Covering Planar Graphs with a Fixed Number of Balls

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Abstract. In this note we prove that there exists a constant ρ such that any planar graph G of diameter $\leq 2R$ can be covered with at most ρ balls of radius R , a result conjectured by Gavaille, Peleg, Raspaud, and Sopena in 2001.

1. Introduction

Let $G = (V, E)$ be a finite connected planar graph. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v) -path. The largest distance D in G is called the *diameter* of G . For a graph G of diameter $D \leq 2R$, the *ball* of center v and radius R consists of all vertices of G at distance at most R from v and is denoted by $B(v)$. Let $\mathcal{B}_R(G)$ be the set of all balls of radius R of such a graph G . Denote by $\rho(G) = \rho(\mathcal{B}_R(G))$ the minimum number of balls of radius R covering G . In this note we prove the following conjecture communicated to us by Gavaille in 1999 and formulated by Gavaille et al. in [4].

Theorem 1. *There exists a constant ρ such that any planar graph G of diameter $\leq 2R$ can be covered with at most ρ balls of radius R (i.e., $\rho(G) \leq \rho$).*

The main ingredients in the proof of Theorem 1 are the fact that the VC-dimension of $\mathcal{B}_R(G)$ for planar graphs G is at most four, the recent result of Matoušek [7] showing that every set system with dual VC-dimension $q - 1$ has the Hadwiger–Debrunner (p, q) -property, and the result of Pach et al. [9] and Agarwal et al. [1] that any topological graph with n vertices and without k pairwise crossing edges has $O(n \log^{2k-6} n)$ edges (the last result is used to prove that $\mathcal{B}_R(G)$ satisfies the $(p, 5)$ -property for a certain integer $p \geq 5$). We do not know how large ρ can be; a planar graph G of diameter $2R$ with $\rho(G) \geq 4$ is presented in [4].

2. Definitions and Preliminary Results

In this section we present a few definitions and notations on set systems and topological graphs (see [6] and [8] for additional information) and formulate the results from [7], [1], and [9] used in the proof.

Let $\mathcal{F}$ be a set system on a finite ground set V . A subset $T \subseteq V$ is called a *transversal* of $\mathcal{F}$ if $T \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. The *transversal number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is the smallest cardinality of a transversal of $\mathcal{F}$. A subfamily $\mathcal{F}' \subseteq \mathcal{F}$ is called a *covering* if $\bigcup\{F \in \mathcal{F}'\} = V$. The *covering number* of $\mathcal{F}$, denoted by $\rho(\mathcal{F})$, is the minimum number of sets of $\mathcal{F}$ required to cover the ground set X . The *dual set system* to $(V, \mathcal{F})$ is the set system $(W, \mathcal{F}^*)$ defined as follows [6]: The ground set is $W = \{w_F : F \in \mathcal{F}\}$, where the w_F are $|\mathcal{F}|$ distinct points, and for each $v \in V$ we have the set $\{w_F : F \in \mathcal{F}, v \in F\}$ in $\mathcal{F}^*$. Notice the well-known equalities $\tau(\mathcal{F}) = \rho(\mathcal{F}^*)$ and $\rho(\mathcal{F}) = \tau(\mathcal{F}^*)$. In the particular case of balls of a graph G , we have $\mathcal{B}_R(G) = \mathcal{B}_R^*(G)$. Indeed, if for a ball $B(x)$ we set $w_{B(x)} = x$, then for any $v \in V$ we obtain

$$\{w_{B(x)} : v \in B(x)\} = \{x \in V : v \in B(x)\} = \{x \in V : x \in B(v)\} = B(v).$$

The *Vapnik–Chervonenkis dimension* (or *VC-dimension* for short) of a set system $\mathcal{F}$ on a ground set V is the maximum size of a set $X \subseteq V$ that is *shattered* by $\mathcal{F}$, meaning that $\{X \cap F : F \in \mathcal{F}\} = 2^X$. It is well known [6] that the VC-dimension of the dual of a set system of VC-dimension k is at most 2^k . Let $p \geq q \geq 2$ be integers. A family of sets $\mathcal{F}$ satisfies the (p, q) -property if among every p sets of $\mathcal{F}$, some q have a point in common. The key step in our proof is to ensure the conditions of the following result of Matoušek:

Theorem A [7]. *Let $\mathcal{F}$ be a set system such that the VC-dimension of $\mathcal{F}^*$ equals $q - 1$, and let $p \geq q$. Then there exists a constant ρ such that if $\mathcal{F}$ satisfies the (p, q) -property, then $\tau(\mathcal{F}) = \rho(\mathcal{F}^*) \leq \rho$; that is, there exists a ρ -point set intersecting all sets of $\mathcal{F}$.*

The proof of this result is a generalization to set systems of the Alon and Kleitman solution method [2] of the old piercing problem of Hadwiger and Debrunner for convex sets. Namely, Matoušek [7] first establishes a fractional Helly property for set systems of bounded VC-dimension, and then LP duality and the existence of ε -nets for systems of bounded VC-dimension imply the result as in the Alon–Kleitman proof.

