Open Caps and Cups in Planar Point Sets*

Pavel Valtr

Department of Applied Mathematics and
Institute for Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
<http://kam.mff.cuni.cz/~valtr>

Abstract. Points $p_1, p_2, \dots, p_k$ in the plane, ordered in the x -direction, form a k -cap (k -cup, respectively) if they are in convex position and $p_2, \dots, p_{k-1}$ lie all above (below, respectively) the segment $p_1 p_k$. We prove the following generalization of the Erdős–Szekeres theorem. For any k , any sufficiently large set P of points in general position contains k points, $p_1, p_2, \dots, p_k$, that form either a k -cap or a k -cup, and there is no point of P vertically above the polygonal line $p_1 p_2 \dots p_k$. We give double-exponential lower and upper bounds on the minimal size of P . We also give several related results.

1. Introduction

A *configuration* is a finite set of points in the plane such that no three points lie on a line and no two points have the same x -coordinate. Let X be a configuration. We say that a subset of X is in *convex position*, if it is the vertex set of a convex polygon. Let $p_1, \dots, p_k$ be k points of X ordered increasingly with respect to their x -coordinates (i.e., $x(p_1) < x(p_2) < \dots < x(p_k)$). For $i = 1, \dots, k-1$, let s_i be the slope of the line $p_i p_{i+1}$, i.e., $s_i = (y(p_{i+1}) - y(p_i)) / (x(p_{i+1}) - x(p_i))$. The set $P = \{p_1, \dots, p_k\}$ is a k -cap or a k -cup if the sequence $s_1, s_2, \dots, s_{k-1}$ is decreasing or increasing, respectively (see Fig. 1). The set $P = \{p_1, \dots, p_k\}$ is *open (in X)*, if no point $p \in X$ with $x(p_1) < x(p) < x(p_k)$ lies above the polygonal line $p_1 p_2 \dots p_k$.

In 1935 Erdős and Szekeres proved the following classical result:

Theorem 1 [3]. *For any $n \geq 3$, there is a (least) integer $F(n)$ such that any set of at least $F(n)$ points in general position in the plane contains n points in convex position.*

* This research was supported by projects LN00A056 and 1M0021620808 of The Ministry of Education of the Czech Republic.

Fig. 1. A 7-cap and a 6-cup; each of them is open if the unbounded shaded region contains no points of X .

Erdős and Szekeres [3] proved Theorem 1 by proving the following result:

Theorem 2 [3]. *For any $k, l \geq 2$, there is a (least) integer $F'(k, l)$ such that any configuration at least $F'(k, l)$ points contains a k -cap or an l -cup.*

Erdős and Szekeres [3] proved that

$$F'(k, l) = \binom{k+l-4}{k-2} + 1 \quad \text{for any } k, l \geq 2.$$

It follows that

$$F(n) \leq F'(n, n) = \binom{2n-4}{n-2} + 1 \quad \text{for any } n \geq 3.$$

It is known that $F(3) = 3$, $F(4) = 5$, and $F(5) = 9$. For $n > 5$, the best known bounds are

$$2^{n-2} + 1 \leq F(n) \leq \binom{2n-5}{n-2} + 1$$

(see [4] and [12]).

Here we show the following generalization of Theorem 2:

Theorem 3. *For any $k, l \geq 2$, there is a (least) integer $f(k, l)$ such that any configuration of size at least $f(k, l)$ contains an open k -cap or an open l -cup.*

Theorem 3 is a “generic” result with some interesting corollaries mentioned below. The function $f(k, l)$ is double-exponential, as shown in Theorem 4 below.

If X is a configuration, then a subset $P \subseteq X$ in convex position is called an *empty polygon (in X)* if the interior of $\text{conv } P$ contains no point of X . Answering a question of Erdős [2], Horton [6] constructed arbitrarily large configurations with no empty 7-gon. Gerken [5] has shown very recently that any sufficiently large configuration contains

an empty hexagon [5]. A modification of Gerken's proof can be found in [16]. Another proof is given in [11].

