0$ an ε-scaled copy of P can be removed so that the resulting annulus can be covered by strips of total width strictly less than the minimal width of P.*
(ii) *If there is no such minimal-width chord, then the removal of any set of sufficiently small diameter gives an annulus that still needs strips of total width at least the minimal width of P to cover it.*

In particular, regular polygons with an odd number of sides are different in this respect from those with an even number of sides.

Part (i) is straightforward and is proved in Section 1. Part (ii) is proved in Section 2 using a method similar to that of Bezdek, but avoiding the use of the incircle via results of Eggleston [4].

1. Proof of Part (i)

The basic idea is to take holes lying along such a minimal-width chord and approaching the vertex as $\varepsilon \to 0$.

Proof of Theorem 2(i). Without loss of generality, P has minimal width 1, the minimal-width chord in the hypothesis is vertical, and divides the angle at the vertex into angles $\theta_1, \theta_2 \in (0, \pi/2)$ as shown in Fig. 1. We remove from the body an ε-scaled copy of P placed at distance h from the vertex and with its minimal-width chord (which has

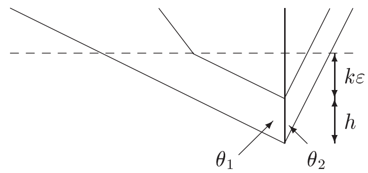

Fig. 1. The proof of Theorem 2 part (i).

length ε) coinciding with that of P. Since θ_1 and θ_2 are both acute, the part of this hole lying below some horizontal line is a triangle as in Fig. 1. Its altitude is $k\varepsilon$ where $k \le 1$ is an absolute constant depending only on P. Cover the annulus by a large horizontal strip of width $1 - h - k\varepsilon$ and two sloping strips of widths $h \sin\theta_1$ and $h \sin\theta_2$. The total width of this cover is less than the minimal width of the polygon if

$$1 - h - k\varepsilon + h\sin\theta_1 + h\sin\theta_2 < 1 \quad \Leftrightarrow \quad h(\sin\theta_1 + \sin\theta_2 - 1) < k\varepsilon,$$

which we can achieve simply by letting h be small enough, dependent on ε. $\square$

Note that if $\sin\theta_1 + \sin\theta_2 \le 1$ then P must be an equilateral triangle, and as expected from the proof we have a counterexample that does not rely on h tending to zero as ε tends to zero. In fact, we can remove an arbitrarily small equilateral triangle *anywhere* and the total strip-width needed will decrease, as shown in Fig. 2. As noted by the referee, this follows from the elementary fact that the sum of the distances from any point in an equilateral triangle to its three sides is equal to the height of the triangle.

Clearly the proof of (i) also works if the body is not a polygon, but merely has at least two flat sides forming the vertex in question. Also, this method can be extended to other non-polygonal bodies, such as the Reuleaux triangle, where a minimal-width chord passes through a point of the boundary which is not smooth and has tangents satisfying the angle criterion.

2. Proof of Part (ii)

The main idea is the same as in Bezdek's work, namely that a certain polygon, constructed from strips that purport to cover, can be translated inside P. Our main lemma achieves something similar, but with our weaker hypothesis.

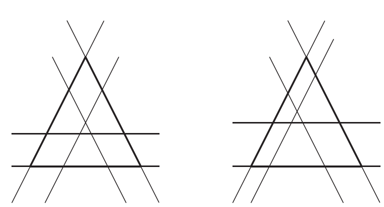

Fig. 2. Strip-coverings of two equilateral triangular annuli.

Lemma 3. *Let P be a plane convex polygon such that no minimal-width chord cuts a vertex into two acute angles. Then there exists an $\varepsilon > 0$ such that every centrally symmetric plane convex body of perimeter strictly less than twice the minimal width of P can be translated in some direction parallel to a side of P by a distance ε while lying wholly within the interior of P.*

Along with this, we need the following result, first stated by Eggleston [4] but which is implicit in Bang's proof [1].

