Homogeneity Properties of Some ℓ_1 -Spaces

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Abstract. Let $\mathcal{M}$ be a family of subspaces of a metric space X . The space X is $\mathcal{M}$ -homogeneous if any isometry between two subspaces in $\mathcal{M}$ can be extended to an isometry of X . In the case when $\mathcal{M}$ is the family of all subspaces of X , the space X is said to be *fully homogeneous*. We show that the space L^n , where L is a fully homogeneous subspace of $\mathbb{R}$, is $\mathcal{R}$ -homogeneous where $\mathcal{R}$ is the family of subspaces that are products of subsets of L .

1. Introduction

A metric space X is said to be *fully homogeneous* if, for every two subspaces $A, B \subseteq X$, and an isometry $f: A \rightarrow B$, f can be extended to an isometry of the entire space X [2, p. 166]. Examples of fully homogeneous spaces include Euclidean spaces, Riemannian manifolds of constant curvature, and finite metric spaces with the discrete metric. On the other hand, an infinite metric space with the discrete metric is not fully homogeneous.

Clearly, fully homogeneous spaces are rare. We say that a metric space X is $\mathcal{M}$ -homogeneous if, for a given nonempty family $\mathcal{M}$ of subspaces of X and every two subspaces $A, B \in \mathcal{M}$, any isometry from A onto B can be extended to an isometry of X . (This is what is sometimes called “ultrahomogeneity” to distinguish it from the weaker notion of homogeneity (see [3]).) The families $\mathcal{M} = 2^X$ and $\mathcal{M} = \{\{x\}: x \in X\}$ illustrate this definition; the former one gives the definition of full homogeneity whereas the latter family is used in the definition of (simple) homogeneity. Witt’s theorem [5] asserts that a finite-dimensional normed vector space E is $\mathcal{M}$ -homogeneous with respect to the family $\mathcal{M}$ of vector subspaces of E . Another example is Urysohn’s universal Polish space U [8] which is $\mathcal{M}$ -homogeneous for the family $\mathcal{M}$ of all finite subspaces of U . In other words, U satisfies Helmholtz’s “free mobility” principle [4].

In the paper we study homogeneity properties of some subspaces of the space $\mathbb{R}_1^n$, the coordinate space $\mathbb{R}^n$ endowed with the ℓ_1 metric:

$$d(x, y) = \sum_{1 \leq k \leq n} |x_k - y_k|, \quad (1)$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Namely, we consider subspaces in the form L^n where L is a subset of $\mathbb{R}$ containing more than one point. Generally speaking, L^n is not fully homogeneous. A typical example is the space $\mathbb{R}_1^n$ itself (for $n > 1$). The isometry group of this space is generated by reflections and permutations of coordinate axes, and translations (see p. 46 of [7] and Corollary 3.1 below). Consider, for instance, the space $\mathbb{R}_1^2$ and let

$$A = \{(x_1, x_2) \in \mathbb{R}_1^2: x_1 = 0, x_2 \geq 0 \text{ or } x_2 = 0, x_1 > 0\}.$$

The mapping $\alpha: A \rightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} -x_2, & \text{if } x_1 = 0, \\ x_1, & \text{otherwise,} \end{cases}$$

is an isometry. Let $\tau: \mathbb{R} \rightarrow \mathbb{R}$ be a nontrivial translation. Then $\alpha^{-1}\tau\alpha$ is an isometry of A onto itself which cannot be extended to an isometry of $\mathbb{R}_1^2$.

On the other hand, as follows from our results, $\mathbb{R}_1^n$ is $\mathcal{R}$ -homogeneous where $\mathcal{R}$ is the family of all *rectangular* subspaces of $\mathbb{R}_1^n$ that are, by definition, products of nonempty subsets of $\mathbb{R}$. In fact, we show that L^n is $\mathcal{R}$ -homogeneous provided that L is a fully homogeneous subspace of $\mathbb{R}$.

The rest of the paper is organized as follows.

In Section 2 we give a complete description of fully homogeneous subspaces of $\mathbb{R}$. They are subspaces isometric to subgroups of the additive group $\mathbb{R}$ and two-point subspaces.