Our results show some structural properties of configurations containing no empty polygons with many vertices. We say that a configuration X is k -convex if the interior of every triangle determined by X contains at most k points of X . Theorem 3 implies the following result which shows that large k -convex configurations contain large empty polygons:

Corollary 1 [15], [9]. *For any $k \geq 1$ and $l \geq 3$, there is a (least) integer $N(k, l)$ such that any k -convex configuration of size at least $N(k, l)$ contains an empty l -gon.*

Corollary 1 in the special case $k = 1$ was first proved by Károlyi *et al.* [7]. In case $k = 1$, $N(1, l)$ is exponential in l [9] and its exact value is determined in [8]. For general k and l , our proof of Corollary 1 gives a double-exponential upper bound $N(k, l) \leq 2^{\binom{k+l}{k+2}-1} + 1$, which is slightly better than the previous bound $N(k, l) \leq (k+2)^{(k+2)^l-1} \leq 2^{(k+2)^{l+1}}$ of Kun and Lippner [9]. No lower bound on $N(k, l)$ better than exponential in $k + l$ is known.

We asked in [15] if any large configuration with no k -cap contains a large empty polygon. This is answered affirmatively by the following direct consequence of Theorem 3:

Corollary 2. *For any $k, l \geq 2$, there is a least integer $m(k, l)$ such that any configuration of size at least $m(k, l)$ with no open k -cap contains an empty l -gon.*

A k -moon is a k -point configuration M with a specific point $a(M) \in M$ (called the apex of M) such that any 4-point subset of M is in convex position if and only if it does not contain $a(M)$. The combinatorial structure of a k -moon is unique for each $k \geq 3$ (see Fig. 2). A k -moon M is empty (in X), if the interior of the region $\text{conv } M \setminus \text{conv}(M \setminus \{a(M)\})$ contains no point of X (see Fig. 2). Theorem 3 is equivalent to the following corollary:

Corollary 3. *For any $k, l \geq 3$, there is a least integer $z(k, l)$ such that any point p in any configuration X of size at least $z(k, l)$ is the apex of an empty k -moon in X or it is one of the vertices of an empty l -gon in X .*

We obtain double-exponential lower and upper bounds on $f(k, l)$ and on two related functions:

Fig. 2. A 7-moon; it is empty in X if the shaded region contains no points of X .

Theorem 4. Let $k, l \geq 2$ and let $f(k, l)$ be the number given by Theorem 3. Then:

- (i) If $f'(k, l)$ denotes the minimum integer such that any configuration of size at least $f'(k, l)$ contains a k -cap or an open l -cup, then

$$2^{\binom{\lfloor k/2 \rfloor + \lfloor l/2 \rfloor - 2}{\lfloor k/2 \rfloor - 1}} \leq f'(k, l) \leq f(k, l) \leq 2^{\binom{k+l-2}{k-1}-1}.$$

- (ii) If $f''(k, l)$ denotes the minimum integer such that any configuration of size at least $f''(k, l)$ contains an open k -cap or an l -cup, then

$$2^{\binom{\lfloor k/2 \rfloor + \lfloor (l-1)/2 \rfloor - 2}{\lfloor k/2 \rfloor - 1}} \leq f''(k, l) \leq f(k, l).$$

If neither the k -cap nor the l -cup are required to be open, then the extremal function is exponential (see Theorem 2 and the remark below it).

Our proof of Theorem 3 uses some of the ideas of Kun and Lippner [9]. It also has some similarities with one of the original proofs of the Erdős–Szekeres theorem (Theorem 1). A different (and shorter) proof of Theorem 3 has been very recently proved by Černý [1].

Open Problems. Corollaries 1–3 can be seen as a little step towards an answer to the following somewhat loosely stated problem:

Problem 1. Which types of subconfigurations are contained in any sufficiently large configuration with no empty l -gon ($l \geq 7$ fixed)?

