Tiling a Polygon with Two Kinds of Rectangles

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Abstract. We fix two rectangles with integer dimensions. We give a quadratic time algorithm which, given a polygon F as input, produces a tiling of F with translated copies of our rectangles (or indicates that there is no tiling). Moreover, we prove that any pair of tilings can be linked by a sequence of local transformations of tilings, called flips. This study is based on the use of Conway's tiling groups and extends the results of Kenyon and Kenyon (limited to the case when each rectangle has a side of length 1).

1. Introduction

In 1990 Conway and Lagarias [3] introduced the notion of a *tiling group*. They created a way to study tiling problems with an algebraic approach. This work has been prolonged by Thurston [17], who introduced the notion of *height function* (also introduced independently in the statistical physics literature (see [2] for a review)). The tools cited above have been used and generalized by different authors [5]–[7], [9]–[13], [15], [16] and some very strong algorithmic and structural results about tilings have been obtained.

In this paper we consider the problem of tiling a rectilinear polygon (i.e. a not necessarily convex (but simply connected) polygon whose sides consist of horizontal and vertical lines) with rectangular tiles. This problem was previously considered by Kenyon and Kenyon [5]. These authors obtained a linear time algorithm for tiling a rectilinear polygon by translates of $m \times 1$ and $1 \times n'$ rectangles, m and n' denoting positive fixed integers. For this, they used Conway–Lagarias-type tiling groups, finding an appropriate generalization of “height function” as used by Thurston for their groups. They also

considered tiling using 2×3 and 3×2 rectangles, where it was asserted that a quadratic time algorithm exists (a brief sketch of an approach without full details is given in the paper). It is also claimed that the method can be extended to the case of tilings by $k \times l$ and $l \times k$ rectangles, for any pair (k, l) of positive integers.

The present paper generalizes the previous paper, considering tiling by two types of tiles, which are translates (we do not allow rotations) of $m \times n$ and $m' \times n'$ rectangles, for an arbitrary fixed sequence (m, m', n, n') of positive integers. Our analysis also uses Conway–Lagarias-type tiling groups. The key new idea needed for the general case is in Proposition 2: we introduce a new technique to reduce the problem by cutting the polygon, using algebraic arguments.

The main results of the paper are:

- *Flip connectivity* (Theorem 1): any pair of tilings of a same polygon can be linked by a sequence of local transformations of tilings, called flips.
- A decision algorithm for tileability of a (rectilinear) polygon P that is quadratic in the area $(A(P))$ (Theorem 2).

An important restriction in this work is the fact that tiled regions are assumed to contain no interior holes. We recall that it is NP-complete to decide if a general connected region (with holes allowed) is tileable by translates of $m \times 1$ and $1 \times n'$ rectangles, with m and n' being integers such that $m \geq 3$ and $n' \geq 2$ [1].

The paper is divided as follows: we first give the definitions and basic concepts (Section 2). Afterwards, we explain our new techniques to cut tilings (Section 3) and we show how to combine them with the machinery of height functions (Section 4) to get a quadratic tiling algorithm and prove the flip connectivity. We finish with the particular cases where at least one side of a rectangle is 1 (some simplifications appear, Section 5), and we show how the method can be used for rectangles of non-integer dimensions (Section 6).

2. Basic Tools

2.1. The Square Lattice

Let Λ be the square lattice of the Euclidean plane $\mathbb{R}^2$. A *cell* of Λ is a square of unit sides whose vertices have integer coordinates. The *vertices* and *edges* of Λ are those of the cells of Λ . Two vertices are *neighbors* if they are linked by an edge.

A *path* P of Λ is a finite sequence of vertices such that two consecutive vertices are neighbors. A *corner* of a path P is the origin or final vertex, or a vertex v of P which is not equal to the center of the line segment linking the predecessor and the successor of v in P . A corner v is *weak* if its predecessor and its successor are equal, otherwise the corner is *strong* (the origin and final vertices of the path are strong corners). A path can be defined by the sequence $(v_0, v_1, \dots, v_p)$ of its corners. In this paper we encode paths in this way. A *move* of a path is the sub-path linking two consecutive corners. We have two kinds of moves: horizontal and vertical ones. The *algebraic length* of a horizontal (respectively vertical) move from $v_i = (x_i, y_i)$ to $v_{i+1} = (x_{i+1}, y_{i+1})$ is the difference $x_{i+1} - x_i$ (respectively $y_{i+1} - y_i$).

A (finite) figure F of Λ is a (finite) union of (closed) cells of Λ . Its boundary is denoted by δF . The *lower boundary* is the union of open horizontal line segments $]v, v'[,$ included in δF , such that the interior of F is above $]v, v'[,$ (i.e. more formally, F contains a rectangle whose bottom side is $]v, v'[,$). One defines in a similar way the *left, right and upper boundaries* of F . A figure F is *simply connected* if F and its complement $\mathbb{R}^2 \setminus F$ are both connected. A finite simply connected figure F is called a *polygon* of Λ . The boundary of a polygon F canonically induces a cycle in Λ , which is called the boundary cycle of F . In this paper we only consider polygons, except when explicitly stated otherwise.

2.2. Tilings

We fix a pair $\{R, R'\}$ of rectangles with vertical and horizontal sides of integer dimensions. The *width* (i.e. horizontal dimension) of R is denoted by m , the width of R' is denoted by m' ; the *height* (i.e. vertical dimension) of R is denoted by n , the height of R' is denoted by n' . An *R-tile* (respectively *R'-tile*) is a translated copy of the rectangle R (respectively R'). A *tiling* T of a figure F is a set of tiles, with pairwise disjoint interiors (i.e. there is no overlap), such that the union of the tiles of T equals F (i.e. there is no gap). A tile t is *cut* by a path P if there exists an open line segment $]v, v'[,$ linking two consecutive vertices of P , included in the interior part of the tile t . The sub-graph $G_T = (V_T, E_T)$ of Λ such that V_T (respectively E_T) is the set (respectively edges) of vertices of Λ which are on the boundary of a tile of T . Obviously, the set V_T (respectively E_T) contains all the vertices (respectively edges) of Λ which are in δF .

