Rearrangement Inequalities and the Alternahedron*

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Abstract. We formulate and prove a rearrangement type inequality and use it to give a description of the face lattice of a certain polytope that is naturally associated to the alternating group A_n .

1. Introduction

The permutahedron is a well-known polytope that may be described as follows. Let $y = (y_1, y_2, \dots, y_n)$ be a point in $\mathbb{R}^n$ with pairwise distinct components. Then $\Pi_{n-1}(y)$ is the convex hull of the set $\{\sigma y: \sigma \in S_n\}$, where the symmetric group S_n acts on the left of $\mathbb{R}^n$ via the standard permutation representation. This is a very well understood polytope (see [1]). In particular, it is a simple $(n - 1)$ -dimensional polytope and its face lattice is isomorphic to the lattice of ordered partitions of the set $\{1, 2, \dots, n\}$ —in Section 2 we review the proof of this fact. A consequence of this characterisation is that the combinatorial type of $\Pi_{n-1}(y)$ is independent of the choice of y and thus the combinatorially minded may unambiguously refer to the permutahedron Π_{n-1} .

A natural question to consider is the following. Suppose that we replace S_n by a proper subgroup in the above definition, what kind of polytope do we obtain in this case? Of course, we can generalise further by considering representations other than the permutation representation. Such orbit polytopes have been considered by various authors—Hood and Perkinson [5] and Onn [7] for example. In general, it seems to be quite difficult to understand the combinatorics of these polytopes. We should also point out that apart from being a natural generalisation of a classical construction in polytope theory, orbit polytopes have applications to the cohomology theory of finite groups.

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Fig. 1. The alternahedron $\Pi_3(A_4, (1, 2, 3, 4))$.

In recent work [2], Ellis et al. have used these orbit polytopes as the basic ingredient for constructing efficient (in a computational sense) resolutions of certain finite groups. Indeed, it was primarily conversations with these authors, together with some computer-aided exploration using polymake [3] that inspired the present paper.

We restrict our attention to the following question. What can we say about the convex hull of the set $\{\sigma y : \sigma \in A_n\}$ where A_n is the group of even permutations on n symbols and y is a point with pairwise distinct components? This is the alternahedron referred to in the title and we denote it by $\Pi_{n-1}(A_n, y)$. If $n = 4$ then we obtain the three-dimensional polytope shown in Fig. 1, which is of course an icosahedron.

The alternahedron has been studied before. For example, in [9] a complete set of inequalities describing the polytope is given. This, of course, does not make it easy to describe the face lattice of the polytope. In [8] Steinkamp presents some computer-aided investigation of low-dimensional alternahedra and makes some conjectures about higher dimensions.

It turns out that in order to understand satisfactorily the combinatorics of $\Pi_{n-1}(A_n, y)$ for large values of n , we must prove a refinement of the rearrangement inequality of Hardy et al. [4]. This is not so surprising as the classical rearrangement inequality essentially characterises the face lattice of the permutahedron.

The paper is structured as follows. In Section 2 we review some basic facts, notation and terminology concerning polytopes. We also outline the proof of the equivalence of the face lattice of Π_{n-1} and the lattice of ordered partitions of $\{1, 2, \dots, n\}$. Of course this result and its proof are standard. However, their inclusion here serves two purposes. First, it allows us to fix notation and terminology for later use and, second, our presentation is intended to highlight the ideas that we later adapt to the case of the alternahedron. In Section 3 we formulate and prove the appropriate rearrangement type inequalities. We feel that some of these inequalities (Lemma 3.2 in particular) are of intrinsic interest

and will possibly have applications to other problems. In Section 4 we give the desired description of the face lattice and as a byproduct we confirm a conjecture of Steinkamp and Perkinson [8]. Finally in Section 5 we make some observations concerning the complexity of the membership problem for the alternahedron and we make some remarks about the relationship between the combinatorial structure of an orbit polytope and the choice of orbit in its construction.

2. Preliminaries

2.1. Polytopes

We assume that the reader is familiar with the basic theory of polytopes as presented in [10], for example. Thus a polytope, P , is the convex hull of a finite set of points in $\mathbb{R}^n$ —we write $P = \text{conv}(X)$ (note that we do not assume that it is necessarily full-dimensional). We use the convention that if x is a point of $\mathbb{R}^n$, then x_i is the i th component of x . Also $\langle, \rangle$ denotes the standard euclidean inner product on $\mathbb{R}^n$. Thus $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$.

In particular, we are concerned with describing the faces of certain polytopes. For $a \in \mathbb{R}^n$ and $z \in \mathbb{R}$, let $H(a, z) = \{x \in \mathbb{R}^n: \langle a, x \rangle \leq z\}$. We say that the halfspace $H(a, z)$ supports the polytope P if $P \subset H(a, z)$ and $F = P \cap \partial H(a, z)$ is nonempty, where $\partial H(a, z) = \{x \in \mathbb{R}^n: \langle a, x \rangle = z\}$. We say that F is the face corresponding to the halfspace $H(a, z)$, and that $\langle a, x \rangle \leq z$ is a face-defining inequality for F . If we also think of the empty set and the entire polytope P as faces then the set of faces of P forms a graded lattice known as the face lattice of P .