Since the so-called Horton sets (e.g., see [13], [14], and [10]) contain no empty 7-gons, Problem 1 may be restricted to types of configurations contained in all sufficiently large Horton sets.

2. Proof of Theorem 3

We write $C = c_1 c_2 \cdots c_k$ if $C = \{c_1, c_2, \dots, c_k\}$ is a k -cap (or a k -cup) with $x(c_1) < x(c_2) < \cdots < x(c_k)$. If $x(p) < x(q)$ then we say that p lies *to the left of* q and q lies *to the right of* p . If $x(p) < x(q) < x(r)$ then we say that q lies *between* p and r . The *left strip* of an l -cup $D = d_1 d_2 \cdots d_l$ is the vertical strip $L(D) = \{p \in \mathbb{R}^2 : x(d_1) < x(p) < x(d_2)\}$ (see Fig. 3). A *right strip* of a k -cap $C = c_1 c_2 \cdots c_k$ is any vertical strip $R(C, w) = \{p \in \mathbb{R}^2 : x(c_k) < x(p) < x(c_k) + w\}$, where $w \in \mathbb{R}^+$. We further define $R^-(C, w)$ as the set of points of $R(C, w)$ lying below the line $c_{k-1}c_k$ (see Fig. 3).

For $\varepsilon > 0$ and for a configuration X , we say that an l -cup D , $D \subseteq X$, is ε -good (in X) if it is open and $|X \cap L(D)| \geq \varepsilon|X| - 1$ (see Fig. 4). We say that a k -cap C , $C \subseteq X$, is ε -good (in X) if it is open and there is a $w > 0$ such that $|X \cap R(C, w)| = |X \cap R^-(C, w)| \geq \varepsilon|X| - 1$ (see Fig. 4).

For $k, l \geq 2$, we recursively define a parameter $\varepsilon(k, l) \in (0, \frac{1}{2}]$ as follows:

- (i) $\varepsilon(i, 2) = \varepsilon(2, i) = 1/i$ for $i \geq 2$,
- (ii) $\varepsilon(k, l) = \varepsilon(k-1, l) \cdot \varepsilon(k, l-1)/2$ for $k, l \geq 3$.

Fig. 3. The regions $R(C, w)$, $R^-(C, w)$, and $L(D)$.

We prove Theorem 3 by proving the following statement:

(*) For any $k, l \geq 2$, any configuration X of size at least $1/\varepsilon(k, l)$ contains an $\varepsilon(k, l)$ -good k -cap or an $\varepsilon(k, l)$ -good l -cup.

First we verify (*) for $k = 2, l \geq 2$ and for $l = 2, k \geq 2$, and then continue by induction on $k + l$. The case $m = l$ in the following lemma gives (*) for $k = 2, l \geq 2$.

Lemma 1. *Let X be a configuration of size at least $l \geq 2$ with no $1/l$ -good 2-cap. Then for each $m = 2, 3, \dots, l$, the set X contains a $1/l$ -good m -cup $D_m = d_1 d_2 \cdots d_m$ such that fewer than $m|X|/l$ points of X lie to the left of d_m .*

Proof. First let $m = 2$. Let d_1 be the leftmost point of X , and let X_0 be the set of $\lceil 2|X|/l \rceil$ leftmost points of X . Further, let d_2 be the point viewed from d_1 as the highest

Fig. 4. An ε -good 5-cap and an ε -good 5-cup (the shaded regions contain no point of X).

Fig. 5. The points d_1 and d_2 (the shaded region contains no point of X).

point of $X_0 \setminus \{d_1\}$. Thus, $d_2 \in X_0 \setminus \{d_1\}$ and no point of X_0 lies above the line $d_1 d_2$ (see Fig. 5). Since the open 2-cap $d_1 d_2$ is not $1/l$ -good in X , fewer than $|X|/l - 1$ points of X_0 lie to the right of d_2 . It follows that at least $|X|/l - 1$ points of X_0 lie between d_1 and d_2 (recall that the size of X_0 is $\lceil 2|X|/l \rceil$). Thus, the 2-cup $D_2 = d_1 d_2$ is $1/l$ -good in X and fewer than $2|X|/l$ points of X lie to the left of d_2 . This settles the case $m = 2$.