2.3. The Preprocessing

The problem of tiling can easily be reduced to the case when $\gcd(m, m') = \gcd(n, n') = 1$ as follows: consider the only tiling Θ of the figure F with translated copies of $\gcd(m, m') \times \gcd(n, n')$ rectangles, called *bricks*. The tiling Θ is a refinement of any “classical” tiling T with R -tiles and R' -tiles, i.e. each brick of Θ is included in a tile of T .

Two bricks θ and θ' of Θ are called *equivalent* if they respectively are the first and last elements of a finite sequence of bricks such that two consecutive bricks of the sequence share a whole side. Note that two bricks included in the same tile of T are equivalent (i.e. the boundary of an equivalence class cuts no tile of T), and the union of all the bricks of an equivalence class form a polygon (if such a union had a hole, then a brick placed on a corner of the hole would create a contradiction). Thus, the problem can be simplified as follows: We first compute the equivalence classes (this can easily be done in a linear time in the area of the polygon). Afterwards, we have to study tilings on each equivalence class. However, since, in each class, bricks are equivalent, we can change the units in order to consider a brick as the new unit rectangle.

This transformation makes us consider that $\gcd(m, m') = \gcd(n, n') = 1$. In the rest of the paper we assume that this simplification has been done and thus $\gcd(m, m') = \gcd(n, n') = 1$.

Fig. 1. The graph of G_1 (with $m = 4$ and $n' = 3$).

3. Tiling Groups and Cut Lines

The two following sections are devoted to the main case, called the *non-degenerate case*, when we have $m > 1$, $n > 1$, $m' > 1$ and $n' > 1$.

We introduce the groups $G_1 = \mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n'\mathbb{Z}$, i.e. the free product (see, for example, [8] for details about group theory) of groups $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n'\mathbb{Z}$, and $G_2 = \mathbb{Z}/m'\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. We only give properties and notations for G_1 , since they are symmetric for G_2 . We recall that there exists a canonical surjection (denoted by s_1) from G_1 to the Cartesian product $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z}$, and that each element g of G_1 can be written in a unique way: $g = \Pi_{i=1}^p g_i$, with the following properties:

- for each integer i such that $1 \leq i \leq p$, g_i is an element of the set $(\mathbb{Z}/m\mathbb{Z})^* \cup (\mathbb{Z}/n'\mathbb{Z})^*$,
- for each integer i such that $1 \leq i < p$, g_i is an element of $\mathbb{Z}/m\mathbb{Z}$ if and only if g_{i+1} is an element of $\mathbb{Z}/n'\mathbb{Z}$.

This expression is called the *canonical expression* of g and the integer p is called the *length* of g (denoted by $l(g)$). In this way an order relation is created on G_1 : we say that $g \leq g'$ if the canonical expression of g is a prefix of g' .

Two distinct elements g and g' of G_1 are *H-equivalent* if there exists an element x of $\mathbb{Z}/m\mathbb{Z}$ such that $g' = gx$, they are *V-equivalent* if there exists an element y of $\mathbb{Z}/n'\mathbb{Z}$ such that $g' = gy$, they are *neighbors* if they are *H-equivalent* or *V-equivalent*. Each element of G_1 is exactly the intersection of its *H-equivalence* class and its *V-equivalence* class.

The *graph* of G_1 (Fig. 1) is the symmetric graph whose vertices are elements of G_1 , and for each pair (g, g') of distinct elements of G_1 , (g, g') is an edge if and only if g and g' are neighbors.¹

Note that each path of this graph linking an element g to an element g' has to pass through each element g'' such that $1_{G_1} \leq g^{-1}g'' \leq g^{-1}g'$. Informally, we can say that the structure of graph elements is an infinite tree of equivalent classes.

¹ In the language of group theory, the graph of elements is the Cayley graph of G_1 when the set of generators is $(\mathbb{Z}/m\mathbb{Z})^* \cup (\mathbb{Z}/n'\mathbb{Z})^*$ and labels have been removed.

3.1. Tiling Function

Let $P = (v_1, \dots, v_p)$ (according to our convention, the vertices of the sequence are the corners of P). This path canonically induces an element g_P of G_1 as follows: each horizontal (respectively vertical) move gives an element of $\mathbb{Z}/m\mathbb{Z}$ (respectively $\mathbb{Z}/n'\mathbb{Z}$) equal to its algebraic length; the element g_P is obtained successively making the product of elements given by successive moves. Notice that any path describing the contour of a tile induces the unit element 1_{G_1} . Thus, tilings can be encoded using the groups G_1 and G_2 , as we see below.

Definition 1 (Tiling Function). Let T be a tiling of a polygon F . A *tiling function* induced by T is a mapping $f_{1,T}$ from the set V_T to G_1 such that, for each edge $[v = (x, y), v' = (x', y')]$ cutting no tile of T :

- if $x = x'$, then $f_{1,T}(v') = f_{1,T}(v)(y' - y)$, with $y' - y$ seen as an element of $\mathbb{Z}/n'\mathbb{Z}$,
- if $y = y'$, then $f_{1,T}(v') = f_{1,T}(v)(x' - x)$, with $x' - x$ seen as an element of $\mathbb{Z}/m\mathbb{Z}$.

Proposition 1 (Conway). Let F be a figure, let T be a tiling of F , let v_0 be a vertex of the boundary of F and let g_0 be an element of G_1 . If F is connected (respectively is a polygon), then there exists at most one (respectively exactly one) tiling function $f_{1,T}$ induced by T such that $f_{1,T}(v_0) = g_0$.