2.2. The Permutahedron

The symmetric group S_n acts on $\mathbb{R}^n$ by permuting the components of points in $\mathbb{R}^n$ —the so-called permutation representation of the symmetric group. We use the standard notation for cycles in S_n . For example, if $(a, b, c, d) \in \mathbb{R}^4$ then the cycle $(1, 2, 4) \in S_4$ acts on this point and $(1, 2, 4)(a, b, c, d) = (d, a, c, b)$. Let y be a point in $\mathbb{R}^n$ with pairwise distinct components and let $\Pi_{n-1}(y) = \text{conv}(\{\sigma y: \sigma \in S_n\})$. Clearly, $\Pi_{n-1}(\pi y) = \Pi_{n-1}(y)$ for any $\pi \in S_n$ so we may as well assume that y has strictly increasing components i.e. $y_i < y_{i+1}$ for $i = 1, \dots, n-1$.

In order to describe the face lattice of $\Pi_{n-1}(y)$ we need some terminology and notation concerning ordered partitions. For any positive integer n let $[n] = \{1, \dots, n\}$. An ordered partition of $[n]$ is an ordered k -tuple $(S_1, S_2, \dots, S_k)$ where the S_i 's are pairwise disjoint nonempty subsets of $[n]$ such that $\bigcup_{i=1}^k S_i = [n]$. We refer to the S_i 's as the blocks of $\mathcal{P}$. Let $\mathcal{P} = (S_1, S_2, \dots, S_k)$ and $\mathcal{Q} = (T_1, T_2, \dots, T_l)$ be ordered partitions of $[n]$. We say that $\mathcal{P} \leq \mathcal{Q}$ if two conditions are satisfied. First, we require that every S_i is a subset of some T_j . Second, if $j_1 < j_2$ and if $S_{i_1} \subset T_{j_1}$ and $S_{i_2} \subset T_{j_2}$, then we require that $i_1 < i_2$. With this partial order, the set of ordered partitions of $[n]$ forms a graded lattice—the rank of an ordered partition $(S_1, S_2, \dots, S_k)$ of $[n]$ is defined to be $n - k$.

If $\mathcal{P} = (S_1, S_2, \dots, S_k)$ is an ordered partition, we associate to $\mathcal{P}$ a point $a_{\mathcal{P}}$ in $\mathbb{R}^n$ by setting $a_{\mathcal{P}} = \sum_{i=1}^k \sum_{j \in S_i} i e_j$, where e_j is the j th standard basis vector of $\mathbb{R}^n$. In

other words, all the components of $a_{\mathcal{P}}$ corresponding to the elements of S_j are j . Let $z_{\mathcal{P}} = \max\{\langle a_{\mathcal{P}}, \sigma y \rangle : \sigma \in S_n\}$. By construction, $H_{\mathcal{P}} = H(a_{\mathcal{P}}, z_{\mathcal{P}})$ is a supporting half-space for $\Pi_{n-1}(y)$. Let $F_{\mathcal{P}} = \Pi_{n-1}(y) \cap H_{\mathcal{P}}$ be the corresponding face. We can easily characterise the vertices that belong to $F_{\mathcal{P}}$. Observe that associated to any point x in $\mathbb{R}^n$ there is an ordered partition $\mathcal{P}_x$ whose blocks are the maximal subsets of $[n]$ on which the corresponding components of x are equal and these are arranged in the order that corresponds to increasing values of the components of x . More formally, we can define $\mathcal{P}_x$ as follows: Let $X_1 = \{r \in [n] : x_r = \min\{x_j : j \in [n]\}\}$. Suppose that $X_1, \dots, X_i$ have been defined and $X_1 \cup \dots \cup X_i \neq [n]$, then $X_{i+1} = \{r \in [n] : x_r = \min\{x_j : j \in [n] - (X_1 \cup \dots \cup X_i)\}\}$. If $X_1, \dots, X_i$ have been defined and $X_1 \cup \dots \cup X_i = [n]$ then let $\mathcal{P}_x = (X_1, X_2, \dots, X_i)$.

Lemma 2.1. *If $\mathcal{P}$ is an ordered partition of $[n]$ then*

$$F_{\mathcal{P}} = \text{conv}(\{\sigma y : \sigma \in S_n \text{ and } \mathcal{P}_{\sigma y} \leq \mathcal{P}\}).$$

This is a consequence of the following well-known inequality, due to Hardy et al. [4]. If u and v are points in $\mathbb{R}^n$ we say that u and v are similarly ordered if, whenever $u_i < u_j$ then $v_i \leq v_j$ and whenever $v_i < v_j$ then $u_i \leq u_j$. Observe that if y has pairwise distinct components then σy and $a_{\mathcal{P}}$ are similarly ordered if and only if $\mathcal{P}_{\sigma y} \leq \mathcal{P}$.

Theorem 2.2 (The Rearrangement Inequality). *Let u and v be two similarly ordered points in $\mathbb{R}^n$. Then for any $\sigma \in S_n$, we have*

$$\langle u, \sigma v \rangle \leq \langle u, v \rangle. \quad (1)$$

Moreover, equality holds in (1) if and only if u and σv are also similarly ordered.

The rearrangement inequality also tells us that all faces of the permutahedron $\Pi_{n-1}(y)$ arise as $F_{\mathcal{P}}$ for some ordered partition. To see this, suppose that $H(a, z)$ is a supporting halfspace for $\Pi_{n-1}(y)$ and that $F_a = \Pi_{n-1}(y) \cap \partial H(a, z)$. One can use Theorem 2.2 to show that $F_a = F_{\mathcal{P}_a}$. Now, one may readily check that this correspondence between ordered partitions and faces of the permutahedron is in fact an isomorphism of graded lattices.