Proposition 4. *Given n strips in the plane, construct the convex centrally symmetric $2n$-gon which has, for each strip, a pair of sides in direction perpendicular to it and with length equal to the width of the strip. Then, no matter where in the plane this polygon is placed, there is a point of it that is not in the union of the interiors of the strips.*

With these tools, the theorem is deduced in the same way as in Bezdek's paper. For the convenience of the reader, and to take care of some technicalities, we include a proof.

Proof of Theorem 2(ii). We are given a polygon P satisfying the hypothesis, and we may assume that its minimal width is 1. Let ε be as in the lemma, let H denote some set of diameter at most ε lying in its interior and suppose that we can cover the annulus $P \backslash H$ by a family $\mathcal{F}$ of n strips having total width strictly less than 1. Certainly $n > 1$. By slightly perturbing and enlarging the strips if necessary we may certainly assume that no two are parallel, and that no three strip edges concur—we say that such strips are in *general position*. For technical reasons we need slightly more than this. For each direction perpendicular to a side of P, fix another strip (not one of our family $\mathcal{F}$) that lies in this direction, has width exactly ε, and is placed so as to cover the hole H. Call this set of strips $\mathcal{H}$. We now perturb and enlarge the strips of our original family $\mathcal{F}$ so that not only are they in general position, but that adding any one of the strips in $\mathcal{H}$ does not destroy the generality of position. This we can do, since it merely involves avoiding finitely many fixed directions for the strips, and ensuring that the intersections of the strip edges avoid finitely many fixed lines.

From the family $\mathcal{F}$ of n strips, construct the $2n$-gon as in Proposition 4 and denote it by K. Its perimeter is twice the total width of the strips and so is less than 2. By the lemma, K can be translated in some direction by an amount ε while lying wholly within P, and so in fact the convex hull K' of these translates lies within P. Observe that K' has sides with all the lengths and directions that K had, plus an additional pair, each of length ε in some direction parallel to a side of P. So let us introduce an additional strip, namely the member of $\mathcal{H}$ perpendicular to this direction. This new family of strips corresponds precisely to the polygon K'. By Proposition 4, there is a point of K' not covered by the interiors of the strips. Since our new family is still in general position, the union of the interiors is equal to the interior of the union, and so no neighbourhood of this point is entirely covered by the closed strips. Since K' lies in the interior of P, this means there are points of P not covered by this family of $n + 1$ strips. The hole H was explicitly covered by the $(n + 1)$th strip, so the uncovered points must belong to the annulus. This is a contradiction. $\square$

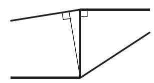

Fig. 3. The perpendicular line joining parallel sides whose projections meet in a single point cannot be a minimal chord.

It is clear from the proof that we could in fact allow a hole H of diameter larger than ε, provided that its widths in directions perpendicular to sides of P were at most ε.

It remains to prove Lemma 3. We begin by reformulating the negation of the angle criterion.

Lemma 5. *Let P be a plane convex polygon with no minimal-width chord dividing the angle at a vertex into two acute angles. Then all minimal-width chords join two parallel sides of P, and moreover, the orthogonal projection of such a side onto its parallel partner intersects it in a line of strictly positive length.*

Proof. Consider two parallel supporting lines to P at minimal distance and a minimal-width chord joining them. If either of the lines only touched P at a vertex, then the chord would divide this vertex into two acute angles. So both supporting lines lie along sides of P. If the projection of one of those sides onto the other met it only in a single point, then the chord would not be minimal, since support lines along whichever adjacent side differs the least in slope would be nearer to each other than the original support lines, as shown in Fig. 3. □

So the polygons we are now dealing with have pairs of parallel sides at minimal distance, perhaps together with some other sides. Note that not all such polygons are covered by Bezdek's work, since equilateral triangles with one vertex snipped off, and the polygons pictured in Figs. 4 and 7, satisfy our hypotheses but no circle of diameter equal to the minimal width can be inscribed.

