Relaxation, New Combinatorial and Polynomial Algorithms for the Linear Feasibility Problem

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Abstract. We consider the homogenized linear feasibility problem, to find an x on the unit sphere satisfying n linear inequalities $a_i^T x \geq 0$. To solve this problem we consider the centers of the inspheres of spherical simplices, whose facets are determined by a subset of the constraints. As a result we find a new combinatorial algorithm for the linear feasibility problem. If we allow rescaling, this algorithm becomes polynomial. We point out that the algorithm also solves the more general convex feasibility problem. Moreover, numerical experiments show that the algorithm could be of practical interest.

1. Introduction

A central problem in optimization is the *linear feasibility problem* F. An instance A of F is to find for n vectors $a_i \in \mathbb{Z}^d$, $i = 1, \dots, n$, and n integers b_i , $i = 1, \dots, n$,

$$x \in \mathbb{R}^d \quad \text{such that} \quad a_i^T x \geq b_i, \quad i = 1, \dots, n, \quad (1)$$

or to show that such an x does not exist.

The *size* L of an instance A is its encoding length, the amount of space to write down the problem (see the books [5], [9] and [12], which provide the background for this paper).

For an instance A of F the set P_A of *feasible points*

$$P_A = \{x \in \mathbb{R}^d \mid a_i^T x \geq b_i, \ i = 1, \dots, n\} \quad (2)$$

is a polyhedron.

The main application of F is the *linear optimization problem* to minimize a linear function over P_A . It is well known how to solve an instance of the linear optimization problem by solving one or more instances of F. The linear feasibility problem is a

particular case of the convex feasibility problem. Here we want to find a point in a convex set K . Clearly the problem depends on the representation of K . A very general way of representing K is by a separation oracle [5]. For every $x \in \mathbb{R}^d$ this oracle either confirms that $x \in K$ or gives a hyperplane separating x from K . A survey on algorithms for the convex feasibility problem is [2].

There are numerous algorithms for F. Well known algorithms are *phase I* of the *simplex algorithm*, *Khachiyan's algorithm* [7] and *Karmarkar's algorithm* [6]; for a fairly recent survey of Karmarkar's algorithm see, e.g. [10].

A class of algorithms for which the worst case behavior is not too bad are the *polynomial algorithms*. We say that an algorithm is (time)-polynomial if there is a polynomial p such that, for all instances with size less than or equal to L , the algorithm gives a solution in a time at most $p(L)$. For details see again [5], [9] and [12]. Khachiyan's and Karmarkar's algorithms are polynomial while the simplex algorithm is not.

A different class of algorithms are *combinatorial algorithms*. We say that an algorithm for the linear feasibility problem or LP is combinatorial if, for each instance, there is a finite set of points, where each point is determined by a subset of the constraints with cardinality at most the dimension of the problem, such that all iterates are in this set. The standard example is the simplex algorithm, where all iterates are basic solutions which are determined by a subset whose cardinality is exactly the dimension of the instance. Here we present polynomial as well as combinatorial algorithms.

For our purposes Khachiyan's algorithm and an algorithm given by Agmon [1] and Motzkin and Schoenberg [8] are of particular interest as they are *relaxation algorithms* for F which may be described by the following simple prototype:

Algorithm 1.1.

1. Choose an arbitrary x^0 .
2. Choose a "simple" convex set C_k and $x^k \in C_k$ depending on x^{k-1} and other data obtained in the course of the algorithm, such that $P_A \subset C_k$ and x^k are in some way "better" than C_{k-1} , x^{k-1} . If $x^k \in P_A$ terminate.
3. Set $k := k + 1$ and goto step 2

In the case of Khachiyan's algorithm, C_k is an ellipsoid and x^k is its center. The next ellipsoid is constructed in the following way. Choose a constraint i such that $a_i^T x^k < b_i$. Then the next ellipsoid C_{k+1} is the smallest ellipsoid circumscribed to

$$C_k \cap \{x \mid a_i^T x \geq a_i^T x^k\}.$$

For an instance A of F and $x \notin P_A$ we say that $i \in \{1, \dots, n\}$ is a *most violated constraint* if $a_i^T x - b_i = \min\{a_j^T x - b_j \mid j = 1, \dots, n\}$. In the Agmon–Motzkin–Schoenberg algorithm a most violated constraint, $a_i^T x \geq b_i$, say, for x^k is chosen and x^{k+1} is the projection of x^k on the half-space $\{a_i^T x \geq b_i\}$. This algorithm is a very special case of an algorithm in [3].