Since $\mathcal{B}_R(G) = \mathcal{B}_R^*(G)$, in order to derive Theorem 1 from Theorem A it suffices to show that the VC-dimension of $\mathcal{B}_R(G)$ is bounded by some constant $q - 1$ and that $\mathcal{B}_R(G)$ satisfies the (p, q) -property for some $p \geq q$. In fact, we show that the VC-dimension of $\mathcal{B}_R(G)$ is at most four. In order to establish the existence of a p such that $\mathcal{B}_R(G)$ satisfies the $(p, 5)$ -property, we employ a result from geometrical graph theory which we recall next.

A *topological graph* is a graph drawn in the plane in such a way that its edges are represented by simple Jordan arcs so that (T1) no edge passes through any vertex other than its endpoints, (T2) no three edges cross at the same point, and (T3) no two arcs meet in more than one point (two edges *cross* each other if they have an interior point

in common). The following statement is a direct corollary of the result in [1] and [9] mentioned above (see also [10]):

Theorem B [1], [9]. *Every complete topological graph on n vertices contains at least $c \log n / \log \log n$ pairwise crossing edges.*

3. Proof of Theorem 1

Throughout this section we suppose that $G = (V, E)$ is a connected planar graph of diameter $D \leq 2R$ and that a straight line plane drawing of G is given.

Proposition 1. *The VC-dimension of $\mathcal{B}_R(G)$ is at most four.*

Proof of Proposition 1. Suppose by way of contradiction that there exists a five-point set $A = \{a_1, \dots, a_5\}$ of G that is shattered by $\mathcal{B}_R(G)$. For each pair of distinct vertices a_i, a_j of A , denote by c_{ij} the center of a ball of radius R such that $B(c_{ij}) \cap A = \{a_i, a_j\}$. Let b_{ij} be a vertex of G furthest away from c_{ij} that lies simultaneously on a shortest path between c_{ij} and a_i , and on a shortest path between c_{ij} and a_j . Denote by P_{ij} an (a_i, a_j) -path consisting of a shortest path between a_i and b_{ij} and a shortest path between b_{ij} and a_j .

Claim 1. *For any four distinct vertices $a_i, a_j, a_{i'}, a_{j'} \in A$, the paths P_{ij} and $P_{i'j'}$ are disjoint.*

Proof of Claim 1. Suppose that P_{ij} and $P_{i'j'}$ share a common vertex, say a vertex x which belongs to the subpath of P_{ij} comprised between b_{ij} and a_i and to the subpath of $P_{i'j'}$ comprised between $b_{i'j'}$ and $a_{i'}$ (see Fig. 1(a)). Now, if $d(x, a_i) \leq d(x, a_{i'})$, then

$$\begin{aligned} d(c_{i'j'}, a_i) &\leq d(c_{i'j'}, x) + d(x, a_i) \\ &\leq d(c_{i'j'}, x) + d(x, a_{i'}) \\ &= d(c_{i'j'}, a_{i'}) \leq R, \end{aligned}$$

yielding $a_i \in B(c_{i'j'})$, which is impossible because $B(c_{i'j'}) \cap A = \{a_{i'}, a_{j'}\}$. $\square$

Fig. 1. To the proof of Claims 1 and 2.

Claim 2. *For any three distinct vertices $a_i, a_j, a_r \in A$, the intersection $P_{ij} \cap P_{ir} \cap P_{jr}$ is empty.*

Proof of Claim 2. Suppose not, and let $x \in P_{ij} \cap P_{ir} \cap P_{jr}$ (see Fig. 1(b)). Now, if $d(x, a_i) \leq d(x, a_j) \leq d(x, a_r)$, then a_i belongs to the ball $B(c_{jr})$, because if, say, b_{jr} belongs to the subpath of P_{rj} comprised between x and a_j , then

$$\begin{aligned} d(c_{jr}, a_i) &\leq d(c_{jr}, b_{jr}) + d(b_{jr}, x) + d(x, a_i) \\ &\leq d(c_{jr}, b_{jr}) + d(b_{jr}, x) + d(x, a_r) \\ &= d(c_{jr}, a_r) \leq R. \end{aligned}$$

This contradiction completes the proof of the claim. $\square$

For a vertex $a_i \in A$, let $S_i := \bigcup \{P_{ij} \cap P_{ij'} : a_j, a_{j'} \in A, a_j \neq a_{j'}\}$.