All isometries between rectangular subspaces are described in Section 3. In particular, we show that, for an arbitrary $L \subseteq \mathbb{R}$, the isometry group $Iso(L^n)$ is generated by the group $(Iso(L))^n$ and the group of permutations of axes.

It is shown in Section 4 that, for a fully homogeneous space $L \subseteq \mathbb{R}$, the space L^n is $\mathcal{R}$ -homogeneous.

Since the space $L = \{0, 1\}$ is fully homogeneous, the space $L^n = \{0, 1\}^n$ is $\mathcal{R}$ -homogeneous. It turns out that this space has a richer homogeneity property. We introduce this property in the last section of the paper.

2. Fully Homogeneous Subspaces of $\mathbb{R}$

In this section we describe all fully homogeneous metric subspaces of $\mathbb{R}$. Note that $Iso(\mathbb{R})$ is a semidirect product of the additive group $\mathbb{R}$ (the translation subgroup of $Iso(\mathbb{R})$) and a subgroup (isomorphic to $\mathbb{Z}_2$) generated by the reflection $x \mapsto -x$.

We begin with a well-known property of the metric space $\mathbb{R}$.

Lemma 2.1. *A point $x \in \mathbb{R}$ is uniquely defined by its distances from two distinct points $a, b \in \mathbb{R}$.*

The following lemma gives a typical example of a fully homogeneous subspace of $\mathbb{R}$.

Lemma 2.2. *Suppose that a subspace L of $\mathbb{R}$ is the orbit of a point $c \in L$ under actions of the translation subgroup H of $\text{Iso}(L)$. Then L is fully homogeneous.*

Proof. Without losing generality, we may assume that $c = 0$ is in L , i.e., $L = H$ is a subgroup of the additive group $\mathbb{R}$.

Let $A, B \subseteq \mathbb{R}$ and $\alpha: A \rightarrow B$ be an isometry. We need to show that α is extendable to an isometry $\hat{\alpha}: L \rightarrow L$.

This is trivial if $|A| = 1$, since L is homogeneous. So we may assume that A is not a singleton. Let $a < b$ be two points in A . We define

$$\hat{\alpha}(x) = \begin{cases} \alpha(a) - a + x, & \text{if } \alpha(a) < \alpha(b), \\ \alpha(a) + a - x, & \text{if } \alpha(a) > \alpha(b), \end{cases} \quad x \in L.$$

Since $L = H$ is an additive subgroup of $\mathbb{R}$, $\hat{\alpha}$ is an isometry of L . We have

$$\begin{aligned} |\hat{\alpha}(x) - \alpha(a)| &= |x - a| = |\alpha(x) - \alpha(a)|, \\ |\hat{\alpha}(x) - \alpha(b)| &= |x - b| = |\alpha(x) - \alpha(b)|, \end{aligned}$$

for all $x \in A$. By Lemma 2.1, $\hat{\alpha}(x) = \alpha(x)$ for all $x \in A$, i.e., $\hat{\alpha}$ is an extension of α . $\square$

It follows that a subspace of $\mathbb{R}$ which is isometric to a subgroup of the additive group $\mathbb{R}$ is fully homogeneous. We shall show that, with one exception, all fully homogeneous subspaces are of this kind.

Lemma 2.3. *Let A and B be isometric subspaces of a fully homogeneous subspace L of $\mathbb{R}$ containing more than one point each. An isometry $\alpha: A \rightarrow B$ admits a unique extension to an isometry $\hat{\alpha}: L \rightarrow L$.*

Proof. An extension $\hat{\alpha}$ exists, since L is fully homogeneous. Suppose β is another extension of α . Then $\beta\hat{\alpha}^{-1}$ is an isometry of L which fixes all points in A . By Lemma 2.1, $\beta\hat{\alpha}^{-1}$ is the identity, i.e., $\beta = \hat{\alpha}$. $\square$

The following statement follows immediately from the previous one, since $\mathbb{R}$ is a fully homogeneous metric space.