3. Rearrangement Inequalities

Let u and v be points in $\mathbb{R}^n$. For the sake of convenience, we suppose that both have nondecreasing components, i.e. $u_i \leq u_{i+1}$ and $v_i \leq v_{i+1}$ for $i = 1, \dots, n-1$. Of course, all the results of this section could easily be reformulated in terms of similarly ordered vectors. We wish to study the function $S_n \rightarrow \mathbb{R}$ defined by $\sigma \mapsto \langle u, \sigma v \rangle$ where S_n acts on $\mathbb{R}^n$ via the permutation representation. The rearrangement inequality tells us that this function attains its maximum at the identity element. However, with a little care we can find some more detailed information. First we record the following straightforward observation for later use. The proof is left as an easy exercise.

Lemma 3.1. *Suppose that σ_1 and σ_2 are elements of S_n whose cycle decompositions are mutually disjoint. Then*

$$\langle u, v \rangle - \langle u, \sigma_1 \sigma_2 v \rangle = (\langle u, v \rangle - \langle u, \sigma_1 v \rangle) + (\langle u, v \rangle - \langle u, \sigma_2 v \rangle).$$

Thus, we are particularly interested in finding bounds for $\langle u, \gamma v \rangle$ when γ is a cycle. The following elementary lemma and the ensuing corollaries are the key results that we need in order to analyse the alternahedron. Intuitively it says that the longer the cycle γ is, the smaller the quantity $\langle u, \gamma v \rangle$ is.

Lemma 3.2 (Strong Rearrangement Inequality). *Let $\gamma \in S_n$ be a cycle and suppose that $\gamma = (m_1, m_2, \dots, m_k)$ where $m_k > m_1$ and $m_k > m_{k-1}$. Then*

$$\langle u, \gamma v \rangle \leq \langle u, (m_1, m_2, \dots, m_{k-1})v \rangle. \quad (2)$$

Moreover, equality holds in (2) if and only if either $u_{m_k} = u_{m_1}$ or $v_{m_k} = v_{m_{k-1}}$.

Proof. Let $\delta = (m_1, m_2, \dots, m_{k-1})$. Observe that γv and δv only differ in the m_1 th and m_k th components. Thus

$$\begin{aligned} \langle u, \delta v \rangle - \langle u, \gamma v \rangle &= u_{m_1}(v_{m_{k-1}} - v_{m_k}) + u_{m_k}(v_{m_k} - v_{m_{k-1}}) \\ &= (u_{m_k} - u_{m_1})(v_{m_k} - v_{m_{k-1}}) \end{aligned}$$

and the desired conclusions follow from this. $\square$

We remark that it is of course always possible to write any cycle γ as in the statement of Lemma 3.2. Later we will be concerned with the group of even permutations so it is particularly interesting for us to compare $\langle u, \sigma v \rangle$ for some arbitrary permutation σ to $\langle u, \tau v \rangle$ for certain transpositions τ .

Corollary 3.3. *Let $\sigma \in S_n$ be a nontrivial permutation. Then there is some transposition τ such that*

$$\langle u, \sigma v \rangle \leq \langle u, \tau v \rangle. \quad (3)$$

Moreover, if u and v have strictly increasing components and σ is not a transposition then we can choose τ so that (3) is strict.

Proof. Suppose that $\sigma = \gamma_1 \gamma_2 \cdots \gamma_l$ where the γ_i 's are pairwise disjoint nontrivial cycles. It follows immediately from Lemma 3.1 and Theorem 2.2 that for any $j = 1, \dots, l$,

$$\langle u, \sigma v \rangle \leq \langle u, \gamma_j v \rangle. \quad (4)$$

Then a straightforward induction argument using Lemma 3.2 shows that there is some transposition τ such that

$$\langle u, \gamma_j v \rangle \leq \langle u, \tau v \rangle. \quad (5)$$

Combining inequalities (4) and (5) yields the first part of the corollary. Now suppose that u and v have strictly increasing components. If $l \geq 2$ then inequality (4) is strict and if γ_j is a cycle of length at least 3 then, by Lemma 3.2, inequality (5) is strict. These two observations lead immediately to the second part of the corollary. $\square$

We remark that it is clear from the proof of Corollary 3.3 that τ can be chosen so that it transposes two components that are moved by the permutation σ . However, this fact is not crucial for our purposes.

Corollary 3.4. *There is some $j \in [n - 1]$ such that for every $\sigma \neq e$ in S_n ,*

$$\langle u, \sigma v \rangle \leq \langle u, (j, j + 1)v \rangle.$$

If u and v have strictly increasing components and σ is not itself an adjacent transposition (i.e. of the form $(i, i + 1)$) then the inequality is strict.

