Lattice Points in Large Borel Sets and Successive Minima*

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Abstract. Let B be a Borel set in $\mathbb{E}^d$ with volume $V(B) = \infty$. It is shown that almost all lattices L in $\mathbb{E}^d$ contain infinitely many pairwise disjoint d -tuples, that is sets of d linearly independent points in B . A consequence of this result is the following: let S be a star body in $\mathbb{E}^d$ with $V(S) = \infty$. Then for almost all lattices L in $\mathbb{E}^d$ the successive minima $\lambda_1(S, L), \dots, \lambda_d(S, L)$ of S with respect to L are 0. A corresponding result holds for most lattices in the Baire category sense. A tool for the latter result is the semi-continuity of the successive minima.

1. Introduction and Statement of Results

A lattice L in Euclidean d -space $\mathbb{E}^d$ is the system of all integer linear combinations of d linearly independent vectors in $\mathbb{E}^d$. These vectors form a *basis* of L and the absolute value of their determinant is the *determinant* $d(L)$ of L . $d(L)$ is independent of the particular choice of a basis. To each lattice L there correspond all $d \times d$ matrices, the column vectors of which form a basis of L . Identify each such matrix with a point in $\mathbb{E}^{d^2}$. There is a Borel set $\mathcal{F}$ in $\mathbb{E}^{d^2}$ consisting of such matrices, which has infinite Lebesgue measure and such that to each lattice L there corresponds precisely one matrix in $\mathcal{F}$. Thus there is a one-to-one correspondence between the space $\mathcal{L}$ of all lattices in $\mathbb{E}^d$ and $\mathcal{F}$. The Lebesgue measure on $\mathcal{F}$ then yields a measure ν on $\mathcal{L}$.

Results of Rogers [7] (for $d \geq 3$) and Schmidt [8] (for $d = 2$) show that for a Borel set B in $\mathbb{E}^d$ with Lebesgue measure $V(B) = \infty$, ν -almost all lattices L contain

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infinitely many primitive points in B , where a point $l \in L$ is *primitive* if it is different from the origin o and on the line-segment $[o, l]$ there are no points of L , except o and l . A refinement of this result is as follows.

Theorem 1. *Let B be a Borel set in $\mathbb{E}^d$ with $V(B) = \infty$. Then for ν -almost every lattice $L \in \mathcal{L}$, the set B contains infinitely many, pairwise disjoint d -tuples of linearly independent primitive points of L .*

Tools for the proof are measure-theoretic results of Rogers [7] and Schmidt [8] and a result of Yao and Yao [10] from applied computational geometry on the dissection of sets in $\mathbb{E}^d$.

A *star body* S in $\mathbb{E}^d$ is a closed set with o in its interior such that each ray with endpoint o meets the boundary of S in at most one point. Equivalently, $S = \{x: f(x) \leq 1\}$, where $f: \mathbb{E}^d \rightarrow \mathbb{R}$ is a *distance function*, i.e. it is non-negative, continuous and positively homogeneous of degree 1. The *successive minima* of S or f with respect to a lattice L are defined as follows:

$$\begin{aligned}\lambda_i(S, L) &= \lambda_i(f, L) \\ &= \inf\{\lambda > 0: \lambda S \cap L \text{ contains } i \text{ linearly independent vectors}\} \\ &= \inf\{\max\{f(l_1), \dots, f(l_i)\}: l_1, \dots, l_i \in L \text{ linearly independent}\}\end{aligned}$$

for $i = 1, \dots, d$. Clearly,

$$0 \leq \lambda_1(S, L) \leq \dots \leq \lambda_d(S, L) \leq \infty. \quad (1)$$

Successive minima play an important role in the geometry of numbers, algebraic number theory, Diophantine approximation and computational geometry, see, e.g., [3], [2], [9], [5] and [1]. For a surprising relation to Nevanlinna's value distribution theory see [4].

Let $\mathcal{L}$ be endowed with its natural topology, see [3]. Then $\mathcal{L}$ is locally compact by Mahler's compactness theorem. Thus a version of the Baire category theorem implies that $\mathcal{L}$ is *Baire*. That is, any meager set has dense complement, where a set is *meager* or *of the first Baire category*, if it is a countable union of nowhere dense sets, see [6].

