The Width of the Transversal Strips of $T(3)$ -Families in the Plane*

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Abstract. Eckhoff [1] proved that to every finite $T(3)$ -family of pairwise disjoint translates of a compact convex set there exists a strip of relative width 1 meeting all members. Our goal is to give a better upper bound in terms of the cardinality of the family. This bound tends to zero as the number of the members increases.

1. Introduction

Throughout this paper the term *figure* is used for compact domains with non-empty interior and we consider families of disjoint translates of a convex figure. A *line transversal* to a family of convex figures is a straight line having a non-empty intersection with every member of the family. Alternatively, we also say that this family has property T . A family of at least k figures is said to have property $T(k)$ if every k -member subset of it has property T . To get an overview of the different branches of the geometric transversal theory, especially Helly-type results, the reader is advised to consult the survey paper [2].

The following is a special case of a more general theorem due to Hadwiger [3]:

Theorem H. *If an infinite family $\mathcal{F}$ of pairwise disjoint translates of a convex figure D has property $T(3)$ then it has property T as well.*

For finite families the above statement is easily false.

The *width* $w_d(X)$ of a set X in direction d is the (not necessarily finite) diameter of the orthogonal projection of X on a straight line of direction d . The minimum of

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$w_d(X)$ taken over all possible directions is the *width* of set X . Let D be a convex figure. The D -width, or relative width, of set X in direction d is $w_d^D(X) = w_d(X)/w_d(D)$, i.e. the width of set X in direction d divided by the width of D in the same direction. The horizontal and vertical directions are denoted by h and v , respectively.

The following is a reformulation of a result of Eckhoff [1]:

Theorem E. *If a finite family $\mathcal{F}$ of pairwise disjoint translates of a convex figure D in the plane has property $T(3)$ then there exists a strip S of D -width 1 intersecting all members of the family.*

The *transversal strip* of a family of translates is a closed (parallel) strip intersecting all members of the family. Theorem E guaranties the existence of a transversal strip of relative width 1. Our goal is to generalize Theorem E by giving an upper bound to the relative width of the transversal strips in terms of the cardinality of the family such that this upper bound vanishes as the number of the translates tends to ∞ . In the following the term “narrowest” is used in the sense “of smallest D -width”. The relative width of the narrowest strip is the *transversal width* of the family. Observe that an affine transformation does not change the transversal width of a set.

We are going to prove the following:

Theorem. *For every width $w \in (0, 1]$ there is some number $N(w)$ such that every $T(3)$ -family of at least $N(w)$ pairwise disjoint translates of a convex figure has a transversal strip of relative width w .*

This theorem provides a bridge between Eckhoff’s upper bound 1 (for finite families of any size) and Hadwiger’s “upper bound” 0 (for infinite sets) of the transversal width of $T(3)$ -families of disjoint translates of convex figures. Observe that our upper bound $N(w)$ does not depend on D .

2. The Proof

Two sets X_1 and X_2 are said to be *separated* (strictly separated) by a straight line l if X_i is covered by H_i , $i = 1, 2$, where H_1 and H_2 are different closed (open) half-planes determined by straight line l .

The union of all straight lines intersecting two disjoint translates of a convex figure is a simply connected unbounded domain, the boundary of which consists of parts of the four lines tangent to both figures. This unbounded domain is called the *sheaf* belonging to the two translates (Fig. 1).

In the case of centrally symmetric convex figures the locus of the centers of the translates which have non-empty intersection with the sheaf (belonging to two given translates) is called the *center sheaf*. Two of the (at most) six lines generating the boundary of the center sheaf are non-separating tangent lines to both figures received through a central enlargement of the original translates by linear factor 2 and four lines are tangent to one of these enlarged translates and passing through the center of the other one (for illustration see Fig. 2).

Fig. 1. A sheaf.

To prove the theorem we show that for $w \in (0, 1]$ the inequality

$$n \leq N(w) = 16/w + 50 \quad (1)$$

holds, where n is the number of the members of the family and w denotes the transversal width of the family.

Proof of the Theorem. 1. Due to the standard reductions described by Tverberg [4] we can assume that D is centrally symmetric. As D and $D_{\text{sym}} = \frac{1}{2}(D - D)$ have the same width in every direction d , the substitution of each translate D by D_{sym} does not change the width of the whole family $\mathcal{F}$ in any direction. (If D is centrally symmetric then the D -width of set X can also be considered as the width of X in the Minkowskian plane of unit ball D .)