We further proceed by induction on m . Let $2 \leq m < l$ and let $D_m = d_1 d_2 \cdots d_m$ be a $1/l$ -good m -cup $D_m = d_1 d_2 \cdots d_m$ such that fewer than $m|X|/l$ points of X lie to the left of d_m . Let d_{m+1} be the leftmost point of X lying to the right of d_m and above the line $d_{m-1} d_m$ (see Fig. 6). Such a point d_{m+1} exists, since otherwise the 2-cap $d_{m-1} d_m$ would be $1/l$ -good (at least $|X| - m|X|/l - 1 \geq |X|/l - 1$ points of X lie to the right of d_m). For the same reason fewer than $|X|/l - 1$ points of X lie between d_m and d_{m+1} . It follows that fewer than $m|X|/l + 1 + (|X|/l - 1) = (m+1)|X|/l$ points of X lie to the left of d_{m+1} . Since the $(m+1)$ -cup $D_{m+1} = d_1 d_2 \cdots d_{m+1}$ is $1/l$ -good, this finishes the inductive step from m to $m+1$. $\square$

Fig. 6. The $(m+1)$ -cup $D_{m+1} = d_1 d_2 \cdots d_{m+1}$.

Fig. 7. The upper envelope of X .

We now prove (*) for $l = 2, k \geq 2$. Let $|X| \geq k$, and let $p_1, p_2, \dots, p_t, x(p_1) < x(p_2) < \dots < x(p_t)$, be the vertices of the upper envelope of $\text{conv } X$ (see Fig. 7). If X contains at least $|X|/k - 1$ points p with $x(p_i) < x(p) < x(p_{i+1})$ for some i , then the 2-cup $p_i p_{i+1}$ is $1/k$ -good. Otherwise $t > k$ and there are more than $|X|/k - 1$ points of X to the right of p_k . Thus, $p_1 p_2 \dots p_k$ is a $1/k$ -good k -cap in this case.

It remains to derive (*) for $k, l \geq 3$ from (*) for all $k', l' \geq 2, k' + l' < k + l$. Suppose that $k, l \geq 3$ and that $|X| \geq 1/\varepsilon(k, l)$. We want to prove that X contains an $\varepsilon(k, l)$ -good k -cap or an $\varepsilon(k, l)$ -good l -cup. By the inductive hypothesis, X contains an $\varepsilon(k-1, l)$ -good $(k-1)$ -cap or an $\varepsilon(k-1, l)$ -good l -cup. In the second case we are done, since $\varepsilon(k-1, l) > \varepsilon(k, l)$. Thus, assume that X contains an $\varepsilon(k-1, l)$ -good $(k-1)$ -cap $C = c_1 c_2 \dots c_{k-1}$. Let X_0 be the set of $\lceil \varepsilon(k-1, l)|X| - 1 \rceil$ leftmost points of X lying to the right of c_{k-1} . Since C is $\varepsilon(k-1, l)$ -good, all points of X_0 lie under the line $c_{k-2} c_{k-1}$.

Since $\lceil \alpha - 1 \rceil \geq (\lceil \alpha/4 \rceil - 1) + (\lceil \alpha/4 \rceil - 2) + \lceil \alpha/2 \rceil$ holds¹ for any $\alpha \in \mathbb{R}$, we can partition the set X_0 into three subsets R, S, T of sizes

$$\begin{aligned} |R| &= \left\lceil \frac{\varepsilon(k-1, l)}{4} |X| \right\rceil - 1 > 2 \cdot \varepsilon(k, l-1) \cdot \frac{\varepsilon(k-1, l)}{4} |X| - 1 = \varepsilon(k, l)|X| - 1, \\ |S| &= \left\lceil \frac{\varepsilon(k-1, l)}{4} |X| \right\rceil - 2 > \varepsilon(k, l)|X| - 2, \\ |T| &\geq \left\lceil \frac{\varepsilon(k-1, l)}{2} |X| \right\rceil \end{aligned}$$