Theorem 2. *Let S be a star body in $\mathbb{E}^d$ with $V(S) = \infty$. Then $\lambda_1(S, L) = \dots = \lambda_d(S, L) = 0$ for*

- (i) ν -almost all lattices L in $\mathcal{L}$ and for
- (ii) all lattices L in $\mathcal{L}$, with a meager set of exceptions.

Tools for the proof are Theorem 1 and a semi-continuity result for successive minima which may be described as follows:

Let (S_n) be a sequence of star bodies and let (f_n) be the corresponding sequence of distance functions. Then (S_n) *converges* to a star body S with corresponding distance function f if the sequence (f_n) converges uniformly to f on the solid unit ball $\{x: \|x\| \leq 1\}$ of $\mathbb{E}^d$. A sequence (L_n) of lattices *converges* to a lattice L , if there are

bases $\{b_{n1}, \dots, b_{nd}\}$ of L_n and $\{b_1, \dots, b_d\}$ of L such that $b_{n1} \rightarrow b_1, \dots, b_{nd} \rightarrow b_d$. This notion of convergence induces the topology on $\mathcal{L}$.

Lemma. *Let (S_n) be a sequence of star bodies and let (L_n) be a sequence of lattices in $\mathbb{E}^d$, converging to a star body S and a lattice L , respectively. Then*

- (i) $\limsup_{n \rightarrow \infty} \lambda_i(S_n, L_n) \leq \lambda_i(S, L)$, for $i = 1, \dots, d$, and
- (ii) *if S is bounded, then $\lim_{n \rightarrow \infty} \lambda_i(S_n, L_n)$ exists and is equal to $\lambda_i(S, L)$, for $i = 1, \dots, d$.*

To see that λ_i is *not* continuous, let S be a star body with $V(S) = \infty$ such that there is a lattice L which has only o in common with the interior of S , for example, the star body $\{x: |x_1 \cdots x_d| \leq 1\}$, see p. 28 of [3]. Then $1 \leq \lambda_i(S, L) < \infty$, while by Theorem 2 there is a sequence (L_n) of lattices such that $L_n \rightarrow L$ with $\lambda_i(S, L_n) = 0$ for all n .

2. Proof of Theorem 1

A result of Yao and Yao [10] says that any mass distribution in $\mathbb{E}^d$ with positive, continuous density which tends rapidly to 0 as $\|x\| \rightarrow \infty$, and of total mass V , can be dissected into 2^d disjoint Borel parts, each of mass $2^{-d}V$ and such that no hyperplane meets all these 2^d masses. We need the following version of this result:

Let $A \subset \mathbb{E}^d$ be a bounded Borel set with volume $V(A) > V > 0$. Then A contains 2^d pairwise disjoint Borel subsets, each of volume $2^{-d}V$ and such (2) that no $(d-1)$ -dimensional subspace of $\mathbb{E}^d$ meets each of these 2^d sets.

To see (2), choose a compact set $C \subset A$ with $V(C) > V$. This is possible by the inner regularity of the Lebesgue measure. Next, choose a continuous function $g: \mathbb{E}^d \rightarrow \mathbb{R}^+$ such that

$$g \geq \chi_C, \quad \int_{\mathbb{E}^d} (g - \chi_C) dx < 2^{-d}(V(C) - V), \quad g(x) \rightarrow 0 \text{ rapidly as } \|x\| \rightarrow \infty,$$

where χ_C is the characteristic function of C . This is possible by the outer regularity of the Lebesgue measure and Urysohn's lemma. Let $F_i, i = 1, \dots, 2^d$, be a dissection of $\mathbb{E}^d$ for the density g as described by Yao and Yao such that

$$\int_{F_i} g dx = 2^{-d} \int_{\mathbb{E}^d} g dx \geq 2^{-d} V(C).$$

Then there is no $(d-1)$ -dimensional subspace of $\mathbb{E}^d$ which meets each of the sets $C \cap F_i$, and for the respective volumes of these sets we have the following estimate:

$$\begin{aligned} V(C \cap F_i) &= \int_{F_i} \chi_C dx = \int_{F_i} g dx - \int_{F_i} (g - \chi_C) dx \geq \int_{F_i} g dx - \int_{\mathbb{E}^d} (g - \chi_C) dx \\ &> 2^{-d} V(C) - 2^{-d}(V(C) - V) = 2^{-d} V. \end{aligned}$$

This implies (2).