Fig. 2. A center sheaf.

2. Let $\mathcal{F}$ be a $T(3)$ -family of n disjoint translates of a centrally symmetric convex figure D and let $w > 0$ denote the D -width of the narrowest transversal strip S of $\mathcal{F}$ (or one of the narrowest strips if it is not unique.) By Theorem E we have $0 < w \leq 1$.

Let S^+ (S^{++}) denote the strip of smallest D -width, parallel to S , covering all centers (all translates). Suppose that S is horizontal and $n \geq 66$. Since the D -width of S^+ (S^{++}) is $w + 1$ ($w + 2$), looking for the narrowest strip intersecting all members, for the narrowest strip covering all centers or for the narrowest strip covering the whole family are equivalent problems.

3. Let the line segment $H_1 H_2$ be the horizontal chord through the center of D . It can be assumed that the two parallel support lines of D through H_1 and H_2 are vertical, else an area-preserving affinity could be applied to the plane making the above-mentioned support lines through H_1 and H_2 vertical while keeping the central symmetry of D , the $T(3)$ property of the family, the horizontal direction h and the vertical D -width $w_v^D(D)$ unchanged. (Even after this transformation the vertical direction gives the smallest D -width of the whole family, since for any set X the D -width $w_d(X)/w_d(D)$ and the D' -width $w_{d'}(X')/w_{d'}(D')$ are equal, where X' , D' and d' are the images of X , D and d .)

4. Again with reference to the fact that an affine transformation does not change the relative width of a set we choose $w_v(D) = w_h(D) = 1$. Then each translate can be enclosed into an axis-parallel unit square containing a boundary point of the enclosed translate on each side. Clearly, the area of D is at least $\frac{1}{2}$ (Fig. 3).

From now on it is assumed that the family $\mathcal{F}$ consists of *normal figures*, i.e. D is a centrally symmetric convex figure of vertical and horizontal width 1, it is enclosed in a unit square of vertical and horizontal sides and the midpoints of the vertical sides are boundary points of D , where “vertical” means the direction in which the D -width of family $\mathcal{F}$ attains its minimum. Clearly, on one hand the family of the (not necessarily pairwise disjoint) unit squares covering the translates inherits the $T(3)$ property and on the other hand if a horizontal strip intersects a square it intersects the translate covered by it.

Before proceeding with the proof of the theorem, a lemma is proved:

Lemma. *Let D_1 and D_2 be translates of the normal figure D centered at $C_1(-x_0, -y_0)$ and $C_2(x_0, y_0)$, respectively, and let $w > 0$, $x_0 > y_0 = (w + 1)/2$. Then for some sufficiently small angle $0 < \delta < \pi/4$, whenever a direction d is between the vertical direction v and direction $v + \delta$ then the D -width $w_d^D(D_{12})$ of $D_{12} = D_1 \cup D_2$ is smaller than $w_v^D(D_{12}) = w + 2$.*

Fig. 3. Normal figure enclosed in a square of side length 1.

Fig. 4. $w_h(D_1 \cup D_2) > w_v(D_1 \cup D_2)$ would admit the reduction of the width of the transversal strip.

Proof. Let $P(-x_0 - \frac{1}{2}, -y_0 - \frac{1}{2})$ denote the lower left corner of the unit square Q_1 covering D_1 , and let $l_{d'}$ and $u_{d'}$ denote the lower and upper support lines of D_1 both of direction d' , where $d' = d + \pi/2$, the direction orthogonal to d . We apply a similarity of factor $(w + 2)$ centered at P , enlarging D_1 and Q_1 to D_1^+ , Q_1^+ , and mapping $l_{d'}$ and $u_{d'}$ into $l_{d'}^+$ and $u_{d'}^+$, respectively (Fig. 4). The D -width of D_1^+ is of course $w + 2$ in any direction. Let l be a line through P with positive slope but running below D_2 and denote the angle of the x -axis and l by $\delta > 0$. Clearly, the lower support line $l_{d'}^+$ of D_1^+ also runs below D_2 if its slope is positive but smaller than that of l since D_1^+ has a point on the line $y = -y_0 - \frac{1}{2}$. Because of the symmetry of D_1^+ and $x_0 > y_0$ the upper support line $u_{d'}^+$ of D_1^+ parallel to $l_{d'}^+$ runs strictly above D_2 as D_1^+ has a point on the line $y = y_0 + \frac{1}{2}$ as well. Consequently, since $l_{d'}^+$ and $u_{d'}^+$ represent the (constant) D -width of D_1^+ the D -width of D_{12} is smaller than that of D_1^+ as claimed. $\square$

5. In a properly chosen coordinate system the horizontal boundary lines of S^+ are

$$y = -w/2 - \frac{1}{2}$$

and

$$y = w/2 + \frac{1}{2}.$$

6. Based on the above lemma it is easy to give an upper bound to the area of a rectangle containing all centers of the family.