Furthermore, the simplex algorithm for a problem in standard form may be viewed as a relaxation algorithm for the dual linear feasibility problem, where each iterate x^k is determined by the requirement that it satisfies d of the inequalities in (1) with equality.

Further, we remark that Khachiyan's algorithm solves the problem of constructing a point in a convex set given by a separation oracle. We return to this topic after the presentation of our algorithms.

We proceed as follows: In Section 2 we present a relaxation algorithm for the homogenized form of F and determine its properties. In Section 3 we use additional transformations of the problem to construct a polynomial algorithm for F . Moreover, we shortly discuss its relation to the convex feasibility problem. While in the previous sections we have concentrated on the geometric aspects of the algorithms, in Section 4 we discuss the underlying linear algebra and present the results of some numerical experiments.

2. Algorithms

In a certain way we want to combine the algorithm of Agmon–Motzkin–Schoenberg with Khachiyan's algorithm. This means that we want to use a circumscribed body of P_A and a point, which we may consider as a projection on this body. Furthermore, we want this body as closely related to P_A as possible. A rather obvious choice is a simplex bounded by $d + 1$ of the hyperplanes $\{a_i^T x = b_i\}$ and the center of its inball, as this point may be considered a simultaneous projection on all half-spaces. For several reasons this does not work very well in Euclidean space and thus we first use the well known process of homogenization to transform the problem into a spherical problem.

To proceed we need some more notation. We write $B(c, r) = \{x \in \mathbb{R}^d \mid \|x - c\| \leq r\}$ for the ball with center c and radius r , $S(c, r) = \{x \in \mathbb{R}^d \mid \|c - x\| = r\}$ for the sphere bounding $B(c, r)$ and $S^{d-1} = S(0, 1)$ for the unit sphere in $\mathbb{R}^d$. Then we define the homogenization

$$\Phi : \mathbb{R}^d \rightarrow S^d, \quad \Phi(x_1, \dots, x_d) = (x_1, \dots, x_d, 1) / \|(x_1, \dots, x_d, 1)\|. \quad (3)$$

For $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ we denote by $\bar{x} = (x_1, \dots, x_d)$ its projection on $\mathbb{R}^d$. For an instance A of F we have for the set of feasible points P_A ,

$$\Phi(P_A) = \{x \in S^d \mid a_i^T \bar{x} - b_i x_{d+1} \geq 0, \ i = 1, \dots, n, \ x_{d+1} > 0\}.$$

Thus we obtain an algorithm for the linear feasibility problem F if we have an algorithm for the *spherical feasibility problem* F' , where an instance A of F' is to find

$$x \in S^d \quad \text{such that} \quad a_i^T x \geq 0, \quad i = 1, \dots, n, \quad (4)$$

or to show that such an x does not exist. For an instance A of F' we denote again by P_A the set of its feasible points.

To study F' it is helpful to use the notion of *violation*. For an instance A of F' and $x \in S^d$ we define the violation $v(x)$ by

$$v(x) = \max\{0, \max\{-a_i^T x \mid i = 1, \dots, n\}\}.$$

Thus a point is feasible for an instance A of F if and only if its violation is zero.

For our algorithms it is somehow more appropriate to take a polar point of view, i.e. we work with the normals a_i rather than with the half-spheres $a_i^T x \geq 0$. Moreover, we

construct a sequence of points which show that the instance is close to being infeasible. To be more precise we denote the origin of $\mathbb{R}^{d+1}$ by O , the *convex hull*, respectively *affine hull*, of a set $M \subset \mathbb{R}^{d+1}$ by $\text{conv } M$, respectively $\text{aff } M$, and define

Definition 2.1. Let $Q = \{x^i \in \mathbb{R}^{d+1} \mid i = 1, \dots, k\}$ be a set of points. We say that Q is *positively spanning* if Q is affinely independent and O is contained in the relative interior of $\text{conv } Q$.