(in the estimate of $|R|$ we used that $2 \cdot \varepsilon(k, l-1) < 1$ for any $k, l \geq 3$). The partition is done “from left to right” so that $x(r) < x(s) < x(t)$ holds for any $r \in R, s \in S, t \in T$.

Since

$$|T| \geq \frac{\varepsilon(k-1, l)}{2} |X| \geq \frac{\varepsilon(k-1, l)}{2} \cdot \frac{1}{\varepsilon(k, l)} = \frac{1}{\varepsilon(k, l-1)},$$

the set T contains an $\varepsilon(k, l-1)$ -good k -cap or an $\varepsilon(k, l-1)$ -good $(l-1)$ -cup. In the first case we are done since $|T| \geq (\varepsilon(k-1, l)/2)|X|$ and therefore any k -cap which is $\varepsilon(k, l-1)$ -good in T is $\varepsilon(k, l-1) \cdot (\varepsilon(k-1, l)/2) = \varepsilon(k, l)$ -good in X . So we

¹ The inequality holds with equality for $\alpha = 4m + q, m \in \mathbb{Z}, q \in (0, 1]$. Otherwise the left-hand side is larger than the right-hand side.

Fig. 8. Case 1 (the shaded regions contain no points of X).

may suppose that T contains an $\varepsilon(k, l-1)$ -good $(l-1)$ -cup $D = d_1 d_2 \cdots d_{l-1}$. Since $|T| \geq (\varepsilon(k-1, l)/2)|X|$, D is $\varepsilon(k, l-1) \cdot (\varepsilon(k-1, l)/2) = \varepsilon(k, l)$ -good in X .

Let U be the set of points p of X_0 with $x(p) \leq x(d_1)$. Let a be the point viewed from c_{k-1} as the highest point of U (see Fig. 8). In other words, $a \in U$ and no point of U lies above the line $c_{k-1}a$. We now distinguish three cases.

Case 1: a lies above the line $c_{k-1}d_2$ (see Fig. 8). In this case, $c_1 c_2 \cdots c_{k-1} a$ is an $\varepsilon(k, l)$ -good k -cap in X (recall that D is $\varepsilon(k, l)$ -good in X and thus $|L(D) \cap X| \geq \varepsilon(k, l)|X| - 1$).

Case 2: a lies below the line $c_{k-1}d_2$ and in R (see Fig. 9). In this case, $c_1 c_2 \cdots c_{k-1} a$ is an $\varepsilon(k, l)$ -good k -cap in X ($S \cup \{d_1\}$ lies entirely below the line $c_{k-1}a$).

Case 3: a lies below the line $c_{k-1}d_2$ and in $U \setminus R$ (see Fig. 10). In this case, $c_{k-1} a d_2 d_3 \cdots d_{l-1}$ is an $\varepsilon(k, l)$ -good l -cup in X (R lies entirely below the segment $c_{k-1}a$).

This concludes the proof of Theorem 3. $\square$

3. Proof of Corollaries 1–3

Proof of Corollary 1. Let X be a configuration of size at least $f(k+3, l-1) + 1$. Let p be a vertex of $\text{conv } X$, and let l be a line touching $\text{conv } X$ at the point p . We transform X to a set X' by a projective transformation P which sends the point p to the “point” $(0, \infty)$ and the line l to the line at infinity. The configuration X' has at least $f(k+3, l-1)$ points in the real plane, thus it contains an open $(k+3)$ -cap C or an open $(l-1)$ -cup D .

Fig. 9. Case 2.

Fig. 10. Case 3.