Consider the convex hulls H_u and H_ℓ of the centers on the upper and lower boundary line, respectively. Neither H_u nor H_ℓ is empty. The following holds:

Proposition. *There exist two centers, one in H_u and another in H_ℓ , such that the difference of their x -coordinates is at most $1 + w$.*

Proof. Two cases are distinguished.

Fig. 5. (a) Case A and (b) Case B of the proposition.

Case A: The projections of H_u and H_ℓ on the x -axis have disjoint interiors. (See the outlined segments in Fig. 5(a).) We can assume that the rightmost end of H_ℓ is $C_1(-x_0, -(w+1)/2)$ and the leftmost end of H_u is $C_2(x_0, (w+1)/2)$, $x_0 \geq 0$.

Now, by our lemma, $2x_0 \leq 1+w$ must hold otherwise S^+ would not be the narrowest strip covering the centers. (The parallel lines through C_1 and C_2 with sufficiently small slope would provide a strip with a D -width smaller than $1+w$.)

Case B: The projections of H_u and H_ℓ on the x -axis overlap. (See the outlined segments in Fig. 5(b).) Let then $C_1(x_1, -(w+1)/2)$, $C_2(x_2, (w+1)/2)$ and $C_3(x_3, -(w+1)/2)$ be three centers, such that $x_1 \leq x_2 \leq x_3$. The inequalities $x_2 - x_1 > 1+w$ and $x_3 - x_2 > 1+w$ cannot hold simultaneously since then the x -axis and the vertical lines $x = (x_2 + x_1)/2$ and $x = (x_3 + x_2)/2$ would separate every one of the three translates centered at C_1 , C_2 and C_3 from the other two and this would contradict the $T(3)$ property of the family. $\square$

7. Let $C_1(-x_0, -(w+1)/2)$ and $C_2(x_0, (w+1)/2)$ denote two centers with $0 \leq x_0 \leq (1+w)/2$.

The length $l(R)$ of the smallest axis-parallel rectangle R covering the intersection of the center sheaf assigned to the two squares centered at C_1 and C_2 and strip S^+ lying between the lines $y = (w+1)/2$ and $y = -(w+1)/2$ can be easily bounded from above. This length attains its largest value for $x_0 = (1+w)/2 \leq 1$. (Figure 6 is an illustration for $x_0 = (1+w)/2$.)

Fig. 6. Covering rectangle R for $x_0 = (1+w)/2$.

$$\text{length}(\overline{C_1E})/w = \text{length}(\overline{C_1F})/(1+w),$$

i.e.

$$\text{length}(\overline{C_1F}) = (1+w)(2+w)/w$$

and

$$l(R) \leq 2(\text{length}(\overline{C_1F}) - (1+w)/2) = 4/w + w + 5.$$

The height of R is $h(R) = 1 + w$.

The enlarged rectangle R^+ of length $l(R) + 1$ and height $h(R) + 1$ covers all members of the family. Since the area of a translate is at least $\frac{1}{2}$ and $w \leq 1$ we have

$$n \leq 2(4/w + w + 6)(2 + w) = 16/w + 50,$$

or

$$w \leq W(n),$$

where

$$W(n) = 16/(n - 50), \quad \text{if } n \geq 66, \quad \text{else } W(n) = 1. \quad (2)$$

This implies the claim of our theorem. $\square$

3. Remarks

1. Using more refined calculation the upper bound (2) can be reduced. For example, for a better estimate the role of the smallest axis-parallel rectangle can be substituted by the (real) intersection of the center sheaf and strip S^+ . (Asymptotically, this area is $\text{area}(R)/4$). Unfortunately, these improvements are not enough to provide a sharp result.

2. Nevertheless, to find the width of the narrowest transversal strip for translates of a specific shape might be a realistic program. The first candidates for this kind of research could be the square and the circle.

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