Claim 3. *The sets $S_1, \dots, S_5$ are pairwise disjoint.*

Proof of Claim 3. Suppose by way of contradiction that for distinct vertices a_i, a_j , the sets S_i and S_j share a common vertex x , say $x \in (P_{ir} \cap P_{ir'}) \cap (P_{jt} \cap P_{jt'})$. If $\{i, r, r'\} = \{j, t, t'\}$, say, $i = t', j = r'$, and $r = t$, then we get a contradiction with Claim 2. In all other cases, we obtain a contradiction with Claim 1. Indeed, if $j \in \{r, r'\}$, say $j = r'$, then $x \in P_{ir} \cap P_{jt}$ if $r \neq t$ and $x \in P_{ir} \cap P_{jt'}$ if $r \neq t'$ (the case $i \in \{t, t'\}$ is analogous). Finally, if $i \notin \{t, t'\}$ and $j \notin \{r, r'\}$, then, since $r \neq r'$ and $t \neq t'$, we can pick two different vertices, one in $\{r, r'\}$ and another one in $\{t, t'\}$, say $r \neq t$, yielding $x \in P_{ir} \cap P_{jt}$. This concludes the proof of Claim 3. $\square$

Thus the sets $S_1, \dots, S_5$ induce pairwise disjoint connected subgraphs of G . Contracting each of these subgraphs to a vertex and replacing the non-contracted part of each path P_{ij} by an edge, in view of Claim 1 we will obtain the complete graph K_5 on five vertices. Hence, G has a K_5 -minor, in contradiction with the planarity of G . $\square$

Let p be an integer such that $c \log p / \log \log p \geq 7$. According to Theorem B, every complete topological graph with p vertices contains at least seven pairwise crossing edges.

Proposition 2. *$\mathcal{B}_R(G)$ satisfies the $(p, 5)$ -property, i.e., among any set $\mathcal{B}$ of p balls of radius R of G there are some five that intersect.*

Proof of Proposition 2. Let $C = \{c_1, \dots, c_p\}$ be the set of centers of the balls from $\mathcal{B}$. For each pair of balls $B(c_i), B(c_j)$ denote by P_{ij} a shortest path between their centers c_i and c_j . We can select these shortest paths in such a way that every intersection $P_{ij} \cap P_{rt}$ is either empty or a path: if not, it suffices to replace every subpath of P_{rt} between two consecutive common vertices of P_{ij} and P_{rt} by the subpath of P_{ij} between the same vertices. Since $d(c_i, c_j) \leq D = 2R$, on the path P_{ij} there exists a vertex which belongs to both balls $B(c_i)$ and $B(c_j)$. Denote this vertex by b_{ij} .

Notice that if C contains eight distinct vertices $c_{i_1}, c_{j_1}, c_{i_2}, c_{j_2}, c_{i_3}, c_{j_3}, c_{i_4}, c_{j_4}$ such that $\bigcap_{k=1}^4 P_{i_k j_k} \neq \emptyset$, then among the eight balls centered at these vertices there are five that intersect. Indeed, let $x \in \bigcap_{k=1}^4 P_{i_k j_k}$, and assume without loss of generality that for any $k \in \{1, 2, 3, 4\}$ the vertex $b_{i_k j_k}$ belongs to the subpath of $P_{i_k j_k}$ comprised between x and c_{j_k} . Now, if c_{i_4} is furthest from the x vertex among the centers $c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}$, then the vertex $b_{i_4 j_4}$ belongs to the balls $B(c_{i_1}), B(c_{i_2}), B(c_{i_3}), B(c_{i_4})$, and $B(c_{j_4})$. Thus, we can further assume that $\bigcap_{k=1}^4 P_{i_k j_k} = \emptyset$ for any eight distinct centers of C .