For the proof of Theorem 2 assume first that $d \geq 3$. The following result is an immediate consequence of a result of Rogers [7, p. 286]:

Let $k = 1, 2, \dots$, and let A be a Borel set in $\mathbb{E}^d$ with $0 < V(A) < \infty$. Then the function $\#^*(A \cap \cdot): \mathcal{L} \rightarrow \{0, 1, \dots\}$, which counts the number of primitive points of L in A , is Borel measurable and

$$\int_{\mathcal{L}(k)} \left(\#^*(A \cap L) - \frac{V(A)}{\zeta(d)} \right)^2 d\nu(L) \leq \alpha V(A). \quad (3)$$

Here $\mathcal{L}(k) = \{L \in \mathcal{L}: d(L) \leq k\}$, $\zeta(\cdot)$ denotes the Riemann zeta-function, and $\alpha > 0$ is a constant depending on k and d .

The main step of the proof is to show the following proposition:

Let $k = 1, 2, \dots$. Then for ν -almost every lattice $L \in \mathcal{L}(k)$ the set B contains infinitely many pairwise disjoint d -tuples of linearly independent points of L . (4)

To prove this, let $0 = \varrho_0 < \varrho_1 < \dots$ be such that

$$V(B_n) > 2^d \zeta(d)n, \quad \text{where } B_n = \{x \in B: \varrho_{n-1} < \|x\| \leq \varrho_n\}.$$

By (2),

for $n = 1, 2, \dots$, there are 2^d pairwise disjoint Borel sets $B_{ni}, i = 1, \dots, 2^d$, of B_n such that $V(B_{ni}) = \zeta(d)n$ and no $(d-1)$ -dimensional subspace of $\mathbb{E}^d$ meets each set B_{ni} . (5)

Consequently (3) implies that

$$\int_{\mathcal{L}(k)} (\#^*(B_{ni} \cap L) - n)^2 d\nu(L) \leq \alpha \zeta(d)n. \quad (6)$$

By (5) and (3) the sets $\mathcal{L}_{ni} = \{L \in \mathcal{L}(k): \#^*(B_{ni} \cap L) = 0\}$, $i = 1, \dots, 2^d$, are Borel. It thus follows from (6) that $n^2 \nu(\mathcal{L}_{ni}) \leq \alpha \zeta(d)n$, or

$$\nu(\mathcal{L}_{ni}) \leq \frac{\alpha \zeta(d)}{n}. \quad (7)$$

The set $\mathcal{L}_n = \mathcal{L}_{n1} \cup \dots \cup \mathcal{L}_{n2^d}$ is Borel and consists of all lattices $L \in \mathcal{L}(k)$ such that at least one of the sets B_{ni} contains no primitive point of L . Hence $\mathcal{L}(k) \setminus \mathcal{L}_n$ is the set of all lattices $L \in \mathcal{L}(k)$ such that each set B_{ni} contains a primitive point of L . Hence (5) shows that

for any lattice $L \in \mathcal{L}(k) \setminus \mathcal{L}_n$, the set B_n contains a d -tuple of linearly independent points of L . (8)

By (7),

$$\nu(\mathcal{L}_n) \leq \frac{\alpha 2^d \zeta(d)}{n}. \quad (9)$$

By definition the sets B_n are pairwise disjoint subsets of B . Hence (8) implies that

$$\begin{aligned} & \{L \in \mathcal{L}(k): B \text{ contains infinitely many pairwise disjoint } d\text{-tuples of linearly} \\ & \quad \text{independent primitive points of } L\} \\ & \supset \{L \in \mathcal{L}(k): \text{for infinitely many } n, \text{ the set } B_n \text{ contains a } d\text{-tuple of linearly} \\ & \quad \text{independent primitive points of } L\} \\ & \supset \{L \in \mathcal{L}(k): \text{for infinitely many } n \text{ the lattice } L \text{ is not contained in } \mathcal{L}_n\} \\ & = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} (\mathcal{L}(k) \setminus \mathcal{L}_n) = \mathcal{L}(k) \setminus \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{L}_n. \end{aligned}$$

