Covering n -Segment Unit Arcs Is Not Sufficient*

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Abstract. In 1974 Gerriets and Poole conjectured for $n = 3$ that a convex set in the plane which contains a congruent copy of every n -segment polygonal arc of unit length must be a cover for the family of all unit arcs. We disprove this general conjecture by describing for each positive integer n a convex region R_n that contains a congruent copy of every n -segment unit arc but not a congruent copy of every unit arc.

Nearly 40 years ago, Moser [4] posed the so-called “worm problem”: “What is the region of smallest area which will accommodate every arc of length 1?” Such a set is called a *cover* for the family of all unit arcs. Gerriets and Poole [3] showed in 1974 that the convex region formed by clipping a small isosceles triangle off one of the two 120° angles of a 60° rhombus of side $1/\sqrt{3}$, a convex region with area about 0.2861, is such a cover, but further progress on this problem has been skimpy (see [7]).

It is generally rather difficult to show that a proposed set is a cover. In their 1974 article Gerriets and Poole formulated the following conjecture, which we quote

* C. Panraksa was supported in part by the Development and Promotion of Science and Technology Talents Project of Thailand. W. Wichiramala was supported in part by Grants for the Development of New Faculty Staff from Chulalongkorn University.

Fig. 1. The unit broadworm.

verbatim:

If a convex region contains all “two angle worms” of length L , then the region contains all worms of length L .

By “two angle worms” we mean all arcs formed by joining consecutively, at any angles, three segments whose total length is L .

This conjecture has been widely reported in subsequent literature about the problem.

In this brief note we *disprove* the n -segment version of this conjecture: for each positive integer n we describe a compact, convex set that contains a congruent copy of every n -segment polygonal arc of unit length but not of every unit arc.

Recall that the *breadth* (also called the *thickness* or *minimum width*) $t(\mathcal{H})$ of a compact, convex set $\mathcal{H}$ in the plane is the least distance between two parallel support lines of $\mathcal{H}$. By the *breadth of an arc* γ we mean the breadth of its convex hull.

The shortest arc of unit breadth was described in 1961 by Zalgaller [8] and again, more than 25 years later, by Adhikari and Pitman [1]. (See also [2].) Scaled to unit length, this arc, which in 1968 Schaer [5] aptly named the *broadworm*, is the *unique* arc of unit length whose breadth is as large as possible. Details and references can be found in [2] and also in [6]. The endpoints of the unit broadworm (Fig. 1) are $d \approx 0.4581$ apart, and its breadth is $b_0 \approx 0.4389$. This is the largest possible breadth of any unit arc.

For each positive integer n let $\mathcal{F}_n$ be the family of n -segment unit arcs, and let

$$t_n = \text{lub}\{t(\gamma) : \gamma \in \mathcal{F}_n\}.$$

Lemma. *For each positive integer n there is an n -segment unit arc γ such that $t(\gamma) = t_n$.*

Proof. Suppose $n \geq 3$ is a given, fixed integer. Let $\langle \gamma_i \rangle_{i=1}^\infty$ be a maximizing sequence of n -segment unit arcs, i.e., a sequence in $\mathcal{F}_n$ so that

$$\lim_{i \rightarrow \infty} t(\gamma_i) = t_n.$$

Place each arc γ_i with its initial point at the origin, so that

$$\gamma_i = [V_0^i, V_1^i, V_2^i, \dots, V_n^i]$$

with $V_0^i = 0$, and note that then every vertex V_j^i lies in the closed disk of radius 1 about the origin. To define the desired limit arc γ in $\mathcal{F}_n$, we determine its vertices $V_0, V_1, \dots, V_n$ seriatim. Let $V_0 = 0$. According to the Bolzano–Weierstrass theorem, there is a subsequence of $\langle \gamma_i \rangle_{i=1}^\infty$, which we again denote by $\langle \gamma_i \rangle$, so that the sequence $\langle V_1^i \rangle_{i=1}^\infty$ converges, say to V_1 . Suppose the vertices $V_0, V_1, \dots, V_k$ have been defined and a subsequence found, again denoted by $\langle \gamma_i^n \rangle$, so that for each $j = 1, 2, \dots, k$,

$$V_j = \lim_{i \rightarrow \infty} V_j^i.$$

If $k < n$ there is a subsequence of $\langle \gamma_i \rangle_{i=1}^\infty$, which we again denote by $\langle \gamma_i \rangle$, so that the sequence $\langle V_{k+1}^i \rangle_{i=1}^\infty$ converges, say to V_{k+1} . We iterate this process until $k = n$. The n -segment polygonal arc $\gamma = [V_0, V_1, V_2, \dots, V_n]$ so defined has length one, and its breadth is the desired least upper bound, namely, t_n . $\square$

Corollary. For each positive integer n , $t_n < b_0$.

Proof. If $t_n = b_0$, then the n -segment unit arc of breadth $t_n = b_0$ whose existence is guaranteed by the lemma contradicts the uniqueness of the broadworm. $\square$

Theorem. For each positive integer n there is a compact, convex region R_n that contains a congruent copy of each n -segment unit arc but does not contain a congruent copy of the unit broadworm.

Proof. Every n -segment unit arc fits in a $t_n \times 1$ rectangular region, but the broadworm does not fit in this rectangle because $b_0 > t_n$. $\square$

References

1. Ani Adhikari and Jim Pitman, The Shortest Planar Arc of Width 1, *Amer. Math. Monthly* **96** (1989), 309–327.
2. Steven R. Finch and John E. Wetzel, Lost in a Forest, *Amer. Math. Monthly* **111** (2004), 645–654.
3. John Gerriets and George Poole, Convex Regions Which Cover Arcs of Constant Length, *Amer. Math. Monthly* **81** (1974), 36–41.
4. Leo Moser, Poorly Formulated Unsolved Problems in Combinatorial Geometry, mimeographed, undated, but probably 1966.
5. Jonathan Schaer, The Broadest Curve of Length 1, Research Paper No. 52, Department of Mathematics, Statistics, and Computer Science, University of Calgary, May 1968.
6. Jonathan Schaer and John E. Wetzel, Boxes for Curves of Constant Length, *Israel J. Math.*, **12** (1972), 257–265.
7. John E. Wetzel, The Classical Worm Problem—a Status Report, *Geombinatorics* **XV** (2005), 34–42.
8. Viktor A. Zalgaller, How to Get Out of the Woods? On a Problem of Bellman (in Russian), *Mat. Prosveshchenie* **6** (1961), 191–195.

Received December 30, 2005, and in revised form April 22, 2006. Online publication February 5, 2007.