Proof. Corollary 3.3 tells us that the maximum value of $\langle u, \sigma v \rangle$ as σ varies over nonidentity elements occurs at some transposition. Now suppose that $1 \leq r < s < t \leq n$. Then

$$\begin{aligned} \langle u, v \rangle - \langle u, (r, t)v \rangle &= (u_t - u_r)(v_t - v_r) \\ &\geq (u_s - u_i)(v_s - v_r) \\ &\geq \langle u, v \rangle - \langle u, (r, s)v \rangle. \end{aligned}$$

So $\langle u, (r, t)v \rangle \leq \langle u, (r, s)v \rangle$ and it is clear that equality cannot occur if u and v have strictly increasing components. Therefore the maximum value of $\langle u, \sigma v \rangle$ as σ varies over nonidentity elements must occur at an adjacent transposition. $\square$

4. The Face Lattice of the Alternahedron

Let A_n denote the group of even permutations of an n -set. Let y be a point in $\mathbb{R}^n$ with pairwise distinct components. Throughout this section, we assume that $y_i < y_{i+1}$ for $i = 1, \dots, n - 1$, i.e. that y has strictly increasing components. This assumption merely simplifies the presentation—it does not affect the generality of results at all. To see this suppose that w is a point with pairwise distinct components. If there is some $\pi \in A_n$ such that πw has nondecreasing components, then clearly $\Pi_{n-1}(A_n, w) = \Pi_{n-1}(A_n, \pi w)$. On the other hand, if there is no such π , we observe that $\Pi_{n-1}(A_n, w)$ is affinely isomorphic to the polytope $\text{conv}(\{\sigma w : \sigma \in O_n\})$ where O_n is the subset of S_n consisting of odd permutations and we can interpret the results that follow as describing the latter polytope in this case.

We give a description of the face lattice of $\Pi_{n-1}(A_n, y) = \text{conv}(\{\sigma y : \sigma \in A_n\})$ by first describing a family of faces, each corresponding to an ordered partition of $[n]$. Then we use the results of Section 3 to characterise all other faces of the polytope.

Recall from Section 2 that ordered partitions of $[n]$ correspond to faces of $\Pi_{n-1}(y) = \text{conv}(\{\sigma y : \sigma \in S_n\})$. Similar considerations will give rise to a family of faces of

$\Pi_{n-1}(A_n, y)$. In the following discussion our notation is consistent with that introduced in Section 2. Thus, for each ordered partition $\mathcal{P} = (S_1, \dots, S_k)$ of $[n]$ we have $a_{\mathcal{P}}$ and $z_{\mathcal{P}}$ as defined in Section 2.2—we emphasise that $z_{\mathcal{P}} = \max\{\langle a_{\mathcal{P}}, \sigma y \rangle : \sigma \in S_n\}$ rather than $\max\{\langle a_{\mathcal{P}}, \sigma y \rangle : \sigma \in A_n\}$. Let $F_{\mathcal{P}} = \text{conv}(\{\sigma y : \sigma \in A_n \text{ and } \langle a_{\mathcal{P}}, \sigma y \rangle = z_{\mathcal{P}}\})$ (thus it is possible that $F_{\mathcal{P}}$ is empty). As in Lemma 2.1, $F_{\mathcal{P}} = \text{conv}(\{\sigma y : \sigma \in A_n \text{ and } \mathcal{P}_{\sigma y} \leq \mathcal{P}\})$.

We make two remarks at this point. First, the dimension of the face $F_{\mathcal{P}}$ is not always the same as the rank of the partition—we deal with this explicitly in Theorem 4.4 below. Second, the subfaces of $F_{\mathcal{P}}$ need not be of the form $F_{\mathcal{Q}}$ for some ordered partition $\mathcal{Q}$ —in contrast with the corresponding situation for the permutahedron.

Now we continue with the problem of classifying the faces of the alternahedron $\Pi_{n-1}(A_n, y)$. Suppose that $H(a, z)$ is a supporting halfspace for $\Pi_{n-1}(A_n, y)$ with corresponding face F_a . Then there are two possibilities for the point a . The following lemma deals with the first of these possibilities.

Lemma 4.1. *If πa has nondecreasing components for some $\pi \in A_n$ then $F_a = F_{\mathcal{P}_a}$.*

Proof. Observe that $z = \max\{\langle a, \sigma y \rangle : \sigma \in A_n\}$. However, a and $\pi^{-1}y$ are similarly ordered so, by Theorem 2.2, $z = \langle a, \pi^{-1}y \rangle$. Now, Theorem 2.2 also tells us that

$$F_a = \text{conv}(\{\sigma y : \sigma y \text{ and } a \text{ are similarly ordered}\}).$$

However, since y has pairwise distinct components, the blocks of $\mathcal{P}_{\sigma y}$ are singletons. Consequently, σy and a are similarly ordered if and only if $\mathcal{P}_{\sigma y} \leq \mathcal{P}_a$. $\square$

Now we must deal with the other case—that there does not exist any even permutation π for which πa is nondecreasing. Then there exists a unique odd permutation ρ such that ρa has strictly increasing components. Let $O_n \subset S_n$ be the set of odd permutations. Now

$$\begin{aligned} z &= \max\{\langle a, \sigma y \rangle : \sigma \in A_n\} \\ &= \max\{\langle \rho a, \rho \sigma y \rangle : \sigma \in A_n\} \\ &= \max\{\langle \rho a, \pi y \rangle : \pi \in O_n\}. \end{aligned}$$

However, by Corollary 3.4 we know that $z = \max\{\langle \rho a, \pi y \rangle : \pi \in O_n\} = \langle \rho a, (j, j+1)y \rangle$ for some $j \in [n-1]$. Moreover, Corollary 3.4 also tells us that if $\pi \in O_n$ is not an adjacent transposition, then $\langle \rho a, \pi y \rangle < \langle \rho a, (j, j+1)y \rangle$. So there is some subset D_a of $[n-1]$ such that $\langle \rho a, \pi y \rangle = z$ if and only if $\pi = (i, i+1)$ for some $i \in D_a$. We have proved the first statement of the following lemma.