Proof (sketch). The function $f_{1,T}$ can easily be constructed by successively exploring the contour of each tile: the function $f_{1,T}$ is first defined on the boundary of a tile t_0 which has v_0 on its boundary. Afterwards, $f_{1,T}$ is defined on the boundary t_1 which shares a part of its boundary with t_0 and so on. This method gives the uniqueness of $f_{1,T}$, for F connected (nevertheless, a conflict (i.e. a vertex v such that two distinct values of $f_{1,T}(v)$ are given) can arise if F has some holes, which yields that there is no tiling function).

The existence for polygons is proved by induction on the number of tiles of T . If F is a polygon, then either F is reduced to a tile (and the existence is trivial) or F can be cut into two sub-polygons F_1 and F_2 , in such a way that each tile of T is included in either F_1 or F_2 . One can assume without loss of generality that v_0 is in F_1 , thus, by induction hypothesis, $f_{1,T}|_{F_1}$ exists on F_1 . Now, take a vertex v_1 in the intersection $F_1 \cap F_2$. By induction hypothesis, there exists a tiling function $f_{1,T}|_{F_2}$ on F_2 such that $f_{1,T}|_{F_1}(v_1) = f_{1,T}|_{F_2}(v_1)$. This gives the existence of the tiling function in the whole polygon F . $\square$

We now study tilings of a polygon F . For each tiling T , we use the unique tiling function $f_{1,T}$ such that $f_{1,T}(O)$ is the unit element of G_1 , where O denotes a fixed lower left corner of the boundary of F . Note that, for each vertex v on δF , $f_{1,T}(v)$ does not depend on the tiling T (which allows us to remove the tiling index and write $f_1(v)$) and can be computed without the knowledge of T , following the boundary of F . We also note that, for each vertex v of V_T , the value $s_1 \circ f_{1,T}(v)$ is the pair of coordinates of v (respectively modulo m and modulo n'), and, consequently, does not depend on T .

3.2. Cutting Using Algebraic Constraints

Definition 2. A *special path* P of F is a path defined by a sequence $(v_1, \dots, v_p)$ of successive corners such that:

- the vertex v_1 is on the *lower* boundary of F , the vertex v_p is on the boundary of F and for each integer i such that $1 \leq i < p$, the line segment $[v_i, v_{i+1}]$ is included in F ,
- for each odd integer i such that $1 \leq i < p$, there exists a non-null integer k_i such that $-m' < k_i < m'$ and $v_{i+1} = v_i + (k_i m, 0)$,
- for each even integer i such that $1 < i < p$, there exists a non-null integer k_i such that $-n < k_i < n$ such that $v_{i+1} = v_i + (0, k_i n')$,
- the element $(f_2(v_0))^{-1} f_2(v_p)$ is the element g_P of G_2 induced by the path P , i.e., $\prod_{i=1}^{p-1} r_i k_i$, with $r_i = n'$ for i even and $r_i = m$ for i odd.

The notion of a special path with the proposition below form the main new idea of this paper. There exists an easier proof [4] in the particular case when $m' = n = 2$ since the graph of G_2 has a linear structure.

Proposition 2. Let T be a tiling and let P be a special path of a polygon F . The path P cuts no tile of T .

Proof. We first construct from P another path P_T as follows: for each tile t which is cut by P , let $P_{t,\text{cross}}$ denote the sub-path of P which crosses t . The path P_T is obtained by replacing in P each sub-path $P_{t,\text{cross}}$ by the shortest path $P_{t,\text{surround}}$, on the boundary of t , with the same endpoints as $P_{t,\text{cross}}$ (if two paths are possible, any convention is possible; one can state, for example, that t is surrounded on its left side). Notice that $P_{t,\text{surround}}$ is formed from two or three moves. Informally, the path P_T is an approximation of P which cuts no tile of T .

Let i be an integer such that $1 \leq i \leq p$. We define the vertex v'_i such that v'_i is the last *strong corner* of P_T such that $f_{2,T}(v'_i)$ is equal to the element g_{p_i} induced by the sub-path of P from v_0 to v_i , i.e. $g_{p_i} = \prod_{j=0}^i r_j n_j$ (with $r_j = n'$ for j even and $r_j = m$ for j odd). This corner exists because of the structural properties of the graphs of elements and classes of G_2 (especially the tree structure of the graph of classes). The point v''_i is defined as the last point reached in P_T , before v'_i which also is in P (possibly $v''_i = v'_i$; see Fig. 2 for the notations). We claim the following fact:

Fact. The vertex v_i is before v''_i in P and, if, moreover, $v_i = v'_i$ then no tile of T is cut by P before this vertex.

The above fact is proved by induction. The result is obviously true for $i = 1$. Now assume that it also holds for a fixed integer i such that $1 \leq i < p$.

If v_{i+1} is before v''_i on P , the result is also obvious, since v''_i is before v''_{i+1} . Moreover, note that, for each vertex v , $s_2 \circ f_{2,T}(v'_i)$ is equal to the pair of coordinates of v (respectively modulo m' and n), thus $s_2 \circ f_{2,T}(v_i) \neq s_2 \circ f_{2,T}(v_{i+1})$, because of the first component.

Fig. 2. The main argument used in the proof of Proposition 2. The critical length has to be equal (modulo m') to the length of the line segment $[v_i, v_{i+1}]$ and, on the other hand, this length is the sum of the widths of tiles.

On the other hand, we have $s_2 \circ f_{2,T}(v'_i) = s_2 \circ f_{2,T}(v_i)$. Thus $v'_i \neq v_{i+1}$, which yields that $v_{i+1} \neq v'_{i+1}$.

Otherwise, by induction hypothesis, the point v''_i is on the line segment $[v_i, v_{i+1}]$. We only treat the case when i is odd and k_i is positive, since the proof is similar in the other cases.