Corollary 2.1. *For any subset $L \subseteq \mathbb{R}$, the group $\text{Iso}(L)$ is (isomorphic to) a subgroup of $\text{Iso}(\mathbb{R})$.*

In fact, any subgroup of $\text{Iso}(\mathbb{R})$ is the isometry group of some set L . For instance, L can be chosen as the orbit of a point in $\mathbb{R}$ under actions from the subgroup.

Remark 2.1. The statement of Lemma 2.3 does not hold for an arbitrary subspace L of $\mathbb{R}$. Consider, for instance, spaces $A = \{0, 2, 4\}$ and $L = \{0, 2, 3, 4\}$ with the metric

inherited from $\mathbb{R}$. The reflection about 2 is isometry of A which is not extendable to an isometry of L .

In order to describe fully homogeneous subspaces of $\mathbb{R}$, we first study homogeneous subspaces of $\mathbb{R}$. Thus, in what follows, we assume that L is homogeneous.

By Corollary 2.1, any isometry of L is either a translation or a reflection. Suppose that a nontrivial isometry group $Iso(L)$ has a trivial translation subgroup. Then $Iso(L)$ contains a single reflection (the product of two reflections is a translation). Since L is homogeneous, it must contain exactly two points. Clearly, a two-point subspace of $\mathbb{R}$ is fully homogeneous. Thus two-point subspaces are examples of fully homogeneous subspaces that are not isometric to subgroups of $\mathbb{R}$.

Suppose now that the translation subgroup H of $Iso(L)$ is not trivial. This group H is an additive subgroup of $\mathbb{R}$. It follows that the set L is unbounded below and above. Thus, two-element subspaces are the only finite, fully homogeneous subspaces of $\mathbb{R}$.

If H is nontrivial, then $Iso(L)$ must also contain reflections. Indeed, suppose $Iso(L) = H$. Let a be a point in L and let x be any other point in L . Then $x = a + h$ for a unique $h \in H$. The mapping $x \mapsto a - h$ is a reflection of L onto itself, a contradiction.

The following examples demonstrate two typical cases.

Example 2.1. Let $L = \mathbb{Z}$, the set of integers. Clearly, $H = \mathbb{Z} = L$. The group $Iso(L)$ also contains a reflection $\rho: n \mapsto -n$. In fact, $Iso(L)$ is a disjoint union of H and $H\rho$.

Example 2.2. Let $L = 3\mathbb{Z} \cup (3\mathbb{Z} + 1)$. In this case $H = 3\mathbb{Z} \neq L$. Unlike in the previous example, reflections of the subset $3\mathbb{Z}$ do not belong to $Iso(L)$. On the other hand, the reflection defined by $\rho: n \mapsto 1 - n$ is an isometry of L . Again, $Iso(L)$ is a disjoint union of H and $H\rho$.

In general, any transitive $Iso(L)$ is a disjoint union of H and $H\rho$ for some reflection ρ . A similar description of a homogeneous L is given in the following theorem.

Theorem 2.1. *If L is homogeneous and the translation subgroup H of $Iso(L)$ is not trivial, then either $L = H(a)$ for some $a \in L$ or there are points $a, b \in L$ such that L is the disjoint union of $H(a)$ and $H(b)$.*

Proof. Suppose $H(a) \neq L$ and let $b \in L \setminus H(a)$. Then $b - a \notin H$. If $H(a) \cap H(b) \neq \emptyset$, then $a + h = b + h'$ for some $h, h' \in H$, i.e., $b - a = h - h' \in H$, a contradiction. Hence, $H(a) \cap H(b) = \emptyset$.