If $k = d + 1$, then Q is a *positive basis*. Here we also consider the case that $\dim(\text{aff } Q) < d + 1$.

Definition 2.2. Let $Q = \{x^i \in \mathbb{R}^{d+1} \mid i = 1, \dots, k\}$ be a set of points. We say that Q is *nearly positively spanning* if Q is affinely independent and the orthogonal projection O' of O on $\text{aff } Q$ is contained in the relative interior of $\text{conv } Q$. The distance of O' to the origin is called the *deficiency* of Q and is denoted by $\text{def } Q$.

Thus a nearly positively spanning set is positively spanning if and only if its deficiency is zero. Moreover, for a nearly positively spanning set the deficiency is equal to the distance of the origin to its convex hull.

The evident significance of positively spanning sets for F' is given by

Proposition 2.3. Let an instance A of F' be given and let $Q \subset \{a_1, \dots, a_n\}$ be a positively spanning set. Then every $x \in P_A$ satisfies $a_i^T x = 0$ for all $a_i \in Q$.

If the cardinality of Q is $d + 2$, then A is infeasible. In the other case P_A is contained in the plane $\{a_i^T x = 0 \mid a_i \in Q\}$ and thus the dimension of the instance is reduced. In this case we may change the problem slightly in a way that Q is transformed into a nearly positively spanning set with positive deficiency without changing the feasibility or infeasibility of the instance. This is quite analogous to cases of degeneration in Khachiyan's algorithm and is not discussed further (but see the remarks after Lemma 3.7).

If the points of Q are contained in S^d , then nearly positively spanning sets can be nicely characterized.

Definition 2.4. Let Q be a set of affinely independent points. The sphere $S(C, R)$ is said to be *touching* for Q if $C \in \text{aff } Q$ and $Q \subset S(C, R)$.

$S(C, R)$ is a *circumsphere* of Q if $Q \subset B(C, R)$ and there is no $S(C', R')$ with this property and $R' < R$.

Clearly, every set of affinely independent points has a unique touching sphere. Moreover, we observe that it is easy to compute a touching sphere for a given Q . The relation of touching sphere and circumsphere is given by the observation that a touching sphere $S(C, R)$ of Q is a circumsphere if and only if $C \in \text{conv } Q$.

The relation of touching spheres and nearly positively spanning sets is given by

Proposition 2.5. Let $Q \subset S^d$ be affinely independent, and let $S(C, R)$ be its touching sphere. Then Q is nearly positively spanning if and only if C is contained in the relative interior of $\text{conv } Q$. In this case $\text{def } Q = \sqrt{1 - R^2}$.

By means of nearly positively spanning sets we may describe an algorithm for F' :

Algorithm 2.6.

Input: A set $\{a_1, \dots, a_n \mid a_i \in S^d\}$.

Output: $x \in S^d$ such that $a_i^T x \geq 0$ for $i = 1, \dots, n$ or a subset $\{a_{i_1}, \dots, a_{i_k}\}$, $k \geq 2$, such that every feasible x must satisfy $a_{i_j}^T x = 0$ for $j = 1, \dots, k$.

begin

Let $j \in \{1, \dots, n\}$ be arbitrary;

$x^1 := a_j$;

$k := 1$;

$Q_1 := \{a_j\}$;

$finished := FALSE$;

while ($finished = FALSE$) **do**

begin

if $x^k / \|x^k\|$ is feasible **then**

begin

$finished := TRUE$;

end

else

begin

 Let m be the index of a most violated constraint for $x^k / \|x^k\|$;