Bezdek's version of Lemma 3 relied on placing the centrally symmetric body inside the inscribed circle; we avoid this using the following two results of Eggleston. The first is like a weaker version of the lemma, and is enough to prove Bang's theorem in the plane.

Proposition 6 [4]. *If C is a plane convex body and K is a centrally symmetric plane convex body of perimeter at most twice the minimal width of C, then K may be translated to lie wholly within C.*

Proposition 7 [4]. *Consider a centrally symmetric plane convex body, and a triangle circumscribing it. Then the perimeter of the body is at least twice the minimal width of the triangle.*

Although for the theorem it is not necessary to obtain actual bounds for ε, we do so in order to obtain an explicit estimate for the square.

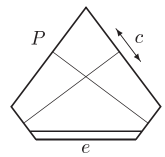

Fig. 4. A polygon with only one non-special edge e. Two minimal-width chords, and $\mathcal{N}_\varepsilon(e)$, are shown.

Proof of Lemma 3. Without loss of generality, let P have minimal width 1. For each side e of P and $\varepsilon > 0$ we define the *ε-neighbourhood* $\mathcal{N}_\varepsilon(e)$ to be the intersection of P with the Euclidean ε-neighbourhood of the infinite line along e. Roughly, our task is to find an $\varepsilon > 0$ such that whenever we take a centrally symmetric K of perimeter less than 2 and place it, by Proposition 6, inside P, we can find a sufficiently long arc of consecutive sides of P whose ε-neighbourhoods K does not meet, so that we can then push K into this arc.

Denote the number of directions in which P has width 1 by m. By Lemma 5 there are m pairs of sides perpendicular to these directions, and for each pair the length of the interval of intersection of one side with the projection of its partner is positive. Let $c > 0$ be the minimum of these lengths, and let $\mathcal{S}$ denote the set of these $2m$ sides—these sides are called *special*. If we take $\varepsilon > 0$ small enough, the polygon P'_ε defined by

$$P'_\varepsilon := P \setminus \bigcup_{e \notin \mathcal{S}} \mathcal{N}_\varepsilon(e)$$

still has minimal width 1 and has the same special edges as P. To be more precise, we need two conditions. One comes from ensuring that the length of each special edge is still positive; we ensure that it is at least λc (for some $\lambda \in (0, 1)$ which we are free to choose) by requiring that

$$\varepsilon \leq \frac{1-\lambda}{2} c \, \min \sin\theta, \tag{1}$$

where the minimum is over all internal angles θ that are adjacent to a special edge. (At this stage, we really only require those angles adjacent to a special *and* a non-special edge, but we will need the rest later.) See Fig. 4. The other condition is to ensure that the widths of the new body in the non-special directions are still greater than 1. These widths are perpendicular to non-special edges (although they may not be attained by chords). With the notation of Fig. 5, it is enough that

$$\varepsilon < \min_{e \notin \mathcal{S}} \frac{\sin(\theta_e/2)}{1 + \sin(\theta_e/2)} (D_e - 1), \tag{2}$$

because the worst case is when the diagram is symmetrical.

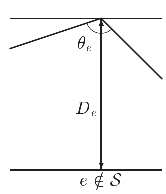

Fig. 5. Ensuring that non-minimal directions stay non-minimal.

By Proposition 6, the body K can still be placed inside the new polygon P'_ε, so from now on we work in P'_ε. If $m = 1$, then K (and even P'_ε) can be translated within P and parallel to the special sides by a distance 2ε, and not just ε, so the proof is complete in the case $m = 1$.