The first case cannot happen, since $P^{-1}(C) \cup \{p\}$ would be an (empty) $(k+4)$ -moon and we would get a contradiction with the k -convexity of X . In the second case the set $P^{-1}(D) \cup \{p\}$ forms an empty l -gon. Thus, $N(k, l) \leq f(k+3, l-1) + 1$. $\square$

Proof of Corollary 2. Since any open l -cup is an empty l -gon, Theorem 3 immediately gives $m(k, l) \leq f(k, l)$. $\square$

Proof of Corollary 3. If a configuration X has at least $2 \cdot f(k-1, l-1)$ points and $p \in X$ is any point in X , then p is a vertex of the convex hull of at least $f(k-1, l-1) + 1$ points of X . We further continue analogously as in the proof of Corollary 1, obtaining $z(k, l) \leq 2 \cdot f(k-1, l-1)$. $\square$

4. Proof of Theorem 4

Proof of Theorem 4. The inequalities $f'(k, l) \leq f(k, l)$, $f''(k, l) \leq f(k, l)$ are trivial, and the proof of Theorem 3 gives the upper bound in (i):

$$f(k, l) \leq \frac{1}{\varepsilon(k, l)} \leq 2^{\binom{k+l-2}{k-1}-1}$$

(the second inequality can be easily proved by induction from the definition of the coefficients $\varepsilon(k, l)$). It remains to prove the lower bounds.

For $k, l \geq 2$ even, we recursively construct a configuration $S_{k,l}$ of size $2^{\binom{k/2+l/2-2}{k/2-1}} - 1$ with no k -cap and no open l -cup. This gives the lower bound in (i). Slight changes in the construction will then give the lower bound in (ii).

If $k = 2$ or $l = 2$ then $S_{k,l} = \{(0, 0)\}$. We continue by induction on $k + l$. Let $k, l \geq 4$ be even, and suppose that the configurations $S_{k-2,l}, S_{k,l-2}$ have already been constructed. We take the configuration $S_{k-2,l}$ and partition the plane into $|S_{k-2,l}| + 1$ regions by the vertical lines through each point of $S_{k-2,l}$. In each region we place a small (shrunk) copy of $S_{k,l-2}$. We denote the copies of $S_{k,l-2}$ by $S^0, S^1, \dots, S^{|S_{k-2,l}|}$. In each S^i we shrink the y -coordinates. Then we rotate and vertically shift each S^i so that any subset of $S^0 \cup \dots \cup S^{|S_{k-2,l}|}$ intersecting each S^i in at most two points is a t -cup for some t . Finally, we shift $S^0 \cup \dots \cup S^{|S_{k-2,l}|}$ vertically so far downwards that the following two conditions hold in the set $S_{k,l} := S_{k-2,l} \cup (S^0 \cup \dots \cup S^{|S_{k-2,l}|})$ (see Fig. 11):

- (C1) If $C = c_1 c_2 \dots c_t$ is a t -cap in $S_{k,l}$, then either it is fully contained in one of the configurations S^i or $c_2 c_3 \dots c_{t-1}$ is a $(t-2)$ -cap in $S_{k-2,l}$.
- (C2) If $D = d_1 d_2 \dots d_t$ is an open t -cup in $S_{k,l}$, then either it is fully contained in $S_{k-2,l}$ or $d_2 d_3 \dots d_{t-1}$ is an open $(t-2)$ -cup in one of the configurations S^i .

It follows from (C1) and (C2) that $S_{k,l}$ contains no k -cap and no open l -cup. Its size is

$$|S_{k,l}| = (|S_{k-2,l}| + 1)(|S_{k,l-2}| + 1) - 1 = 2^{\binom{k/2+l/2-3}{k/2-2}} \cdot 2^{\binom{k/2+l/2-3}{k/2-1}} - 1 = 2^{\binom{k/2+l/2-2}{k/2-1}} - 1.$$

This gives the lower bound in (i).