$y := x^k$;

if $Q_k \cup \{a_m\}$ is positively spanning **then**

begin

$finished := TRUE$;

end

else

begin

 Let C be the center of the touching sphere of $Q_k \cup \{a_m\}$;

while $C \notin \text{conv}(Q_k \cup \{a_m\})$ **do**

begin

 Let y be the point where the line segment $\overline{yC}$ intersects the relative boundary of $\text{conv}(Q_k \cup \{a_m\})$;

 Let F be a facet of $\text{conv}(Q_k \cup \{a_m\})$, which contains y ;

 Let a_j be the point of Q_k , which is not contained in F ;

$Q_k := Q_k \setminus \{a_j\}$;

 Let C be the center of the touching sphere of $Q_k \cup \{a_m\}$;

end;

$k := k + 1$;

$x^k := C$;

$Q_k := Q_k \cup \{a_m\}$;

end;

end;

end;

end.

The iterates x^k of the algorithm are the centers of touching spheres of the subsets Q_k of $\{a_1, \dots, a_n\}$. Clearly there are only finitely many such subsets and each subset has cardinality less than or equal to d . Thus the algorithm is a combinatorial algorithm as described in the Introduction. We have that the deficiency of the Q_k is strictly decreasing and the sets Q_k cannot be repeated. Thus we have

Lemma 2.7. *Algorithm 2.6 solves F' for feasible as well as infeasible problems in finitely many steps.*

As is easy to see and is discussed in more detail in the next section the iterates x^k are the incenters of the simplices $\{x \in S^d \mid a_i^T x \geq 0, a_i \in Q_k\}$. As mentioned before there is a Euclidean analogue of this algorithm. While we have no quantitative information on this algorithm, it shows the close relation of our concepts to the dual simplex algorithm. In the dual simplex algorithm we consider the set X of intersections of d hyperplanes given by the constraints. X is ordered by $c^T x, x \in X$, for a suitable $c \in \mathbb{R}^d$ and searched until a feasible point is found. In the Euclidean algorithm we consider the set X of incenters of simplices defined by the intersection of $d + 1$ half-spaces, given by the constraints. X is ordered by the inradii of the simplices and searched with the help of this ordering.

In contrast to the simplex algorithm and the Euclidean insphere algorithm mentioned above we may modify Algorithm 2.6 slightly to obtain quantitative information on its progress.

Algorithm 2.8. Initialize the point y in the outer while loop of the algorithm by choosing y as the point on the line $x^k a_m$, which is closest to the origin rather than x .

Lemma 2.9. *Algorithm 2.8 solves the feasibility problem F' and it holds that*

$$\text{def } Q_{k+1} \leq \sqrt{\frac{1 - v(x^k)^2}{1 + (\text{def } Q_k)^2 + 2 \text{def } Q_k v(x^k)}} \text{def } Q_k. \quad (5)$$

Proof. The y used to initialize the outer while loop of Algorithm 2.8 has the distance from the origin given by the right side of (5) and by construction the distance of x^{k+1} from the origin is certainly not greater than the distance of y from the origin. $\square$

While the estimate (5) is not linear, it is independent of the dimension and in particular of the size of the problem. The formula may be used to estimate the number of steps that the algorithm needs to reach a given level.

Lemma 2.10. *In Algorithm 2.8 it holds that $\text{def } Q_k \leq t^{-1}$ after at most t^2 steps.*

Proof. It is somewhat easier to consider the quantities $y_k = (\text{def } Q_k)^{-1}$ instead of $\text{def } Q_k$. We have from Lemma 2.9 and using $v(x^k) \geq 0$,

$$y_{k+1}^2 \geq y_k^2 + 1. \quad (6)$$

Together with $y_1 = 1$ this gives inductively $y_{k+1}^2 \geq k + 1$. $\square$

It is not known (and was not investigated) whether the sequence of iterates in Algorithms 2.6 and 2.8 can differ for the same problem. According to Lemma 2.9 Algorithm 2.8 takes large steps if the deficiency or the violation are large, in particular this is the case at the start of the algorithm.