Next, for each special edge $s \in \mathcal{S}$ with parallel partner $\bar{s} \in \mathcal{S}$, let $\mathcal{A}(s)$ denote those edges in $\mathcal{S}$ that lie strictly between s and $\bar{s}$ in a clockwise direction. So $\mathcal{A}(s)$ and $\mathcal{A}(\bar{s})$ are disjoint arcs of $m - 1$ special edges. Define for each $\varepsilon > 0$ the polygon $P_\varepsilon(s)$ given by

$$P_\varepsilon(s) = P'_\varepsilon \setminus \bigcup_{e \in \mathcal{A}(s)} \mathcal{N}_\varepsilon(e).$$

Our aim is to show that if $\varepsilon > 0$ is sufficiently small, then every centrally symmetric convex K with perimeter strictly less than 2 may be translated to lie inside $P_\varepsilon(s)$ for some $s \in \mathcal{S}$. This will complete the proof, since we can then translate K within P by a distance ε parallel to s.

First we claim that, for all $s \in \mathcal{S}$, if K cannot lie inside $P_\varepsilon(s) \cap P_\varepsilon(\bar{s})$ then it also cannot meet both $\mathcal{N}_\varepsilon(s)$ and $\mathcal{N}_\varepsilon(\bar{s})$ simultaneously. To prove this, let K lie in P'_ε (by Proposition 6) and draw the rectangle whose sides are supporting lines of K and are parallel or perpendicular to the side s. Note that we do not claim that this rectangle is included in the polygon. Let x and y be the side lengths of the rectangle parallel to and perpendicular to s respectively, as in Fig. 6. By our choice of ε in (1), the polygon $P_\varepsilon(s) \cap P_\varepsilon(\bar{s})$ includes a vertical rectangle of height 1 and width λc, so that if $x \le \lambda c$ then we can certainly fit the rectangle containing K into $P_\varepsilon(s) \cap P_\varepsilon(\bar{s})$. So assume that $x > \lambda c$. By elementary geometry, the perimeter of K is at least twice the diagonal of the rectangle, so we see that $x^2 + y^2 < 1$. From this, we find $y^2 < 1 - \lambda^2 c^2$ so that if we now also make

$$\varepsilon \le \tfrac{1}{2}\left(1 - \sqrt{1 - \lambda^2 c^2}\right), \tag{3}$$

we have $y < 1 - 2\varepsilon$ and so the rectangle cannot meet both $\mathcal{N}_\varepsilon(s)$ and $\mathcal{N}_\varepsilon(\bar{s})$ simultaneously.

When $m = 2$ the proof is now complete, for if K lies in $P_\varepsilon(s_1) \cap P_\varepsilon(\bar{s_1})$ then it can be translated by 2ε parallel to s_1. Otherwise K cannot meet both $\mathcal{N}_\varepsilon(s_1)$ and $\mathcal{N}_\varepsilon(\bar{s_1})$, so

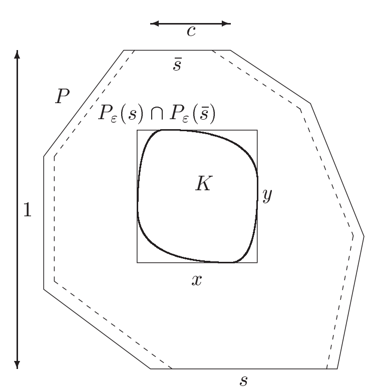

Fig. 6. Shifting K in $P_\varepsilon(s) \cap P_\varepsilon(\bar{s})$.

lies in either $P_\varepsilon(s_2)$ or $P_\varepsilon(\bar{s_2})$ and can be translated by ε in a direction parallel to s_2. If in addition P is the square, then we can actually translate by 2ε in a direction parallel to s_2 in this last case, for some translate of K parallel to s_2 meets neither $P_\varepsilon(s_1)$ nor $P_\varepsilon(\bar{s_1})$.