Lemma 4.2. *Suppose there does not exist any $\sigma \in A_n$ such that σa has nondecreasing components. Then there exists an odd permutation ρ and a set $D_a \subseteq [n-1]$ such that $F_a = \text{conv}(\{\rho^{-1}(i, i+1)y : i \in D_a\})$. Moreover, F_a is a simplex of dimension $|D_a| - 1$.*

Proof. It remains to show that the points of the set $\{\rho^{-1}(i, i+1)y : i \in D_a\}$ are affinely independent. This follows from the observation that these points are all distinct neighbours of the vertex $\rho^{-1}y$ in the permutahedron $\Pi_{n-1}(y)$, which is a simple polytope. $\square$

In order to complete our description of the faces of $\Pi_{n-1}(A_n, y)$ it suffices to observe that, in converse to Lemma 4.2, if D is any nonempty subset of $[n-1]$ and ρ is any odd permutation, then we can obviously find $a \in \mathbb{R}^n$ such that $\langle a, \sigma y \rangle \leq 1$ for all $\sigma \in A_n$ with equality if and only if $\sigma = \rho^{-1}(i, i+1), i \in D$. Thus $H(a, 1)$ is a supporting halfspace for $\Pi_{n-1}(A_n, y)$ and we write $F_{\rho, D}$ for the corresponding face. It is appropriate at this point to record the following straightforward observations which will be useful in Section 5.

Lemma 4.3. *Let ρ be an odd permutation in S_n and suppose that $\langle a, x \rangle \leq c$ is a face-defining inequality for $F_{\rho, D}$. Then:*

- (1) ρa has strictly increasing components.
- (2) If τ is another odd permutation in S_n then $\langle \tau^{-1} \rho a, x \rangle \leq c$ is a face-defining inequality for $F_{\tau, D}$.

We now have a description of the face lattice. Any face of $\Pi_{n-1}(A_n, y)$ is either of the form $F_{\mathcal{P}}$ for some ordered partition $\mathcal{P}$, or $F_{\rho, D}$ for some $\rho \in O_n$ and $D \subseteq [n-1]$. Note that some edges have multiple descriptions but all higher-dimensional faces have unique descriptions in these terms.

4.1. The Grading of the Face Lattice

We complete our description of the face lattice by finding the dimension of all the faces that we have described above. In the case of the simplicial faces, $F_{\rho, D}$, there is obviously nothing more to say. However, we must be slightly careful about the dimension of the face $F_{\mathcal{P}}$ where $\mathcal{P}$ is an ordered partition. In the permutahedron, it turned out that the dimension of the face corresponds to the rank of the partition. In the alternahedron this also turns out to be true except for some partitions of rank 1 or 2 where the dimension is one less than the rank of the partition.

Theorem 4.4. *Let $\mathcal{P} = (S_1, S_2, \dots, S_k)$ be an ordered partition of $[n]$. If $|S_i| = 2$ for precisely one i and $|S_j| = 1$ for all $j \neq i$, then $F_{\mathcal{P}}$ is either empty or has dimension 0. If $|S_{i_1}| = |S_{i_2}| = 2$ for distinct i_1 and i_2 and $|S_j| = 1$ for all $j \neq i_1$ or i_2 , then $\dim(F_{\mathcal{P}}) = 1$. In all other cases $\dim(F_{\mathcal{P}}) = n - k$.*

Proof. We begin with four relevant observations:

- (1) For every ordered partition $\mathcal{P}$, $F_{\mathcal{P}}$ is contained in the $(n - k)$ -dimensional face of the permutahedron corresponding to the partition $\mathcal{P}$, so we already know that $\dim(F_{\mathcal{P}}) \leq n - k$.
- (2) Clearly, $\dim(\Pi_1(A_2, y)) = 0$. However, for all $n \geq 3$, $\Pi_{n-1}(A_n, y)$ has dimension $n - 1$. This follows, for example, by observing that the set $\{(1, 2)(i, i+1)y, i = 1, \dots, n-1\} \cup \{(1, 3, 2)y\}$ is an affinely independent n -set.

- (3) Let $\mathcal{Q} = (T_1, \dots, T_l)$ be an ordered partition of $[n]$. Suppose that for some j , $|T_j| \geq 2$ and $\mathcal{Q}^{(j)}$ is an ordered partition such that $\mathcal{Q}^{(j)} \leq \mathcal{Q}$ and

$$\mathcal{Q}^{(j)} = (\{m_1\}, \dots, \{m_r\}, T_j, \{m_{r+1}\}, \dots, \{m_s\}).$$

In other words, to obtain $\mathcal{Q}^{(j)}$, we refine $\mathcal{Q}$ by breaking all the blocks other than T_j into singletons. Then $F_{\mathcal{Q}^{(j)}}$ is itself an alternahedron and is a subface of $F_{\mathcal{Q}}$.