Suppose there is $c \in L$ such that $c \notin H(a)$ and $c \notin H(b)$. Since $c \notin H(a)$, there is a reflection α in $Iso(L)$ such that $c = \alpha(a)$. Similarly, $c = \beta(b)$ for a reflection β in $Iso(L)$. Thus, $\beta^{-1}\alpha(a) = b$, a contradiction, since $\beta^{-1}\alpha$ is a translation. $\square$

A fully homogeneous metric space $L \subseteq \mathbb{R}$ is of course homogeneous. Suppose $L \neq H(a)$ and let a and b be two points such that L is a disjoint union of $H(a)$ and $H(b)$. There is a reflection $\rho \in Iso(L)$ such that $b = \rho(a)$. For a nonzero $h \in H$,

consider two sets $A = \{a, a + h\} \subset L$ and $B = \{b, b + h\} \subset L$. We define an isometry $\alpha: A \rightarrow B$ by $\alpha(a) = b$ and $\alpha(a + h) = b + h$. Suppose it is expandable to $\hat{\alpha}: L \rightarrow L$. Then $\hat{\alpha}$ is a reflection, since $b \notin H(a)$. Therefore $\hat{\alpha} = \rho$, since $\rho^{-1}\hat{\alpha}$ is a translation fixing a . We have $\rho(x) = -x + b + a$. Therefore $\hat{\alpha}(a + h) = -a - h + b + a = b - h$. Since $\alpha(a + h) = b + h$, we have $h = 0$, a contradiction. Thus, α is not extendable.

In summary, we have the following theorem.

Theorem 2.2. *A fully homogeneous subspace L of $\mathbb{R}$ is either a two-point subspace or a subspace isometric to a subgroup of the additive group $\mathbb{R}$.*

3. Partial Isometries in L^n

In this section we describe isometries $\alpha: A \rightarrow B$ where A and B are rectangular subspaces of L^n . For a rectangular subspace $A = \prod_{i=1}^n A_i$ we define

$$A^{(k)} = \prod_{\substack{i=1 \\ i \neq k}}^n A_i \quad \text{and} \quad L_k^A(a) = \{x \in A: x_i = a_i, i \neq k\}, \quad \text{for } a \in A^{(k)}.$$

Distinct points in each set $L_k^A(a)$ differ in exactly one coordinate. Clearly, $L_k^A(a)$ is isometric to A_k . Geometrically, $L_k^A(a)$ is a “line” in A “parallel” to the k th coordinate axis.

The following proposition follows immediately from definition (1) of the distance d in L^n .

Proposition 3.1. *For two rectangular subspaces*

$$A = \prod_{i=1}^n A_i \quad \text{and} \quad B = \prod_{i=1}^n B_i$$

of L^n , let σ be a permutation of the set of indices $\{1, \dots, n\}$ and let $\{\alpha_i\}_{i=1}^n$ be a family of isometries $\alpha_i: A_i \rightarrow B_{\sigma(i)}$. Then α defined by

$$\alpha(x) = (\alpha_{\sigma^{-1}(1)}(x_{\sigma^{-1}(1)}), \dots, \alpha_{\sigma^{-1}(n)}(x_{\sigma^{-1}(n)})), \quad \text{for } x \in A, \quad (2)$$

is an isometry from A onto B .

It turns out that any isometry of a rectangular subspace onto another one can be represented in the form (2). We obtain this result after proving three technical lemmas.

In what follows, A and B are two rectangular subspaces of L^n and α is a given isometry from A onto B .

Lemma 3.1. *The image of $L_i^A(a)$ under α is $L_j^B(b)$ for some j and $b \in B^{(j)}$.*

Proof. Consider two cases.

(i) $|L_i^A(a)| = |A_i| > 1$. For distinct points $x, y \in L_i^A(a)$, let $u = \alpha(x)$, $v = \alpha(y)$. Suppose that points u and v differ in more than one coordinate. We may assume that

$$u = (u_1, u_2, \dots, u_n), \quad v = (v_1, v_2, \dots, v_n),$$

for $u_1 \neq v_1$, $u_2 \neq v_2$. Consider the following four distinct points:

$$\begin{aligned} u &= (u_1, u_2, u_3, \dots, u_n), & v &= (v_1, v_2, v_3, \dots, v_n), \\ u' &= (u_1, v_2, u_3, \dots, u_n), & v' &= (v_1, u_2, v_3, \dots, v_n). \end{aligned}$$

Clearly, $u', v' \in [u, v]$ and $u, v \in [u', v']$. Then, for $x' = \alpha^{-1}(u')$ and $y' = \alpha^{-1}(v')$, we must have $x', y' \in [x, y]$ and $x, y \in [x', y']$ for four distinct points in $L_i^A(a)$. Since $L_i^A(a)$ is isometric to $A_i \subseteq \mathbb{R}$, we have a contradiction.