Now let $m \geq 3$ and, for a contradiction, suppose that for every $s \in \mathcal{S}$, the body K cannot lie in $P_\varepsilon(s)$. Then for every $s \in \mathcal{S}$ the body K cannot meet both $\mathcal{N}_\varepsilon(s)$ and $\mathcal{N}_\varepsilon(\bar{s})$ and yet there are not $m-1$ consecutive special edges whose neighbourhoods it does not meet. This implies that there are three special edges $s_1, s_2, s_3 \in \mathcal{S}$ with K meeting $\mathcal{N}_\varepsilon(s_1)$, $\mathcal{N}_\varepsilon(s_2)$, $\mathcal{N}_\varepsilon(s_3)$ and the parallel partner of each edge lying in between the other two edges. (More precisely, $s_{i+1} \in \mathcal{A}(s_i)$ and $s_{i+2} \in \mathcal{A}(\bar{s_i})$ for $i = 1, 2, 3$ modulo 3.)

Consider the triangle T containing P and obtained from it by extending these three sides. Its minimal width is attained in a direction that is also minimal for P, but the vertex lies further out (since one of $\bar{s_1}, \bar{s_2}, \bar{s_3}$ lies in between), so the minimal width of T is clearly greater than 1. This will continue to hold for the slightly smaller triangle formed by the ε-neighbourhoods of the three lines provided we place the following final condition on ε:

$$\varepsilon \leq \frac{c}{2} \min \frac{\sin\theta}{1-\cos\theta}, \tag{4}$$

where the minimum is over all angles θ between non-parallel non-adjacent special sides. See Fig. 7. With this choice, the triangle T_K whose sides are the supporting lines of K in these three directions certainly has minimal width greater than 1. On the other hand, this triangle circumscribes K, so by Proposition 7, the perimeter of K is at least 2, which is a contradiction. □

We remark that if P is a square of side 1, then $c = 1$, there are no non-special edges so inequality (2) does not apply, we have $m = 2$ so (4) does not apply, while inequalities (1) and (3) become

$$\varepsilon \leq \frac{1-\lambda}{2}, \qquad \varepsilon \leq \tfrac{1}{2}\left(1-\sqrt{1-\lambda^2}\right),$$

which, if we optimally choose $\lambda = 1/\sqrt{2}$, give $\varepsilon = \frac{1}{2}(1-1/\sqrt{2})$. As remarked in the proof, we can shift K by a distance of 2ε rather than just ε, thus a square hole of side

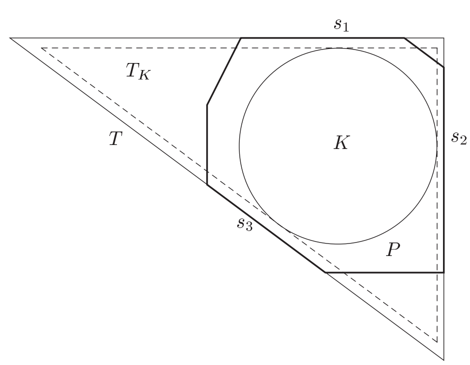

Fig. 7. If K met $\mathcal{N}_\varepsilon(s_1), \mathcal{N}_\varepsilon(s_2), \mathcal{N}_\varepsilon(s_3)$ then its perimeter would be too large.

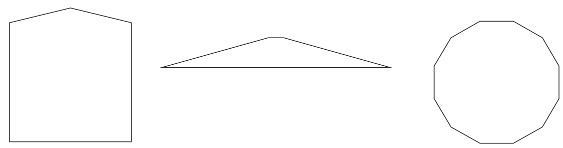

Fig. 8. Inequality (2) matters most for the first polygon; (1) and (3) for the second; and (4) for the third.

$1 - 1/\sqrt{2}$ does not reduce the width of strips needed to cover the square. This is the same bound obtained by Bezdek, and the question of whether this holds for a hole of side $\frac{1}{2}$ remains open.

It is perhaps interesting to observe which of our four bounds on ε is most significant for various polygons. A selection of polygons is shown in Fig. 8.

The proof naturally extends to bodies all of whose minimal chords meet pairs of flat sides with the required positive projection lengths. However, it is evident from Lemma 3 that we cannot treat a circular annulus by these methods, since a very long thin K can only be moved within a circle by a distance comparable with $1 - \operatorname{diam}(K)$, rather than a distance independent of K.

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