We do not have information on the global behavior of the algorithm beyond Lemma 2.10. We may obtain some more insight into its working by analyzing one step of Algorithm 2.6. Using an appropriate system of coordinates we may assume that Q_k spans the affine plane $H = \{(x_1, 0, \varepsilon) \mid x_1 \in \mathbb{R}^l\}$ for some fixed $\varepsilon = \text{def } Q_k > 0$ and appropriate l . Then we have $x^k = (0, 0, \varepsilon)$. Further, we have that the constraint which is chosen in step 2 has the form $a_m = (x_1, x_2, \varepsilon + \eta)$ with $x_1 \in \mathbb{R}^l$, $x_2 \in \mathbb{R}^{d-l}$ and $\eta < -\varepsilon$. We find for the center C of the touching sphere of $Q_k \cup \{a_m\}$ in step 4 that

$$C = (0, 0, \varepsilon) + \frac{-\eta\varepsilon}{\|x_2\|^2 + \eta^2}(0, x_2, \eta).$$

From this we find $\|C\| \approx \|x^k\|$ if $|\eta| \ll \|x_2\|$. In this case the algorithm can only increase the cardinality of Q_k , but not significantly decrease $\text{def } Q_k$. Clearly the crucial point of the algorithm is whether we may expect $C \in \text{conv}(Q_k \cup \{a_m\})$. To see what happens, we denote by a'_m the orthogonal projection of a_m onto H . By construction we have that x^k is the orthogonal projection of C onto H . Finally, let $y \in \text{relbd}(\text{conv } Q_k)$ be the point that $x^k \in \overline{ya'_m}$. Now an examination of the triangle y, a'_m, a_m shows that $C \notin \text{conv}(Q_k \cup \{a_m\})$ if $\|y - x^k\| \ll \|a'_m - x^k\|$. Clearly this may happen and in a single step the algorithm may make very little progress. Fortunately this is not the complete story.

In the algorithm we gather information about a_i . If we define for $x \in S^d$,

$$M(x) = \{a \in S^d \mid a^T x \geq -v(x)\},$$

then we have for every iterate x^j that $a_i \in M(x^j)$, $i = 1, \dots, n$. Consequently all a_i satisfy for all k ,

$$a_i \in \bigcap_{j=1}^k M(x^j).$$

Unfortunately the sets $M(x^k)$ are not convex and this information is hard to use.

If we want to repeat the bad step from above, the set $M(x^k)$ tells us that either for the new ε', η' it holds that $|\varepsilon' + \eta'| \ll |\varepsilon + \eta|$ or the angle between a_m and $a_{m'}$ must be large. Thus we can hope that bad steps cannot be repeated too often. A precise analysis of the phenomena appears to be difficult.

3. A Polynomial Algorithm

In view of the last remark in the previous section it is natural to use periodically a rescaling to increase the deficiency in order to speed up the convergence. While it is simple to give a transformation of the sphere which takes polyhedra to polyhedra and increases the deficiency of a set Q_k , it appears difficult to work out the consequences of iterating such transformations.

Fortunately it turns out that, for $\text{def}(Q_k)$ sufficiently small, it is possible to find a transformation which increases the volume of the sets P_A . To work out our ideas it is necessary to recall some concepts of spherical convexity. A set $M \subset S^d$ is called convex if there exists a half-sphere which contains M and for every pair $x, y \in M$ there exists a great circle C such that $C \cap M$ is connected. The intersection of convex sets is itself convex. By this we may define, for every $M \subset S^d$ which is contained in a half-sphere, the *spherical convex hull* $\text{co } M$ as the intersection of all convex subsets of S^d which contain M . Here we have chosen the notation $\text{co } M$ to distinguish the spherical convex hull from the Euclidean convex hull used in the previous section. As every half-sphere is convex we have that the intersection of half-spheres is convex. We say that the intersection of finitely many half-spheres is a *(spherical) polyhedron*. An example of a spherical polyhedron is the set P_A of feasible points of an instance A of problem F' .