Since v'_i is a strong corner of P_T , v'_i is on the left side of a tile t_1 used to define P_T . The vertex v''_i is also on the left side of t_1 , thus they have the same horizontal coordinate. Assume that v''_{i+1} is before v_{i+1} in P . In this case, v'_{i+1} is a right vertex of a tile of T , and the vertical line passing through v'_{i+1} cuts the line segment $[v_i, v_{i+1}]$ in a vertex v'''_{i+1} .

Now consider the line segment $[v'_i, v'''_{i+1}]$. This segment is not reduced to a vertex (since v'_i and v'_{i+1} do not have the same horizontal component modulo m'). Moreover, there exists a sequence $(t_1, t_2, \dots, t_q)$ of tiles of T such that the left side of t_1 contains v'_i and v''_i , the right side of t_q contains v'_{i+1} and v'''_{i+1} , and for each integer j such that $0 \leq j < q$, the intersection of the right side of t_j and the left side of t_{j+1} is a line segment not reduced to a single point. Let μ (respectively μ') denote the number of R -tiles (respectively R' -tiles) on the sequence $(t_1, t_2, \dots, t_q)$. We have

$$\mu m + \mu' m' \leq k_i m \quad (1)$$

and, moreover, from the definition of v''_i and v'''_{i+1} , using the horizontal coordinates,

$$\mu m + \mu' m' = k_i m \quad (\text{modulo } [m']). \quad (2)$$

Thus, we necessarily have $(\mu, \mu') = (k_i, 0)$ (since $\gcd(m, m') = 1$, equality (2) gives that $\mu = k_i + km'$ with $k \in \mathbb{Z}$; the integer k is null since $0 < \mu < m'$ from inequality (1); thus $\mu' = 0$). This means that $v''_i = v_i$, $v'''_{i+1} = v_{i+1}$ and the line segment $[v_i, v_{i+1}]$ only cuts R -tiles. Thus v'_i and v_i are both on the left side of the R -tile t_1 which yields that $v'_i = v_i$ since their vertical coordinates are equal modulo n . With the same argument, we necessarily have $v'_{i+1} = v_{i+1}$. Thus we have

$$f_{2,T}(v_{i+1}) = f_{2,T}(v_i)(k_i m). \quad (3)$$

Now consider the path starting in v_i by a horizontal move which follows the boundary of the polygon formed by the k_i R -tiles of the sequence until v_{i+1} is reached. This path directly induces a word, without possible simplification in G_2 (since the (absolute) length of each vertical move is lower than n , and the length of each horizontal move is km with k integer such that $0 < k \leq k_i < m'$). This word is necessarily $k_i m$, from equality (3). This means that no tile is cut by the segment $[v_i, v_{i+1}]$. This finishes the proof of the fact above, using induction.

Now, to get the result, it suffices to apply the fact for $i = p$. We necessarily have $v'_p = v_p$ since v_p finishes the path, thus the whole path cuts no tile. $\square$

3.3. Application to Stairs of Blocks

Definition 3. A *horizontal block* (respectively *vertical block*) of F is a (closed) $mm' \times n'$ rectangle (respectively $m \times nn'$ rectangle).

A horizontal (respectively vertical) block B is *covered* by a block B' if the (open) top (respectively right) side of B and the (open) bottom (respectively left) side of B' share a line segment of length km , where k denotes a positive integer.

A *stair of blocks* (see Fig. 3) is a finite sequence $(B_1, B_2, \dots, B_p)$ of blocks of the same type such that, for each integer i such that $1 \leq i < p$, B_{i+1} covers B_i , and there exists no subsequence $(B_{i+1}, B_{i+2}, \dots, B_{i+k})$ such that $\bigcup_{j=1}^k B_{i+j}$ is an $mm' \times nn'$ rectangle.

Notice that the figure formed by the union of blocks of a fixed stair has exactly one tiling which contains a unique type of tiles.

Proposition 3. Assume that there exists a stair $(B_1, B_2, \dots, B_p)$ of horizontal blocks such that the intersection of B_1 with the lower boundary of F contains a horizontal line segment S of length at least m , the intersection of B_p with the upper boundary of F

Fig. 3. A stair of blocks which crosses the polygon and an associated special path.

contains a horizontal line segment S' of length at least m , and the tiles of the tiling of $\bigcup_{i=1}^p B_i$ can be placed in F without inducing contradiction in the definition of the tiling function. Each tiling T of F contains the tiling of $\bigcup_{i=1}^p B_i$.

Proof. From the hypothesis, there exists a vertex v of S such that the distance between v and the lower right corner of B_1 is km , for an integer k such that $0 < k < m'$. In a similar way, there exists a vertex v' of S' such that the distance between v' and the upper right corner of B_p is $k'm$, for an integer k' such that $0 < k' < m'$.

The path from v to v' following the boundary of $\bigcup_{i=1}^p B_i$ counterclockwise is a special path and, from the previous proposition, no tile of T cuts this path. The same argument can be used following the boundary of $\bigcup_{i=1}^p B_i$ clockwise. This gives that no tile of T is cut by the boundary of $\bigcup_{i=1}^p B_i$. $\square$

Of course, we have a similar proposition for stairs of vertical blocks.

4. Height Function and Order

We now use the same height function as in [5]. We extend it, defining the height function for tiles and tilings, in order to obtain an order on the set of tilings of the polygon.

Definition 4 (Height, Predecessor, Order on G_1). The height $h_1(g)$ of an element g of $G_1 = \mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n'\mathbb{Z}$ is defined by the following axioms :

- If there exists an integer k such that the canonical expression of g is $(m'n)^k$ or $(m'n)^k m'$, then $h_1(g) = -l(g)$ (with m' seen as an element of $\mathbb{Z}/m\mathbb{Z}$ and n seen as an element of $\mathbb{Z}/n'\mathbb{Z}$).
- For each element g of G_1 , there exists a unique element $pr(g)$ (called the predecessor of g), neighbor of g , such that $h_1(pr(g)) = h_1(g) - 1$.
- For each element g' of G_1 , neighbor of both g and $pr(g)$, we have $h_1(g') = h_1(g)$.
- For each other neighbor g'' of g , we have $pr(g'') = g$.