It follows that all points in the image of $L_i^A(a)$ under α differ in just one coordinate. Thus α maps $L_i^A(a)$ to some $L_j^B(b)$. Since the inverse of α is an isometry from B onto A , we can apply the above argument to show that α maps $L_i^A(a)$ isometrically onto $L_j^B(b)$.

(ii) $|L_i^A(a)| = 1$. Suppose that $|B_j| > 1$ for all j . By applying (i) to α^{-1} , we obtain a contradiction. Thus there is j for which $|B_j| = 1$. $\square$

Now we prove that j in the previous lemma does not depend on $a \in A^{(i)}$, provided that $|L_i^A(a)| = |A_i| > 1$.

Lemma 3.2. *Let $\alpha(L_i^A(a)) = L_j^B(b)$ and $\alpha(L_i^A(a')) = L_k^B(b')$. If $|L_i^A(a)| > 1$ then $k = j$.*

Proof. Suppose first that all three indices i, j , and k are distinct (thus we assume that $n \geq 3$). Without losing generality, we may assume that $i = 1$, $j = 2$, and $k = 3$. Let

$$\begin{aligned} x &= (x_1, a_2, \dots, a_n) \in L_1^A(a), & y &= (y_1, a_2, \dots, a_n) \in L_1^A(a), \\ x' &= (x_1, a'_2, \dots, a'_n) \in L_1^A(a'), & y' &= (y_1, a'_2, \dots, a'_n) \in L_1^A(a') \end{aligned}$$

be four distinct points in A . The images of these points under α are

$$\begin{aligned} \alpha(x) &= (b_1, u_2, b_3, b_4, \dots, b_n) \in L_2^B(b), & \alpha(y) &= (b_1, v_2, b_3, b_4, \dots, b_n) \in L_2^B(b), \\ \alpha(x') &= (b'_1, b'_2, r_3, b'_4, \dots, b'_n) \in L_3^B(b'), & \alpha(y') &= (b'_1, b'_2, s_3, b'_4, \dots, b'_n) \in L_3^B(b'), \end{aligned}$$

where $u_2, v_2 \in B_2$, $r_3, s_3 \in B_3$, and $b_i, b'_i \in B_i$ for $i \neq 2, 3$.

Since $d(x, y') = d(x, y) + d(y, y')$, we have

$$d(\alpha(x), \alpha(y')) = d(\alpha(x), \alpha(y)) + d(\alpha(y), \alpha(y')),$$

implying

$$|u_2 - b'_2| = |u_2 - v_2| + |v_2 - b'_2|. \quad (3)$$

Since $d(x', y) = d(x', x) + d(x, y)$, we have

$$d(\alpha(x'), \alpha(y)) = d(\alpha(x'), \alpha(x)) + d(\alpha(x), \alpha(y)),$$

implying

$$|v_2 - b'_2| = |u_2 - b'_2| + |u_2 - v_2|. \quad (4)$$

By adding (3) and (4), we have $u_2 = v_2$, a contradiction, since $x \neq y$.

A similar argument shows that the assumption $i = j \neq k$ also leads to a contradiction. Hence, we must have $j = k$. $\square$

It follows that α defines a partial function $i \mapsto j$. This function is defined for those indices i for which $|L_i^A(a)| = |A_i| > 1$ and is a one-to-one function; indeed, by the previous lemmas, the image of $L_j^B(b)$ under the inverse isometry α^{-1} is some $L_i^A(a)$ and i does not depend on b .

We extend the partial function $i \mapsto j$ to a permutation σ_α of the set of indices $\{1, \dots, n\}$, which in turn defines in an obvious way an isometry of L^n onto itself. We use the same symbol σ_α for this isometry. Clearly, the isometry

$$\beta = \sigma_\alpha^{-1} \alpha$$

maps each $L_i^A(a)$ onto some $L_i^C(c)$, where $C = \sigma_\alpha^{-1}(B)$.