We denote for two points $x, y \in S^d$ the spherical distance by $\angle(x, y)$, i.e. $\cos(\angle(x, y)) = x^T y$. Next for $x \in S^d$ and $0 \leq \rho \leq \pi$,

$$K_\rho(x) = \{y \in S^d \mid \angle(x, y) \leq \rho\}$$

denotes the *spherical cap* with center x and radius ρ . For closed $M \subset S^d$ we may now define the spherical counterparts of inradius, circumradius and diameter. The *spherical inradius* $r'(M)$ and the *spherical incenter* $c(M)$ are the radius and center, respectively, of a largest spherical cap $K_{r'(M)}(c(M)) \subset M$. We remark that in contrast to the Euclidean case for convex M the spherical incenter is always well defined. The *spherical circumradius* $R'(M)$ and *spherical circumcenter* $C(M)$ are the radius and center, respectively, of the smallest spherical cap $K_{R'(M)}(C(M)) \supset M$. Here the notions R', r' were chosen to distinguish the spherical circumradius and inradius from their Euclidean counterparts. For $M \subset S^d$ we denote the boundary of $K_{R'(M)}(C(M))$ as the *circumsphere* of M . For any M contained in a half-sphere the *spherical diameter* $D(M)$ is given by

$$D(M) = \sup\{\angle(x, y) \mid x, y \in M\}.$$

For $M \subset S^d$, $a_1, a_2 \in S^d$ satisfying $a_i^T x \geq 0$ for all $x \in M$, $a_i^T x = 0$ for at least one $x \in M$ and $a_1 \neq a_2$ we denote by $\angle(a_1, a_2)$ a *width* of M . The *breadth* $d(M)$ is then defined as the minimum over all widths of M .

For $M \subset S^d$ the *polar set* M^* is defined by

$$M^* = \{x \in S^d \mid m^T x \geq 0 \text{ for all } m \in M\}.$$

In our context it is important to observe that the polar set of a spherical polyhedron is again a polyhedron and that inclusions are reversed, i.e. for $M \subset M'$ we have $M^* \supset M'^*$. For $M = \{x^1, \dots, x^k\}$, $x^1, \dots, x^k$ linearly independent, we have that M^* is a polyhedron with k facets, which we denote as a $(k-1)$ -cosimplex.

Further for $M \subset S^d$ convex we have the relations

$$r'(M) + R'(M^*) = \pi, \quad d(M) + D(M^*) = 2\pi.$$

In this terminology we have for an instance A of F' and set $Q = \{a_{j_1}, \dots, a_{j_l}\}$ from Algorithm 2.6 with $P_A \neq \emptyset$ that $M = \text{co } Q$ is an $(l-1)$ -simplex with $M \subset P_A^*$ and

$\cos(R'(M)) = \text{def } Q$. Correspondingly M^* is an $(l-1)$ -cosimplex with $P_A \subset M^*$ and $r'(P_A) \leq r'(M^*) = \pi - R'(M)$.

Moreover, for $M \subset S^d$ the quantities $R'(M)$ and $D(M)$ are not independent as the following theorem by Santaló [11] shows:

Theorem 3.1 (Santaló). *Let $M \subset S^d$ be contained in a half-sphere. Then:*

1. For $\cos R' \geq 1/\sqrt{d+1}$,

$$\cos 2R' \leq \cos D \leq \frac{(d+1) \cos^2 R' - 1}{d}.$$

2. For $0 \leq \cos R' \leq 1/\sqrt{d+1}$,

$$\begin{aligned} \cos 2R' \leq \cos D &\leq \frac{(d+1) \cos^2 R' - 1}{1 + (d+1) \cos^2 R'} \quad \text{for } d \text{ odd,} \\ \cos 2R' \leq \cos D &\leq \frac{(d+1) \cos^2 R' - 1}{(1 + (d+1) \cos^2 R')^{1/2} (1 + ((d+1)(d-2))/(d+2) \cos^2 R')^{1/2}} \\ &\quad \text{for } d \text{ even.} \end{aligned}$$

Using Santaló's theorem we may estimate the breadth $d(P_A)$ of P_A by computing $R'(Q)$. After a suitable transformation of coordinates we have for all $(x_1, \dots, x_{d+1})^T \in P_A$ that $|x_1| \leq \sin d(P_A)$, i.e. all points of P_A are close to the equator $\{(x_1, \dots, x_{d+1}) \in S^d \mid x_1 = 0\}$, and P_A can be enlarged by a transformation. If we know that in the beginning P_A is not too small, we must find a feasible x^k after a number of steps, for otherwise the transformed P_A would not fit on the sphere.