Since by (9),

$$\nu \left(\bigcap_{n=m}^{\infty} \mathcal{L}_n \right) = 0 \quad \text{for } m = 1, 2, \dots, \quad \text{and thus } \nu \left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{L}_n \right) = 0,$$

the proof of (4) is complete.

Since $\mathcal{L} = \bigcup \mathcal{L}(k)$, Theorem 1 for $d \geq 3$ is an immediate consequence of (4).

Assume now that $d = 2$ and let ξ be the measure on $\mathcal{L}$ used by Schmidt [8, p. 525]. A result of Schmidt [8, pp. 526–527] shows that (3) continues to hold, but with the following weaker inequality:

$$\int_{\mathcal{L}(k)} \left(\#^*(A \cap L) - \frac{V(A)}{\zeta(2)} \right)^2 d\xi(L) \leq \beta V(A) \log_2 V(A).$$

Using this, we see that in the case $d = 2$ the proof is, in essence, the same as the above proof for $d \geq 3$. Finally, note that the sets of measure 0 with respect to ξ and ν coincide. $\square$

3. Proof of the Lemma

Let f_n and f be the distance functions of S_n and S , respectively. Since distance functions are positively homogeneous of degree 1, and $f_n \rightarrow f$ uniformly for $\|x\| \leq 1$, we have that $f_n \rightarrow f$ uniformly on each bounded set in $\mathbb{E}^d$. This yields the following statement:

$$\text{Let } l_n, l \in \mathbb{E}^d \text{ be such that } l_n \rightarrow l. \text{ Then } f_n(l_n) \rightarrow f(l). \quad (10)$$

The following claims are simple consequences of the convergence $L_n \rightarrow L$, see pp. 178–179 of [3]:

$$\text{Given } l \in L, \text{ there are } l_n \in L \text{ such that } l_n \rightarrow l. \quad (11)$$

$$\text{If } l_n \in L_n \text{ and } l \in \mathbb{E}^d \text{ such that } l_n \rightarrow l, \text{ then } l \in L. \quad (12)$$

(i) Let $\varepsilon > 0$. By the definition of successive minima one can show that there are linearly independent lattice points $l_1, \dots, l_d \in L$ such that

$$\max\{f(l_1), \dots, f(l_d)\} \leq \lambda_i(S, L) + \varepsilon. \quad (13)$$

By (11) we may choose points $l_{nj} \in L_n$, $j = 1, \dots, d$, such that

$$l_{nj} \rightarrow l_j. \quad (14)$$

Since $l_1, \dots, l_d$ are linearly independent, it follows that

$$l_{n1}, \dots, l_{nd} \in L_n \text{ are also linearly independent for all sufficiently large } n. \quad (15)$$

Hence, by the definition of λ_i together with (15), (14), (10) and (13) we obtain that

$$\begin{aligned} \lambda_i(f_n, L_n) &\leq \max\{f_n(l_{n1}), \dots, f_n(l_{ni})\} \leq \max\{f(l_1), \dots, f(l_{ni})\} + \varepsilon \\ &\leq \lambda_i(f, L) + 2\varepsilon \quad \text{for all sufficiently large } n, \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof of claim (i).