Given a pair (g, g') of elements of G_1 , we say that $g \leq g'$ if there exists a sequence $(g_1, g_2, \dots, g_p)$ of elements of G_1 such that $g_1 = g$, $g_p = g'$ and, for each integer i such that $1 \leq i < p$, g_i is the predecessor of g_{i+1} .

Notice that the predecessor relation induces a tree structure on the group G_1 . Also notice that the canonical expression of any element g can be written $w_1 w_2$, where w_1 is a prefix word of maximal length such that $w_1 = (m'n)^k$ or $w_1 = (m'n)^k m'$, for an integer k . From this decomposition, $h_1(g)$ can be computed easily:

- $h_1(g) = l(w)$ if w_1 is the empty word,
- $h_1(g) = -l(w)$ if w_2 is the empty word,
- $h_1(g) = l(w_2) - l(w_1) - 1$ otherwise.

Notice that the heights of vertices which are on the same side $[v', v'']$ of a tile form either a singleton $\{h\}$ or a pair $\{h, h - 1\}$. With these notations, we say that h is the

Fig. 4. A weak flip.

height of $[v', v'']$ in T (denoted by $h_{1,T}([v', v''])$). With the convention $f_{1,T}(O) = 1_{G_1}$, the height of horizontal sides is even, and the height of vertical sides is odd. Both vertical sides of an R -tile t have the same height, say h , and the height of each horizontal tile is either $h - 1$ or $h + 1$ (a symmetric situation holds for R' -tiles). The height $h_1(t)$ of the tile t is defined by $h_1(T) = h$ (i.e. the height of an R -tile (respectively R' -tile) is the height of its vertical (respectively horizontal) sides). The height $h_1(T)$ of the tiling T is defined by $h_1(T) = \sum_{t \in T} a(t)h_{1,T}(t)$ where $a(t)$ denotes the area of the tile t .

4.1. Local Flips

4.1.1. Weak Flips. An $mm' \times (n + n')$ rectangle admits two tilings. Let T be a tiling of F such that an $mm' \times (n + n')$ rectangle is exactly covered by tiles of T . The replacement of those tiles of T (which cover the $mm' \times (n + n')$ rectangle) by tiles of the other tiling of the same rectangle is called a *weak vertical flip* (see Fig. 4). In this way a new tiling T_{flip} is obtained.

The flip is said to be *pushing up* if R' -tiles of T are in the bottom part of the $mm' \times (n + n')$ rectangle (and R' -tiles of T_{flip} are in the top part of the $mm' \times (n + n')$ rectangle). Otherwise, the flip is said to be *pushing down*.

If the height of both horizontal sides of the $mm' \times (n + n')$ rectangle differ by two units, then $h_1(T)$ and $h_1(T_{\text{flip}})$ are not equal. Otherwise $h_1(T) = h_1(T_{\text{flip}})$ and the flip is said to be *height stable*.

In a symmetric way we have another type of weak flips: horizontal flips which are based on the two tilings of an $(m + m') \times nn'$ rectangle. We use some similar definitions (pushing left flip, pushing right flip, height stable flip).

4.1.2. Strong Flips. We have another type of flip (called *strong flip*) based on the fact that we have two ways to tile an $mm' \times nn'$ rectangle (see Fig. 5). Such a flip always changes the value of $h_1(T)$.

Flips allow us to introduce an equivalence and an order relation on tilings as follows:

Definition 5 (Equivalence, Order on the Set of Tilings). Two tilings are said to be *height equivalent* if one can pass from one to the other by a sequence of height stable flips.

Given a pair (T, T') of tilings, we say that T *covers* T' if there exists a pair $(T_{\text{aux}}, T'_{\text{aux}})$ of tilings of F such that T is equivalent to T_{aux} , T' is equivalent to T'_{aux} , $h_1(T_{\text{aux}}) > h_1(T'_{\text{aux}})$ and T_{aux} is deduced from T'_{aux} by one flip. The order relation which is the transitive closure of the relation of covering is denoted by $>$.

Fig. 5. A strong flip.

4.2. Critical Tiling

For the order defined above, there exists at least a minimal tiling (since the set of tilings is finite) and if a tiling is minimal, then all the height equivalent tilings are also minimal.

Definition 6. A *critical tiling* T_{crit} , is a minimal tiling such that neither a height equivalent pushing down flip nor a height equivalent pushing left flip can be done.

To prove that there exists a critical tiling, it suffices to consider the potential function “pot” defined as follows: for each tiling T , $\text{pot}(T)$ is the sum of the vertical coordinates of the horizontal sides of R' -tiles of T and the horizontal coordinates of the vertical sides of R' -tiles. Notice that a pushing down (respectively left) flip removes $2mn'$ (respectively $2m'n$) units from the function “pot”. Thus, for any weak flip $\text{pot}(T)$ and $\text{pot}(T_{\text{flip}})$ are comparable. A minimal tiling which has the lowest potential value among the minimal tilings is necessarily critical.