A suitable quantity to measure the size of P_A is the volume and the estimate of the size of P_A at the start of the algorithm is standard from the theory of linear programming. This leaves the determination of the hyperplane. While the proof of Santaló's theorem is constructive, the construction of the hyperplane needs the computation of about $\binom{d}{d/2}$ distances and is therefore not practicable for a polynomial algorithm. Thus we substitute Santaló's theorem by a weaker one, for which we can easily compute all quantities.

Definition 3.2. Let $a_1, \dots, a_{d+1} \in S^d$, such that $Q = \text{co}\{a_1, \dots, a_{d+1}\}$ is a spherical simplex. For each a_i let a'_i be the intersection of the circle through a_i and the circumcenter $C(Q)$ with $\text{co}\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{d+1}\}$. Then the *vertex diameter* $D_v(Q)$ is defined by

$$D_v(Q) = \max\{\angle(a_1, a'_1), \dots, \angle(a_{d+1}, a'_{d+1})\}.$$

Clearly the vertex diameter can be computed easily. While [11] contains no explicit estimate of the vertex diameter, the methods can be adopted. To be complete we present the changes in the proof. First we have for the vertex diameter of the regular simplex

Lemma 3.3 (Santaló). *Let $Q \subset S^d$ be a regular simplex with circumradius R' and vertex diameter D_v , then*

$$\cos D_v = \frac{(d+1) \cos^2 R' - 1}{(1 + (d-1)(d+1) \cos^2 R')^{1/2}}. \quad (7)$$

Proof. The formula is a simple combination of formulas (2.3) and (2.13) in [11]. $\square$

Lemma 3.4 (Santaló). *Let $Q = \text{co}\{a_1, \dots, a_{d+1}\} \subset S^d$ be a spherical simplex, such that all vertices are contained in its circumsphere. Let $C = \sum_{i=1}^{d+1} \mu_i a_i$ be the circumcenter of Q . Then for the circumradius R' of Q it holds that*

$$\cos R' = \left(\sum_{i=1}^{d+1} \mu_i \right)^{-1}, \quad \frac{1}{\cos^2 R'} \leq (d+1) \sum_{i=1}^{d+1} \mu_i^2.$$

Proof. We have

$$\sum_{i=1}^{d+1} \mu_i \cos R' = \sum_{i=1}^{d+1} \mu_i a_i^T C = C^T C = 1.$$

From the equation we obtain

$$\frac{1}{\cos^2 R'} = \sum_{i=1}^{d+1} \mu_i^2 + 2 \sum_{1 \leq i < j \leq d+1} \mu_i \mu_j.$$

The inequality is an immediate consequence of $2\mu_i \mu_j \leq \mu_i^2 + \mu_j^2$. $\square$

Lemma 3.5 (Santaló). *Among all spherical d -simplices on S^d with circumradius R' and $\cos R' < 1/\sqrt{d+1}$, which contain all vertices in their circumsphere, the regular simplex has the minimal vertex diameter.*

Proof. Assume that there is a simplex $Q \subset S^d$ with circumradius R' , circumcenter C and vertex diameter Δ_v , which has all vertices on its circumsphere and has a smaller vertex diameter than the regular simplex with circumradius R' . Let D_v be the vertex diameter of the regular simplex with circumradius R' . From (7) we have $\cos D_v < 0$. Let $a_1, \dots, a_{d+1}$ be the vertices of Q . We have

$$C = \sum_{i=1}^{d+1} \mu_i a_i \quad \text{with} \quad \mu_i > 0.$$

and

$$a'_1 = \sum_{i=2}^{d+1} \mu_i a_i / \left\| \sum_{i=2}^{d+1} \mu_i a_i \right\|.$$