- (4) Let $\mathcal{Q}$ and $\mathcal{Q}^{(j)}$ be as above. Suppose that

$$\mathcal{R} = (T_1, \dots, T_{j-1}, \{n_1\}, \{n_2\}, \dots, \{n_t\}, T_{j+1}, \dots, T_l)$$

so that $\mathcal{R} \leq \mathcal{Q}$. Then $F_{\mathcal{Q}} \supseteq F_{\mathcal{Q}^{(j)}} \times F_{\mathcal{R}}$ and as a consequence $\dim(F_{\mathcal{Q}}) \geq \dim(F_{\mathcal{Q}^{(j)}}) + \dim(F_{\mathcal{R}})$.

Now, if $|S_j| \geq 3$ for all j , then we can choose $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(k)}$ so that

$$F_{\mathcal{P}} \supseteq F_{\mathcal{P}^{(1)}} \times \dots \times F_{\mathcal{P}^{(k)}}.$$

Therefore, $\dim(F_{\mathcal{P}}) \geq \sum_{j=1}^k (|S_j| - 1) = n - k$. So $\dim(F_{\mathcal{P}}) = n - k$ in this case. It is also clear that if all the S_i 's are singletons then either $F_{\mathcal{P}}$ is empty or $\dim(F_{\mathcal{P}}) = 0$. It follows that $\dim(F_{\mathcal{P}}) = n - k$ when none of the S_j 's is a 2-set.

To deal with the cases where some of the S_j 's are 2-sets we may suppose, without loss of generality, that $\mathcal{P} = (S_1, \dots, S_l, S_{l+1}, \dots, S_k)$ where $|S_j| \neq 2$ for all $j \geq l+1$. First we observe that if $l = 1$ and if $|S_j| = 1$ for all $j \geq 2$, then it is possible that $F_{\mathcal{P}}$ is empty. This happens precisely when there is no $\sigma \in A_n$ such that $\mathcal{P}_{\sigma y} \leq \mathcal{P}$. In all other cases $F_{\mathcal{P}}$ is nonempty. Thus, from now on we assume that there exists some $\sigma \in A_n$ such that $\mathcal{P}_{\sigma y} \leq \mathcal{P}$. Let $\mathcal{Q}$, respectively $\mathcal{R}$, be partitions obtained from $\mathcal{P}$ by refining all the 2-sets, respectively the non-2-sets, to singletons. We may suppose, again without loss of generality, that

$$\mathcal{Q} = (\{1\}, \{2\}, \dots, \{2l\}, S_{l+1}, \dots, S_k),$$

$$\mathcal{R} = (S_1, \dots, S_l, \{2l+1\}, \{2l+2\}, \dots, \{n\}).$$

We claim that if $l \geq 3$ then $F_{\mathcal{R}}$ contains a simplex of dimension l . This follows from the observation that the set

$$\{(1, 2)(i, i+1)y : i = 1, 3, 5, \dots, 2l-1\} \cup \{(3, 4)(5, 6)y\}$$

is affinely independent and therefore its convex hull is a simplex of dimension l (this simplex is not in general a face of $\Pi_{n-1}(A_n, y)$). Now, we already know that $\dim(F_{\mathcal{Q}}) = n - k - l$. Therefore, if $l \geq 3$ we have $\dim(F_{\mathcal{P}}) = n - k$.

Finally, we must deal with the cases $l = 1, 2$. We need the following technical lemma.

Lemma 4.5. *Let $\mathcal{P}$ be an ordered partition as above with $l \geq 1$ and let $\mathcal{Q}_0 = (\{1\}, \{2\}, S_2, \dots, S_k)$. If $|S_j| \geq 2$ for at least one $j \in \{2, \dots, k\}$ and if $\dim(F_{\mathcal{Q}_0}) = n - k - 1$ then $\dim(F_{\mathcal{P}}) = n - k$.*

Proof. Let $\mathcal{Q}_1 = (\{2\}, \{1\}, S_2, \dots, S_k)$. The hypothesis of the lemma implies that $\dim(F_{\mathcal{Q}_1}) = n - k - 1$ also. Moreover, both $F_{\mathcal{Q}_0}$ and $F_{\mathcal{Q}_1}$ are full-dimensional subsets of the corresponding faces of the permutahedron. Thus, if $\text{aff}(X)$ denotes the affine hull of a set X , we have

$$\begin{aligned} \text{aff}(F_{\mathcal{Q}_1}) &= \{(1, 2)x : x \in \text{aff}(F_{\mathcal{Q}_0})\} \\ &= \{x + (y_2 - y_1, y_1 - y_2, 0, \dots, 0) : x \in \text{aff}(F_{\mathcal{Q}_0})\} \\ &= \text{aff}(F_{\mathcal{Q}_0}) + (y_2 - y_1, y_1 - y_2, 0, \dots, 0). \end{aligned}$$

Now, $\text{aff}(F_{\mathcal{P}}) = \text{aff}(\text{aff}(F_{\mathcal{Q}_0}) \cup \text{aff}(F_{\mathcal{Q}_1}))$ and the result follows from this. $\square$

Now, it follows that if $l = 1$ or 2 and at $|S_j| \geq 3$ for at least one $j \geq l + 1$, then $\dim(F_{\mathcal{P}}) = n - k$. Finally, we are left with the cases that $l = 1$ or 2 and $|S_j| = 1$ for $j = l + 1, \dots, k$. It is clear that if $l = 1$, then $F_{\mathcal{P}}$ is a single point (recall we have already dealt with the possibility that $F_{\mathcal{P}}$ is empty). If $l = 2$, then either $F_{\mathcal{P}}$ is the line segment $[\sigma y, \sigma(1, 2)(3, 4)y]$ where $\sigma \in A_n$ or it is the line segment $[\pi(1, 2)y, \pi(3, 4)y]$ where $\pi \in O_n$. $\square$

Now, we have a complete description of the graded face lattice of the alternahedron. For example, we have the following enumeration of the facets.