Claim 4. *The set $\mathcal{B}$ contains 14 different balls $B(c_1), B(c_{\overline{1}}), \dots, B(c_7), B(c_{\overline{7}})$ such that the paths $P_{1\overline{1}}, \dots, P_{7\overline{7}}$ pairwise intersect in G .*

Proof of Claim 4. Consider the drawing Γ of the complete graph on p vertices having C as the vertex set and the Jordan arcs $P_{ij}, c_i, c_j \in C$, as the set of edges. Γ is not a topological graph, however, it can be transformed into a topological graph Γ' having the same set of vertices C and such that any pair of crossing arcs $Q_{ij}, Q_{rt}, \{i, j\} \cap \{r, t\} = \emptyset$, in Γ' corresponds to a pair of intersecting paths P_{ij}, P_{rt} in Γ . For this, we use the standard operations that are indicated in Fig. 2. To ensure the condition (T1), we use a sequence of operations of type (1), (T2) is obtained by using the standard transformation (2) and the fact that no four arcs of Γ share a common point, and, finally, since the intersection of two paths is either empty or a path, condition (T3) can be satisfied by employing a sequence of transformations (3a) and (3b). From Theorem B and the definition of p we conclude that the topological graph Γ' contains seven pairwise crossing edges, which give rise to seven pairwise intersecting paths $P_{1\overline{1}}, \dots, P_{7\overline{7}}$ of the initial graph G . The endvertices $c_1, c_{\overline{1}}, \dots, c_7, c_{\overline{7}}$ of these paths are centers of 14 different balls from $\mathcal{B}$. This concludes the proof of Claim 4. $\square$

Let $P_{1\overline{1}}, \dots, P_{7\overline{7}}$ be the seven pairwise intersecting paths defined in Claim 4. For a path $P_{i\overline{i}}, i = 1, \dots, 7$, and a vertex $v \in P_{i\overline{i}}$, denote by $P_i(v)$ the subpath of $P_{i\overline{i}}$ comprised between v and c_i minus v and denote by $P_{\overline{i}}(v)$ the subpath of $P_{i\overline{i}}$ comprised between v and $c_{\overline{i}}$ minus v . In other terms, the paths $P_i(v)$ and $P_{\overline{i}}(v)$ are obtained from $P_{i\overline{i}}$ by removing the vertex v . For two different paths $P_{i\overline{i}}$ and $P_{j\overline{j}}$, let x be their common vertex. Then the vertex $b_{i\overline{i}}$ does not belong to one of the paths $P_i(x), P_{\overline{i}}(x)$. Analogously, the vertex $b_{j\overline{j}}$ does not belong to one of the paths $P_j(x), P_{\overline{j}}(x)$. If $\alpha \in \{i, \overline{i}\}$ and $\beta \in \{j, \overline{j}\}$ are such that $b_{i\overline{i}} \notin P_\alpha(x)$ and $b_{j\overline{j}} \notin P_\beta(x)$, then we call $\Phi(i, j) = \{c_\alpha, c_\beta\}$ a Φ -pair for the paths $P_{i\overline{i}}$ and $P_{j\overline{j}}$; see Fig. 3 for an illustration. Notice that if $x \notin \{b_{i\overline{i}}, b_{j\overline{j}}\}$, then among the four pairs $\{c_i, c_j\}, \{c_i, c_{\overline{j}}\}, \{c_{\overline{i}}, c_j\}, \{c_{\overline{i}}, c_{\overline{j}}\}$ exactly one constitutes a Φ -pair. If $x = b_{i\overline{i}}$ or $x = b_{j\overline{j}}$, then we have exactly two Φ -pairs, and if $b_{i\overline{i}} = x = b_{j\overline{j}}$, then all four pairs are Φ -pairs. In the last two cases we can take anyone of these Φ -pairs as $\Phi(i, j)$. Given $\Phi(i, j) = \{c_\alpha, c_\beta\}$ with $\alpha \in \{i, \overline{i}\}$ and $\beta \in \{j, \overline{j}\}$, define the binary relation $\prec$ by setting $i \prec j$ if and only if i is distinct from j and $d(c_\alpha, x) \leq d(c_\beta, x)$.

Claim 5. *If $i \prec j$, then $b_{j\overline{j}} \in B(c_\alpha)$.*

Fig. 2. To the proof of Claim 4.