Proposition 4. Let T_{crit} be a critical tiling. Let M denote the highest value of $h_{1,T_{\text{crit}}}$ in $V_{T_{\text{crit}}}$ and let M' denote the highest value of $h_{1,T_{\text{crit}}}$ in δF . We have $M' \geq M - 1$. Moreover, when $M' = M - 1$ and M is odd:

- There exists a set of blocks with pairwise disjoint interiors (called *critical blocks*) whose tilings are included in T_{crit} , for each vertex v , $h_{1,T_{\text{crit}}}(v) = M$ if and only if v is in the interior part of a line segment $[v', v'']$ which is the common side of two R' -tiles of the same block.
- Each critical block of T_{crit} whose bottom side does not share a line segment with δF covers another critical block of T_{crit} .
- There is no set of n critical blocks whose union is an $mm' \times nn'$ rectangle.
- The value M' is reached on both lower and upper boundaries of F .

Proof. Assume that $M > M'$ and M is odd. Let v be a vertex such that $h_{1,T}(v) = M$. The point v cannot be the corner of a tile of T , and for each side $[v', v'']$ of a tile t of T containing v , we necessarily have $f_{1,T}(v') = f_{1,T}(v'') = \text{pr}(f_{1,T}(v))$. On the other hand, this side is vertical, since M is odd. This yields that t is an R' -tile. Moreover, since

v is an interior point of F , then $[v', v'']$ is shared by two R' -tiles, whose horizontal sides have height $M - 1$.

Assume that t is the R' -tile whose left side is $[v', v'']$. Let k denote the lowest integer such that $f_{1,T}(v')(km') = \text{pr}(f_{1,T}(v))$ (with m' seen as an element of $\mathbb{Z}/m\mathbb{Z}$). The vertex $v' + (m', 0)$ is a right corner of t and $f_{1,T}(v' + (m', 0)) = f_{1,T}(v')m'$ (with m' seen as an element of $\mathbb{Z}/m\mathbb{Z}$). Thus, if $k > 1$, then the height of the vertices on the (open) right side of t is M , thus t shares its right side with another R' -tile t_1 . If $k > 2$, then the argument can be repeated. The same type of argument can also be used on the left side of t . In this way we have a set $\{t_{k-m}, t_{k-m+1}, \dots, t_{-1}, t, t_1, \dots, t_{k-1}\}$ of R' -tiles of T_{crit} whose union forms a critical block. This proves the first item of the proposition.

We now take a critical block of T_{crit} and are interested in what happens below its bottom side $[u', u'']$. Assume that $[u', u'']$ shares no sub-segment with the boundary δF . If the top of an R' -tile touches $[u', u'']$, then the height of (at least) one of its vertical sides is again M , which implies the existence of another critical block which contains the new R' -tile. Moreover, the new critical block is necessarily covered by the previous one.

The only remaining possibility is that $[u', u'']$ is the top side of an $mm' \times n$ rectangle, tiled by R -tiles of T_{crit} (since M cannot be reached on a vertical side of an R -tile, the height of vertical sides of R -tiles of T_{crit} touching $[u', u'']$ is $M - 2$, which gives the precise position of R -tiles). However, in this last possibility, the union of the block and the $mm' \times n$ rectangle seen above form an $mm' \times (n + n')$ rectangle on which a pushing down flip can be done, which is a contradiction (either this flip is height stable, or it creates a tiling covered by T_{crit}). This last case cannot hold.

This study implies that there exists a block whose bottom side reaches δF (since otherwise an infinite sequence of blocks would be generated), which yields that $M' = M - 1$ and M' is reached on the lower boundary of F .

It remains to prove that M' is reached on the upper boundary of F . To do it, from T_{crit} , while it is possible, make a sequence of height stable pushing up flips. A new minimal tiling T_{subcrit} is obtained in this way, and the maximal height reached by T_{subcrit} is also M . The arguments used for T_{crit} can be used in a symmetric way for T_{subcrit} to prove that $M - 1$ is reached on the upper boundary of F . This achieves the proof (the third item is obviously satisfied since otherwise a downwards strong flip could be done on the critical tiling). $\square$

Of course, there exists a symmetric proposition when M is even.

Corollary 1. *We keep the notations above. Assume that M' is even. For each vertex v of δF such that $h_1(v) = M'$, at least one of the following alternatives holds:*

- *The vertex v is on the boundary of an R -tile of T_{crit} with two vertical sides of height $M' - 1$.*
- *The vertex v is on the boundary of the horizontal side of a critical block of T_{crit} . Moreover, if v is on the upper boundary of F , then there exists a stair $(B_1, B_2, \dots, B_p)$ of critical blocks such that the block B_1 meets the lower boundary of F and v is on the top side of B_p .*

Proof. Obvious, since any other situation contradicts the above proposition. $\square$

4.3. Algorithm of Tiling

We present the algorithm below:

Initialization: Construct f_1 and f_2 on δF (if a contradiction holds, then stop: there is no tiling).

Main loop: Consider the set $S_{M'}$ of vertices such that $h_1(v) = M'$. We assume that M' is even (the case when M' is odd is treated in a symmetric way).

If $S_{M'}$ is included in the lower boundary of F , then place the R -tile t in the neighborhood of a vertex v of $S_{M'}$ in such a way that the maximal height reached in the boundary of t is M' (at most one possibility with no contradiction for $f_{1,T}$ and $f_{2,T}$). Otherwise, take a vertex v of $S_{M'}$ in the upper boundary of F . If there exists a stair $(B_1, B_2, \dots, B_p)$ of blocks such that the block B_1 meets the lower boundary of F , v is on the top side of B_p and the stair creates no contradiction for $f_{1,T}$ and $f_{2,T}$, then place the tiles on the tiling the stair of blocks. If such a stair does not exist, then place the R -tile t in the neighborhood of the vertex v in such a way that the maximal height reached in the boundary of t is M' (at most one possibility with no contradiction for $f_{1,T}$ and $f_{2,T}$).

Remove the area tiled from F . Update M' and $S_{M'}$. Repeat the main loop until the remaining figure is empty or a contradiction appears about tiling functions.