Proposition 4.6. *For $n \geq 5$ the alternahedron $\Pi_{n-1}(A_n, y)$ has $n!/2 + 2^n - 2$ facets.*

Proof. For $n \geq 5$, there are two disjoint classes of facets. First there are facets of the form $F_{\mathcal{P}}$ where $\mathcal{P}$ is an ordered partition of rank $n - 2$. There are $2^n - 2$ such ordered partitions. Second there are the simplicial facets of the form $F_{\rho, [n]}$ where ρ is an odd permutation, and there are $n!/2$ odd permutations. $\square$

We remark that the proof of Proposition 4.6 shows that for $n \geq 5$ the $(n - 1)$ -dimensional alternahedron has exactly $n!/2$ simplicial facets since none of the facets $F_{\mathcal{P}}$ is a simplex for $n \geq 5$. This was conjectured by Steinkamp and Perkinson in [8].

4.2. An Example

We conclude this section by using our results to give an explicit description of the vertex figure, V , of the vertex y in the four-dimensional polytope $\Pi_4(A_5, y)$ —we assume, as before, that $y_i < y_{i+1}$ for $i = 1, \dots, 4$. The vertex figure of a vertex is essentially the polytope that we obtain by slicing through a given polytope with a hyperplane sufficiently close to the given vertex (see [10] for a more precise definition). Thus, in this particular example, V is a three-dimensional polyhedron. It has 9 vertices (corresponding to edges of the alternahedron containing y), 15 edges and 8 faces. Table 1 summarises the structure of the faces of $\Pi_4(A_5, y)$ that contain the vertex y , or equivalently the face lattice of V . Note that we can specify a face of V by specifying the corresponding (one-dimensional higher) face of $\Pi_{n-1}(A_n, y)$. Also note that many of the vertices of V (i.e. edges of

Table 1. Faces of $\Pi_4(A_5, y)$ that contain y .

Dimension	Faces $F_{\mathcal{P}}$	Simplicial faces $F_{\rho, D}$
3	Two alternahedra corresponding to $\mathcal{P} = (\{1\}, \{2, 3, 4, 5\})$ and $(\{1, 2, 3, 4\}, \{5\})$, and 2 octahedra corresponding to $\mathcal{P} = (\{1, 2\}, \{3, 4, 5\})$ and $(\{1, 2, 3\}, \{4, 5\})$.	Four tetrahedra corresponding to $\rho = (i, i + 1)$ and $D = \{1, 2, 3, 4\}$.
2	Three triangles corresponding to $\mathcal{P} = (\{1, 2, 3\}, \{4\}, \{5\})$, $(\{1\}, \{2, 3, 4\}, \{5\})$ and $(\{1\}, \{2\}, \{3, 4, 5\})$	Twelve triangles corresponding to $\rho = (i, i + 1)$ and D a 3-subset of $\{1, 2, 3, 4\}$ containing i
1	Three edges corresponding to $\mathcal{P} = (\{1, 2\}, \{3, 4\}, \{5\})$, $(\{1, 2\}, \{3\}, \{4, 5\})$ and $(\{1\}, \{2, 3\}, \{4, 5\})$	Twelve edges corresponding to $\rho = (i, i + 1)$ and D a 2-set containing i

$\Pi_4(A_5, y)$ are counted more than once in the table. For example, $F_{(\{1,2\},\{3,4\},\{5\})} = F_{(\{1,2\},\{1,3\})} = F_{(\{3,4\},\{1,3\})}$. One readily checks that there are indeed only nine distinct vertices in V . In contrast all the edges and facets have unique descriptions. Thus there are 15 edges and 8 facets. It is not hard to see that V is in fact combinatorially equivalent to the polyhedron depicted in Fig. 2. It has two pentagonal faces corresponding to the three-dimensional alternahedra and two quadrilateral faces corresponding to the octahedra.

5. Further Remarks

5.1. The Membership Problem for the Alternahedron

We observe that as a consequence of our characterisation of the face lattice of the alternahedron we have an efficient algorithm for deciding whether or not a given point belongs to $\Pi_{n-1}(A_n, y)$. For $n \geq 3$ we know that $\Pi_{n-1}(A_n, y)$ has dimension $n - 1$ —it is a full-dimensional subset of the affine hyperplane $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Now suppose that w is some point in this hyperplane. We must decide whether or not w satisfies all the facet-defining inequalities. By Proposition 4.6 we know that for $n \geq 5$ there are $n!/2 + 2^n - 2$ such inequalities. However, we can easily reduce the number of inequalities that need to be checked to at most $2n - 2$. First we observe that $w \in \Pi_{n-1}(A_n, y)$ if and only if $\sigma w \in \Pi_{n-1}(A_n, y)$ for any $\sigma \in A_n$; thus by applying an appropriate σ (which

Fig. 2. The vertex figure of y in $\Pi_4(A_5, y)$.

depends only on the order of the components of w to w we can assume that either w has nondecreasing components or that $(1, 2)w$ has strictly increasing components.