Let $p_i: L^n \rightarrow L$ be a projection of the space L^n to the i th coordinate defined by

$$p_i(x) = x_i, \quad \text{for } x = (x_1, \dots, x_n) \in L^n.$$

The restriction of this projection to $L_i^A(a)$ (resp. $L_i^C(c)$) is an isometry between $L_i^A(a)$ and A_i (resp. $L_i^C(c)$ and C_i). We define an isometry $\alpha_i: A_i \rightarrow C_i$ by

$$\alpha_i = p_i \beta (p_i|_{L_i^A(a)})^{-1}.$$

Lemma 3.3. *The isometry α_i does not depend on the choice of $a \in A^{(i)}$.*

Proof. It suffices to consider the case $i = 1$. The statement of the lemma is trivial if $|A_1| = 1$. Thus we assume that $|A_1| > 1$. Suppose that for

$$x = (x_1, a_2, \dots, a_n) \in L_1^A(a), \quad x' = (x_1, a'_2, \dots, a'_n) \in L_1^A(a'),$$

we have

$$\begin{aligned} y &= \beta(x) = (y_1, c_2, \dots, c_n) \in L_1^C(c), \\ y' &= \beta(x') = (y'_1, c'_2, \dots, c'_n) \in L_1^C(c'). \end{aligned}$$

Let $y'' = (y'_1, c_2, \dots, c_n) \in L_1^C(c)$. There is $x'' = (x'_1, a_2, \dots, a_n) \in L_1^A(a)$ such that $y'' = \beta(x'')$. We have

$$d(x', x) + d(x, x'') = d(x', x'').$$

Since β is an isometry, we have

$$d(y', y) + d(y, y'') = d(y', y'').$$

On the other hand,

$$d(y, y'') + d(y'', y') = d(y, y').$$

It follows that $d(y, y'') = 0$, i.e., $y_1 = y'_1$. Thus the first coordinate of $\beta(x)$ depends only on the first coordinate of x ; therefore α_1 does not depend on $a \in A^{(1)}$. $\square$

It follows that, for $x \in A$,

$$\beta(x) = (\alpha_1(x_1), \dots, \alpha_n(x_n)).$$

Hence,

$$\alpha(x) = \sigma_\alpha \beta(x) = (\alpha_{\sigma_\alpha^{-1}(1)}(x_{\sigma_\alpha^{-1}(1)}), \dots, \alpha_{\sigma_\alpha^{-1}(n)}(x_{\sigma_\alpha^{-1}(n)})), \quad \text{for } x \in A,$$

which is the same representation as in (2).

In summary, we have the following theorem.

Theorem 3.1. *Let $A = \prod_{i=1}^n A_i$ and $B = \prod_{i=1}^n B_i$ be two rectangular subspaces of L^n . A mapping α from A onto B is an isometry if and only if there is a permutation σ of the set $\{1, \dots, n\}$ and a family $\{\alpha_i\}_{i=1}^n$ of isometries $\alpha_i: A_i \rightarrow B_{\sigma(i)}$ such that*

$$\alpha(x) = (\alpha_{\sigma^{-1}(1)}(x_{\sigma^{-1}(1)}), \dots, \alpha_{\sigma^{-1}(n)}(x_{\sigma^{-1}(n)})), \quad \text{for } x \in A.$$

Clearly, L^n is a rectangular space. Thus we have the following corollary.

Corollary 3.1. *The group $\text{Iso}(L^n)$ is generated by the group $(\text{Iso}(L))^n$ and the group of permutation of axes (the symmetric group S_n).*

4. $\mathcal{R}$ -Homogeneity of the Space L^n

Let $\mathcal{R}$ be the family of all rectangular subspaces of L^n . Generally speaking, for an arbitrary subspace L , the space L^n is not $\mathcal{R}$ -homogeneous. For instance, if $n = 1$ then $\mathcal{R}$ -homogeneity is just the full homogeneity of L .