Corollary 2. *Assume that F can be tiled. The above algorithm gives a critical tiling. In particular, there exists a unique tiling T_{crit} .*

Proof. From Proposition 4, if $S_{M'}$ is included in the lower boundary of F , then $M = M'$, which guarantees the correctness of the placement of the tile in this alternative. On the other hand, Corollary 1 guarantees the correctness of the placement of tiles when $S_{M'}$ contains a vertex of the upper boundary of F . $\square$

From the above corollary, one can state the theorems below.

Theorem 1 (Connectivity). *Let m, m', n and n' be fixed integers, each of them being larger than 2. Let (T, T') be a pair of tilings of a polygon F by translated copies of $m \times n$ and $m' \times n'$ rectangles. One can pass from T to T' by a sequence of local flips.*

Proof. Obvious. $\square$

Theorem 2 (Quadratic Algorithm). *Let m, m', n and n' be fixed integers, each of them being larger than 2. There exists a quadratic-time tiling algorithm (in the area $A(F)$ of F) which, given a polygon F , either produces a tiling of F by translated copies of $m \times n$ and $m' \times n'$ rectangles, or claims that there is no tiling (if this is the case).*

Proof. We have seen in Corollary 2 that the algorithm given above is a tiling algorithm. We just have to study its complexity. Each passage through the loop permits us to place at least one tile, thus there are at most $A(F)$ passages through the loop.

Each passage through the loop costs $O(A(F))$ time units. The key point is that the search of a stair of blocks crossing the polygon is similar to the depth first search in a tree which has $O(A(F))$ vertices.

The tree is constructed as follows: the root corresponds to the vertex v whose neighborhood is tiled by the first block, each other vertex is labeled, this label corresponding to the position of a block. The vertices of level 1 correspond to different potential positions for a block which tiles the neighborhood of v . For $i > 1$, the vertices of level i correspond to different potential positions for the i th block. Two vertices of successive levels are joined by an edge if they correspond to blocks with a common part of the boundary. Hence, the sons of a vertex w of the tree are labeled by the possible blocks which cover the block which is the label of w .

Clearly, each stair of blocks which has to be tested is encoded by a path from a root to a leaf. A depth first search allows us to test all these stairs of blocks successively.

This tree has $O(A(F))$ vertices, since each block can cover a fixed bounded number of blocks. Thus, in the tree, the number of vertices having a fixed label is bounded. Thus the time complexity is at most quadratic in the area of F . $\square$

5. Degenerate Cases

These cases are those when a tile has a side of unit length. They are called *degenerate*, since at least one of the tiling groups is degenerate, i.e. reduced to a cyclic group.

5.1. Case: $n = 1, m \geq 2$ and $n' \geq 2$

With this hypothesis, G_2 is degenerate but, fortunately, G_1 is not degenerate. Thus we can apply Proposition 4, with the following modifications:

First modification: there is no possible stair of critical horizontal blocks, since, in this case, a horizontal block (which is an $mm' \times n'$ rectangle) admits two tilings which differ by a strong flip: if a critical block tiled by R' -tiles would appear, then a decreasing strong flip could be done. Thus, when M' is even, we necessarily have $M = M'$.

Second modification: if a vertical critical block B' covers another vertical critical block B , then B' is deduced from B by a horizontal translation of length m . This yields that each stair of vertical critical blocks is formed with at most $m' - 1$ blocks (since we have to avoid subsequences of blocks forming an $mm' \times nn'$ rectangle).

How can the stairs of blocks crossing the polygon be detected? Since G_2 is degenerate, the method of the main case cannot be applied. Fortunately, in this case, stairs of blocks are quite simple. We use this simplicity as follows:

An *isthmus* I of F is a $km \times 1$ rectangle (where k denotes an integer such that $0 < k < m'$) included in F , such that the left and right sides of this rectangle are respectively included in the left and right boundaries of F . For each tiling T of F , there exists a pair (μ, μ') of integers such that I contains μm cells covered by R -tiles and $\mu' m'$ cells covered by R' -tiles. We have $\mu m + \mu' m' = km$, which forces $\mu = k$ and $\mu' = 0$, since $k < m'$. This proves that, in each tiling of F , each isthmus of F is tiled with R -tiles. The above remarks yield the two modifications below on the tiling algorithm of the main case:

First modification: when M' is even, directly place an R -tile, since there cannot exist a vertical critical block.

Second modification: when M' is odd and $S_{M'}$ contains a vertex of the left boundary of F and a vertex of the right boundary of F , take a vertex v of the right boundary. If v is a corner of an isthmus of F , then tile the isthmus with R -tiles. Otherwise, place the unique R' -tile whose highest vertex is v .

With these modifications, each passage through the loop costs a bounded number of time units, which improves the complexity: in this case we have a linear algorithm. We have the following theorem, which contains the result of [5] as a special case when $m' = 1$.

Theorem 3 (Linear Algorithm). *Let m, m' and n' be fixed positive integers, with $m \geq 2$ and $n' \geq 2$. There exists a linear-time tiling algorithm (in the area $A(F)$ of F) which, given a polygon F , either produces a tiling of F with translated copies of $m \times 1$ and $m' \times n'$ rectangles, or claims that there is no tiling (if this is the case).*

We can also see that a pushing down (or pushing up) weak flip can be decomposed into a sequence of two strong flips. Thus, the space of tilings is connected if we only use strong flips, pushing left and pushing up weak flips. In the special case when, moreover, $n = 1$, each weak flip can be decomposed into a sequence of two strong flips. Thus the space of tilings is connected by strong flips.

5.2. Case: $n = n' = 1$

In this case the problem is in fact one-dimensional. It can be reduced to a set of instances of the following problem (called the *small problem*): given an integer k , find a tiling of the $k \times 1$ rectangle. This small problem can be treated by the following algorithm: if k is a multiple of m , then tile the whole rectangle with R -tiles. Otherwise, place an R' -tile on the left part of the rectangle. Update the rectangle which remains to be tiled. Repeat these instructions until either the rectangle obtained is empty or an impossibility appears. Thus the small problem can be solved in $O(k)$ time units, and the original problem can be solved in linear time in the area of the polygon.