Proposition 5.1. *If w has nondecreasing components and $\sum_{i=1}^n w_i = \sum_{i=1}^n y_i$ then $w \in \Pi_{n-1}(A_n, y)$ if and only if it satisfies the following $2n - 2$ inequalities:*

- (1) $\langle a_{\mathcal{P}}, x \rangle \leq \langle a_{\mathcal{P}}, y \rangle$ where $\mathcal{P}$ is an ordered partition of the form $(\{1, \dots, k\}, \{k+1, \dots, n\})$ with $1 \leq k \leq n-1$.
- (2) A face-defining inequality for each of the facets $F_{(i,i+1),[n-1]}$, $1 \leq i \leq n-1$.

Proof. Clearly, if $w \in \Pi_{n-1}(A_n, y)$ then it satisfies the $2n - 2$ inequalities described in the statement, as these are all face-defining inequalities.

Conversely, suppose that w satisfies the $n - 1$ inequalities described in (1). Then by Theorem 2.2, $\langle a_{\mathcal{P}}, w \rangle \leq \langle a_{\mathcal{P}}, y \rangle$ for any ordered partition $\mathcal{P}$ of rank $n - 2$, since w has nondecreasing components. Now suppose that w satisfies the inequalities described in (2) and let $\langle a, x \rangle \leq c$ be a face-defining inequality for a facet $F_{\rho, [n-1]}$ where ρ is some odd permutation. We claim that w satisfies this inequality. Now, by Lemma 4.3, for $1 \leq i \leq n-1$, $\langle (i, i+1)\rho a, x \rangle \leq c$ is a face-defining inequality for the facet $F_{(i,i+1), [n-1]}$. Since w belongs to the $(n-1)$ -dimensional hyperplane $\sum_{i=1}^n w_i = \sum_{i=1}^n y_i$ and satisfies some face-defining inequality for the $(n-2)$ -dimensional face $F_{(i,i+1), [n-1]}$, it satisfies any face-defining inequality for $F_{(i,i+1), [n-1]}$. Therefore $\langle \rho a, (i, i+1)w \rangle \leq c$ for all $i \in [n-1]$. Now it is also true that ρa has strictly increasing components (Lemma 4.3) and that w has nondecreasing components. Therefore, by Corollary 3.4 we know that $\langle \rho a, \rho w \rangle \leq \langle \rho a, (j, j+1)w \rangle$ for some $j \in [n-1]$. Therefore $\langle a, w \rangle = \langle \rho a, \rho w \rangle \leq \langle \rho a, (j, j+1)w \rangle \leq c$ as required. $\square$

In the case where $(1, 2)w$ has strictly increasing components we have the following proposition whose proof is similar to that of Proposition 5.1 and is therefore omitted.

Proposition 5.2. *If $(1, 2)w$ has strictly increasing components and $\sum_{i=1}^n w_i = \sum_{i=1}^n y_i$ then $w \in \Pi_{n-1}(A_n, y)$ if and only if it satisfies the following n inequalities:*

- (1) $\langle (1, 2)a_{\mathcal{P}}, x \rangle \leq \langle a_{\mathcal{P}}, y \rangle$ where $\mathcal{P}$ is an ordered partition of the form $(\{1, \dots, k\}, \{k+1, \dots, n\})$ with $1 \leq k \leq n-1$.
- (2) A face-defining inequality for the facet $F_{(1,2), [n-1]}$.

Thus, in order to check that an arbitrary point w belongs to the $(n-1)$ -dimensional alternahedron $\Pi_{n-1}(A_n, y)$, one first finds a permutation σ such that either σw has nondecreasing components or $(1, 2)\sigma w$ has strictly increasing components and then there are at most $2n - 2$ inequalities to check.

5.2. Orbit Polytopes

The problem of understanding the combinatorial structure of orbit polytopes remains largely open—even in the special case of subgroups of the symmetric group acting

via the permutation representation. The second author has considered the finite reflection groups and their even subgroups in this context [6], where similar results may be obtained.

The results that we have presented here raise some obvious questions concerning orbit polytopes in general. Note that as a consequence of Section 4 we have

Proposition 5.3. *Let y and z be points in $\mathbb{R}^n$ that have a trivial stabiliser with respect to the permutation action of S_n . Then $\Pi_{n-1}(A_n, y)$ and $\Pi_{n-1}(A_n, z)$ have isomorphic face lattices.*

This suggests two obvious questions. First, one may ask what about the case where $y_i = y_j$ for some $i \neq j$? We observe that in this case $\Pi_{n-1}(A_n, y) = \Pi_{n-1}(y)$ since, in this case, the orbit of y under the action of A_n is the same as its orbit under the action of S_n . Thus, this case has already been analysed in [1]—in the section on multipermutahedra. Second, it is natural to wonder whether or not the combinatorial structure of an orbit polytope $P(G, v)$ is invariant for all sufficiently generic points v . This question has been considered before by Onn [7] who exhibits an example of a representation of a subgroup of S_6 for which the combinatorial type of the orbit polytope is dependent on the choice of orbit. Indeed, it is easily seen that the combinatorial type of $\Pi_{n-1}(A_n, y)$ depends on the choice of y . However, as we have just observed, if we restrict y to the set of points that have a trivial stabiliser with respect to S_n , then the face lattice is uniquely determined. This is not true for smaller subgroups of the symmetric groups [8] and it would be interesting (to the authors at least!) to determine which particular subgroups of S_n have this property.

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