To get the lower bound in (ii), we slightly change the recursive step in the construction. For $k, l \geq 2$ even, we now construct a configuration $T_{k,l}$ of size $2^{\binom{k/2+l/2-2}{k/2-1}} - 1$ with no

Fig. 11. The configuration $S_{k,l}$ giving the lower bound in Theorem 4(i).

open k -cap and no $(l+1)$ -cup. Again $T_{k,l} = \{(0,0)\}$ if $\min\{k,l\} = 2$. For $k, l \geq 4$ even, the recursive step indicated in Fig. 12 starts similarly as above but the rotations and vertical shifts of the sets T^i are done so that the following hold for the set $T_{k,l} := T_{k-2,l} \cup (T^0 \cup \dots \cup T^{|T_{k-2,l}|})$:

- (C1') If $C = c_1 c_2 \dots c_t$ is an open t -cap in $T_{k,l}$, then either it is fully contained in one of the configurations T^i or $c_2 c_3 \dots c_{t-1}$ is an open $(t-2)$ -cap in $T_{k-2,l}$, and
- (C2') if $D = d_1 d_2 \dots d_t$ is a t -cup in $T_{k,l}$ and $t \geq 5$, then either it is fully contained in $T_{k-2,l}$ or $d_2 d_3 \dots d_{t-1}$ is a $(t-2)$ -cup in one of the configurations T^i .

Then the set $T_{k,l}$ of size $2^{\binom{k/2+l/2-2}{k/2-1}} - 1$ indeed contains no open k -cap and no $(l+1)$ -cup. The lower bound in (ii) follows. $\square$

Fig. 12. The configuration $T_{k,l}$ giving the lower bound in Theorem 4(ii).

Acknowledgment

I am grateful to Gyula Károlyi, Gábor Lippner, and Helena Nyklová for interesting and stimulating discussions. I also thank two anonymous referees for their helpful comments.

References

1. J. Černý, A simple proof for open cups and caps, *European J. Combin.*, to appear.
2. P. Erdős, On some problems of elementary and combinatorial geometry, *Ann. Mat. Pura. Appl.* (4) **103** (1975), 99–108.
3. P. Erdős and Gy. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 464–470.
4. P. Erdős and Gy. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **3/4** (1960–61), 53–62.
5. T. Gerken, On empty convex hexagons in planar point sets, to appear.
6. J. D. Horton, Sets with no empty convex 7-gons, *Canad. Math. Bull.* **26** (1983), 482–484.
7. Gy. Károlyi, J. Pach, and G. Tóth, A modular version of the Erdős–Szekeres theorem, *Studia Sci. Math. Hungar.* **38** (2001), 245–259.
8. Gy. Károlyi, G. Lippner, and P. Valtr, Empty convex polygons in almost convex sets, *Periodica Math. Hungar.*, to appear.
9. G. Kun and G. Lippner, Large convex empty polygons in k -convex sets, *Period. Math. Hungar.* **46** (2003), 81–88.
10. J. Matoušek, *Lectures on Discrete Geometry*, Springer, Berlin, 2002.
11. C. M. Nicolás, Empty convex hexagons in planar point sets, submitted.
12. G. Tóth and P. Valtr, The Erdős–Szekeres theorem: upper bounds and related results, in: J. E. Goodman, J. Pach, and E. Welzl (eds.), *Combinatorial and Computational Geometry*, MSRI Publications 52, 2005, pp. 557–568.
13. P. Valtr, Convex independent sets and 7-holes in restricted planar point sets, *Discrete Comput. Geom.* **7** (1992), 135–152.
14. P. Valtr, On the minimum number of empty polygons in planar point sets, *Studia Sci. Math. Hungar.* **30** (1995), 155–163.
15. P. Valtr, A sufficient condition for the existence of large empty convex polygons, *Discrete Comput. Geom.* **28** (2002), 671–682.
16. P. Valtr, On the empty hexagon theorem, to appear.

Received May 4, 2004, and in revised form June 8, 2006. Online publication April 17, 2007.