Proof of Claim 5. Indeed, since $d(c_\alpha, x) \leq d(c_\beta, x)$ and $b_{j\bar{j}} \in B(c_\beta)$, we deduce that

$$\begin{aligned}
 d(c_\alpha, b_{j\bar{j}}) &\leq d(c_\alpha, x) + d(x, b_{j\bar{j}}) \\
 &\leq d(c_\beta, x) + d(x, b_{j\bar{j}}) \\
 &= d(c_\beta, b_{j\bar{j}}) \leq R.
 \end{aligned}$$

□

Define a tournament $\mathcal{T}$ with seven vertices $v_1, \dots, v_7$, in which we draw an arc from vertex v_i to vertex v_j if and only if $i < j$. The sum of in-degrees of the vertices of $\mathcal{T}$ is equal to the number of arcs of $\mathcal{T}$. Since $\mathcal{T}$ contains 15 arcs and 7 vertices, necessarily at least one vertex of $\mathcal{T}$, say v_7 , has in-degree larger than or equal to 3. Let v_1, v_2 , and v_3 be the vertices having outgoing arcs to v_7 . Consider the paths $P_{1\bar{7}}, P_{2\bar{7}}, P_{3\bar{7}}$, and the path $P_{7\bar{7}}$ of the graph G . Suppose without loss of generality that $c_1 \in \Phi(1, 7)$, $c_2 \in \Phi(2, 7)$, and $c_3 \in \Phi(3, 7)$. From the definition of the vertex $b_{7\bar{7}}$ we know that it belongs to the balls

Fig. 3. $\Phi(i, j) = \{c_i, c_{\bar{j}}\}$.

$B(c_7)$ and $B(c_{\bar{7}})$. Since $1 < 7$, $2 < 7$, and $3 < 7$, from Claim 5 we infer that $b_{7\bar{7}}$ also belongs to the balls $B(c_1)$, $B(c_2)$, and $B(c_3)$. This shows that $b_{7\bar{7}}$ belongs to five balls of $\mathcal{B}$, thus establishing the $(p, 5)$ -property. This completes the proof of Proposition 2 and Theorem 1.

Remark 1. Proposition 1 generalizes in a straightforward way to the system of all balls of a planar graph, and, more generally, to the system of all balls of a graph without K_{r+1} -minors: in that case, the VC-dimension is at most r . Notice also that the VC-dimension for radius 1 balls was introduced by Haussler and Welzl in [5].

Remark 2. Theorem 1 extends to planar graphs of diameter $\leq 2R + 1$, however, it is no longer true for planar graphs of diameter $\geq 2R + 2$. For this, consider a star with n leaves in which every edge is replaced by a path of length $R + 1$. The resulting tree T has diameter $2R + 2$. Every radius R ball covers at most one leaf of T because the distance between any two leaves is $2R + 2$. Therefore, $\rho(T) \geq n$.

Remark 3. It is known that $\rho(G) = 2$ if G is outerplanar [4] or if G is a plane triangulation with all inner vertices of degree at least six [3]. On the other hand, Gavioille et al. [4] present for any $k \geq 8$ a planar graph G_k of diameter $2k$ for which $\rho(G_k) \geq 4$.

Remark 4. From Theorem 8 of [4] and Theorem 1 it follows that any planar graph of diameter D has an interval routing scheme with dilation at most $\lceil 1.5D \rceil$ and a constant number of intervals per edge.

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References

1. P.K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir, Quasi-planar graphs have a linear number of edges, *Combinatorica* **17** (1997), 1–9.
2. N. Alon and D. Kleitman, Piercing convex sets and the Hadwiger–Debrunner (p, q) -problem, *Adv. Math.* **96** (1992), 103–112.

3. V. Chepoi and Y. Vaxès, On covering planar bridged triangulations with balls, *J. Graph Theory* **44** (2003), 65–80.
4. C. Gavoille, D. Peleg, A. Raspaud, and E. Sopena, Small k -dominating sets in planar graphs with applications, *Proc. 27th International Workshop on Graph-Theoretic Concepts in Computer Science*, 2001, pp. 201–216, Lecture Notes in Computer Science, Vol. 2204, Springer-Verlag, Berlin, 2001.
5. D. Haussler and E. Welzl, Epsilon-nets and simplex range queries, *Discrete Comput. Geom.* **2** (1987), 127–151.
6. J. Matoušek, *Lectures on Discrete Geometry*, Springer, New York, 2002.
7. J. Matoušek, Bounded VC-dimension implies a fractional Helly theorem, *Discrete Comput. Geom.* **31** (2004), 251–255.
8. J. Pach and P. Agarwal, *Combinatorial Geometry*, Wiley, New York, 1995.
9. J. Pach, F. Shahrokhi, and M. Szegedy, Applications of the crossing number, *Algorithmica* **16** (1996), 111–117.
10. J. Pach, J. Solymosi, and G. Tóth, Unavoidable configurations in complete topological graphs, *Discrete Comput. Geom.* **30** (2003), 311–320.

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