In what follows, we assume that L is fully homogeneous. Even under this assumption, it is not true, generally speaking, that L^n is fully homogeneous for $n > 1$. Indeed, by Theorem 2.2, L is either isometric to a subgroup of $\mathbb{R}$ or is a two-point subspace. An argument very similar to the one presented in Section 1 for the case of $\mathbb{R}_1^2$ shows that in the former case L^n is not fully homogeneous if $n > 1$.

In the case when $|L| = 2$ we actually have two options. It can be verified that spaces $\{0, 1\}^n$ are fully homogeneous for $n \leq 3$ (we omit the proof). On the other hand, spaces $\{0, 1\}^n$ are not fully homogeneous for $n > 3$. Consider, for instance, the space $\{0, 1\}^4$ and its two subspaces:

$$\begin{aligned} A &= \{(0, 0, 0, 0), (1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1)\} \quad \text{and} \\ B &= \{(0, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 1, 0)\}. \end{aligned}$$

Clearly, these subspaces are isometric. All distances from point $(0, 0, 0, 1)$ to points in A are equal to 1. On the other hand, it is easy to verify that there is no point in $\{0, 1\}^4$ which is on distance 1 from all points in B . Thus an isometry from A onto B cannot be extended to an isometry of $\{0, 1\}^4$. This argument can be easily modified to show that L^n for $|L| = 2$ and $n > 3$ is not fully homogeneous.

Now we prove that L^n is $\mathcal{R}$ -homogeneous.

Theorem 4.1. *Let L be a fully homogeneous subspace of $\mathbb{R}$ and let $\mathcal{R}$ be the family of rectangular subspaces of L^n . Then the space L^n is $\mathcal{R}$ -homogeneous.*

Proof. Let $A = \prod_{i=1}^n A_i$ and $B = \prod_{i=1}^n B_i$ be two rectangular subspaces of L^n and let $\alpha: A \rightarrow B$ be an isometry. By Theorem 3.1,

$$\alpha(x) = (\alpha_{\sigma^{-1}(1)}(x_{\sigma^{-1}(1)}), \dots, \alpha_{\sigma^{-1}(n)}(x_{\sigma^{-1}(n)})), \quad \text{for } x \in A,$$

for some permutation σ of the set $\{1, \dots, n\}$ and some family $\{\alpha_i\}_{i=1}^n$ of isometries $\alpha_i: A_i \rightarrow B_{\sigma(i)}$. Since L is a fully homogeneous space, each α_i has an extension $\hat{\alpha}_i$ to an isometry of L .

By the same theorem,

$$\hat{\alpha}(x) = (\hat{\alpha}_{\sigma^{-1}(1)}(x_{\sigma^{-1}(1)}), \dots, \hat{\alpha}_{\sigma^{-1}(n)}(x_{\sigma^{-1}(n)})), \quad \text{for } x \in L^n,$$

is an extension of α to an isometry of L^n . □

5. Conclusion

The main result of the paper is Theorem 4.1 which asserts that the metric space L^n is $\mathcal{R}$ -homogeneous if L is a fully homogeneous subspace of $\mathbb{R}$.

It follows from Theorem 2.2 that two-point subspaces are rather special fully homogeneous subspaces of $\mathbb{R}$. In fact, they have a stronger homogeneity property than $\mathcal{R}$ -homogeneity.

Let $X = \{0, 1\}^n$ be a hypercube metric space, i.e., the metric space of the hypercube graph $H(n, 2)$. Let $\mathcal{M}$ be the set of all partial cubes (isometric subspaces) in X . Clearly, the rectangular subspaces of X are just special partial cubes. It is shown (in a different context) in [6] that the space X is $\mathcal{M}$ -homogeneous.

Acknowledgments

The author is grateful to Jean-Paul Doignon for his comments on the original draft of the paper and Alice Devillers for references [3], [4]. He also thanks two anonymous referees for their valuable comments.

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Received December 2, 2002, and in revised form October 16, 2004. Online publication December 19, 2005.