(ii) Let $0 < \varepsilon < 1$. Since $f_n \rightarrow f$ uniformly on $\{x: \|x\| = 1\}$ and $f(x) > 0$ for $\|x\| = 1$ by the boundedness of S , and f_n, f are all continuous and positively homogeneous of degree 1, there is a constant $\alpha > 0$ such that

$$\alpha \|x\| \leq (1 - \varepsilon)f(x) \leq f_n(x) \quad \text{for all } x \in \mathbb{E}^d \text{ if } n \text{ is sufficiently large.} \quad (16)$$

For such n we have that $f(x) > 0$ for $x \neq o$, hence S_n is bounded. The definition of λ_i then yields that

$$\begin{aligned} &\text{for all sufficiently large } n, \text{ there are linearly independent points } l_{n1}, \dots, l_{nd} \\ &\text{in } L_n, \text{ such that } \lambda_i(f_n, L_n) = \max\{f_n(l_{n1}), \dots, f_n(l_{ni})\} \text{ for } i = 1, \dots, d. \end{aligned} \quad (17)$$

Propositions (16), (17), (1) and part (i) together imply that

$$\begin{aligned} \|l_{ni}\| &\leq \frac{1}{\alpha} f_n(l_{ni}) \leq \frac{1}{\alpha} \lambda_i(f_n, L_n) \leq \frac{1}{\alpha} \lambda_d(f_n, L_n) \\ &\leq \frac{1}{\alpha} \lambda_d(f, L) + \varepsilon \quad \text{for all sufficiently large } n. \end{aligned} \quad (18)$$

For all sufficiently large n , the vectors $l_{n1}, \dots, l_{nd}$ are linearly independent by (17). Consequently, $|\det(l_{n1}, \dots, l_{nd})|$ is an integer multiple of $d(L)$. By assumption, $L_n \rightarrow L$. Hence $d(L_n) \rightarrow d(L)$. Combining this, it follows that

$$|\det(l_{n1}, \dots, l_{nd})| \geq d(L_n) \geq (1 - \varepsilon)d(L) \quad \text{for all sufficiently large } n. \quad (19)$$

By (18), all the sequences $(l_{n1}), \dots, (l_{nd})$ are bounded. Fix an index $i = 1, \dots, d$. By considering a suitable subsequence of $1, 2, \dots$, and re-numbering, if necessary, we may suppose that

$$\liminf_{n \rightarrow \infty} \lambda_i(f_n, L_n) \text{ is the same as for the original sequence,} \quad (20)$$

and $l_{n1} \rightarrow l_1, \dots, l_{nd} \rightarrow l_d$, say. By (10), (12) and (19) the latter implies that

$$\begin{aligned} f_n(l_{n1}) &\rightarrow f(l_1), \dots, f_n(l_{nd}) \rightarrow f(l_d), l_1, \dots, l_d \in L, \\ |\det(l_1, \dots, l_d)| &\geq (1 - \varepsilon)d(L) > 0. \end{aligned}$$

In particular, $l_1, \dots, l_d$ are linearly independent. Using (17) and the definition of λ_i , it follows that

$$\lambda_i(f_n, L_n) = \max\{f_n(l_{n1}), \dots, f_n(l_{ni})\} \rightarrow \max\{f(l_1), \dots, f(l_i)\} \geq \lambda_i(f, L).$$

This together with (20) and (i) finally yields (ii). $\square$

4. Proof of Theorem 2

(i) Apply Theorem 1 with $B = (1/k)S$, $k = 1, 2, \dots$, to see that for ν -almost all lattices εS contains a d -tuple of linearly independent primitive points of L for any $\varepsilon > 0$. Hence $\lambda_d(S, L) = 0$ for ν -almost all lattices L . In conjunction with (1), this completes the proof of claim (i).

(ii) Let $\mathcal{M}_n = \{L \in \mathcal{L} : \lambda_d(S, L) \geq 1/n\}$, $n = 1, 2, \dots$. Since $\lambda_d(S, \cdot)$ is upper semi-continuous by the lemma, $\mathcal{M}_n$ is closed. If the interior of $\mathcal{M}_n$ is non-empty, then $\nu(\mathcal{M}_n) > 0$ by the definitions of ν and the topology on $\mathcal{L}$, in contradiction to (i). Hence $\mathcal{M}_n$ has empty interior. Being closed, $\mathcal{M}_n$ is nowhere dense in $\mathcal{L}$. Hence

$$\bigcup_{n=1}^{\infty} \mathcal{M}_n = \{L \in \mathcal{L} : \lambda_d(S, L) > 0\} \text{ is meager.}$$

Now note (1) to conclude the proof of claim (ii). □

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