Flip connectivity is easy: from any tiling of the rectangle, first make the necessary weak flips in order to place the R' -tiles on the left part and the R -tiles on the right part. Afterwards, while there are at least m R' -tiles, make a strong flip on the rectangle tiled with the m rightmost R' -tiles. The tiling obtained after the above flips is always the tiling given by the previous algorithm, which gives the connectivity.

5.3. Case: $m = n = 1$

We now assume that a rectangle is reduced to a cell. In this case we have a trivial algorithm of tiling (which only uses R -tiles) and, obviously, connectivity with strong flips.

All the cases have been treated, up to symmetry.

6. Extensions to Rectangles with Real Dimensions

The arguments given in this paper seem to be extended to the case when dimensions are real numbers. It is rather difficult to speak about algorithms and complexity in this framework, thus our goal in this section is only to find a method to construct a tiling (without speaking about complexity), and a theorem of connectivity.

The tools used in the discrete case can be adjusted. We use the continuous groups $G'_1 = \mathbb{R}/m\mathbb{Z} * \mathbb{R}/n'\mathbb{Z}$ and $G'_2 = \mathbb{R}/m'\mathbb{Z} * \mathbb{R}/n\mathbb{Z}$ instead of G_1 and G_2 above. Thus functions $f_{1,T}$, $f_{2,T}$, $h_{1,T}$, $h_{2,T}$ are directly defined for each point on the boundary of any tile of T .

Given a pair (x, x') of real positive numbers, we say that x and x' are *rationally dependent* if the quotient x/x' is an element of the set $\mathbb{Q}$ of rational numbers. In this case the unique pair (a, a') of relatively prime positive integers such that $x/x' = a/a'$ is called the *integer reduction* of (x, x') .

6.1. First Case: Vertical and Horizontal Dimensions Are Rationally Dependent

In this case we assume that, with the notations used above, the pairs (m, m') and (n, n') are both rationally independent. The treatment is very close to the discrete case.

Let (a, a') and (b, b') respectively denote the integer reductions of (m, m') and (n, n') , and let α and β be the real numbers defined by $\alpha = m/a = m'/a'$ and $\beta = n/b = n'/b'$. In this case we define a brick as a rectangle of width α and height β . Notice that each R -tile can be tiled with ab translated copies of R_u rectangles, and each R' -tile can be tiled with $a'b'$ translated copies of R_u . Thus we can do a simplification preprocessing in a similar way as for the discrete case.

Once this simplification is done, it suffices to change the scale (in order that the bricks tiling the polygon become unit cells) to be reduced to the discrete case. Thus, we have the same results (construction, flip connectivity) as those of the discrete case.

6.2. Second Case: Vertical and Horizontal Dimensions Are Rationally Independent

In this case we assume that the pairs (m, m') and (n, n') are both rationally independent.

Proposition 5. *Let T be a tiling of F and let M be the highest value reached by $h_{1,T}$. There exists a point v of δF such that $h_{1,T}(v) = M$.*

Proof. Let v be a point of V_T such that $h_{1,T}(v) = M$. If v is on the boundary, then we are done. Otherwise, up to symmetry, one can assume without loss of generality that v_0 is on a vertical edge $[v', v'']$ common to two R' -tiles. In this way a sequence $(R'_{0,1}, R'_{0,2})$ of joined tiles is created. In this situation there necessarily exists a vertex v_1 on the left side of $R'_{0,1}$ or on the right side of $R'_{0,2}$ such that $h_{1,T}(v_1) = M$, since the distance between two vertical sides of tiles in the sequence is not in $m\mathbb{Z}$. If v_1 is on the boundary, then we are done. Otherwise, the previous sequence can be extended in

a sequence $(R'_{1,1}, R'_{1,2}, R'_{1,3})$ of joined rectangles. We can repeat this process until the boundary is reached (this arises since the polygon is bounded). $\square$

Theorem 4. *Let F be a polygon. F has at most one tiling.*

Proof. By induction on the area of F . If this area is at most $\min(mn, m'n')$, then the result is obvious. This gives the initialization of the induction.

Otherwise, take a point v of the boundary of maximal height. The above lemma enforces the tile t on the neighborhood of v . Once t has been placed, the remaining figure is a polygon (or a set of polygons if a disconnection arises), and the area has decreased by at least $\min(mn, m'n')$. Thus, by the induction hypothesis, the remaining figure has at most one tiling, which gives the result. $\square$

6.3. *Third Case: Vertical Dimensions Are Rationally Independent and Horizontal Dimensions Are Rationally Dependent*

In this case we assume that (m, m') is rationally dependent and (n, n') is rationally independent. Let (a, a') denote the integer reduction of (m, m') , and let α be defined by $\alpha = m/a = m'/a'$. By a change of horizontal scale (taking α as the new unit), one can assume without loss of generality that m and m' are relatively prime integers.

This case is very similar to the first case, thus we do not give details. We have to work only with weak vertical flips. We also have to relax the definitions of a special path (allowing k_i to be arbitrarily large for i even) and a stair of blocks (removing the condition about the $mm' \times nn'$ rectangles). Once these changes are done, the method of the first case can be applied with no new difficulty arising. We get an algorithm of tiling, and the flip connectivity, using only weak vertical flips.

7. Extensions

A motivating direction of research is a better knowledge of the structure of the space of tilings, i.e. the graph whose vertices are tilings of a polygon F and two tilings are linked by an edge if and only if they differ by a single flip. Some very strong structural results (lattice structures, effective computations of geodesics between fixed tilings) have previously been found in the special case when each tile has a side of unit width [14], [15]. Can we extend these results to the main case?

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