From this we obtain

$$\begin{aligned}\mu_1 \cos R' &= \mu_1 a_1^T C \\ &= \mu_1^2 + \mu_1 \left(\sum_{i=2}^{d+1} \mu_i^2 + 2 \sum_{2 \leq i < j \leq d+1} \mu_i \mu_j a_i^T a_j \right)^{1/2} \cos \angle(a_1, a_1').\end{aligned}\quad (8)$$

By our assumption we have $\cos \angle(a_1, a_1') > \cos D_v$. Thus (8) yields

$$\mu_1 \cos R' - \mu_1^2 - \mu_1 \left(\sum_{i=2}^{d+1} \mu_i^2 + 2 \sum_{2 \leq i < j \leq d+1} \mu_i \mu_j a_i^T a_j \right)^{1/2} \cos D_v > 0. \quad (9)$$

For $a_2, \dots, a_{d+1}$ we obtain inequalities analogous to (9). Summing up all these inequalities, using the Cauchy–Schwarz inequality $x^T y \leq \|x\|^{1/2} \|y\|^{1/2}$ and taking $\|C\| = 1$ into account we obtain

$$\begin{aligned}0 &< \sum_{k=1}^{d+1} \mu_k \cos R' - \sum_{k=1}^{d+1} \mu_k^2 \\ &\quad - \sum_{k=1}^{d+1} \mu_k \left(\sum_{\substack{i=1 \\ i \neq k}}^{d+1} \mu_i^2 + 2 \sum_{\substack{1 \leq i < j \leq d+1 \\ i, j \neq k}} \mu_i \mu_j a_i^T a_j \right)^{1/2} \cos D_v \\ &< \sum_{k=1}^{d+1} \mu_k \cos R' - \sum_{k=1}^{d+1} \mu_k^2 - \left(\sum_{k=1}^{d+1} \mu_k^2 \right)^{1/2} \left(d \sum_{i=1}^{d+1} \mu_i^2 + (d-1) \left(1 - \sum_{i=1}^{d+1} \mu_i^2 \right) \right)^{1/2} \\ &\quad \times \cos D_v \\ &= \sum_{k=1}^{d+1} \mu_k \cos R' - \sum_{k=1}^{d+1} \mu_k^2 - \left(\sum_{k=1}^{d+1} \mu_k^2 \right)^{1/2} \left(\sum_{i=1}^{d+1} \mu_i^2 + (d-1) \right)^{1/2} \cos D_v\end{aligned}\quad (10)$$

Solving (10) for $\cos D_v$ and using the identity and inequality from Lemma 3.4 gives finally

$$\begin{aligned}\cos D_v &< \frac{1 - \sum_{k=1}^{d+1} \mu_k^2}{\left(\sum_{k=1}^{d+1} \mu_k^2 \right)^{1/2} \left(\sum_{i=1}^{d+1} \mu_i^2 + (d-1) \right)^{1/2}} \\ &\leq \frac{(d+1) \cos^2 R' - 1}{(1 + (d+1)(d-1) \cos^2 R')^{1/2}}.\end{aligned}$$

This is a contradiction to (7). $\square$

By Lemmas 3.3 and 3.5 we may determine with the help of a set Q two half-spheres, which contain P_A in their intersection and the angle φ between them is at most

$$\cos \varphi \geq \frac{1 - (d+1) \cos^2 R'(Q)}{(1 + (d-1)(d+1) \cos^2 R'(Q))^{1/2}}. \quad (11)$$

Equation (11) yields, by an easy calculation and further estimation,

$$\sin \varphi \leq (d+1) \cos R'. \quad (12)$$

Next we show that for φ sufficiently small we may find a transformation of the sphere, which maps spherical polyhedra into spherical polyhedra and increases the volume of P_A by a prescribed factor. Any spherical map

$$B : S^d \rightarrow S^d, \quad B(x) = B'x / \|B'x\|$$

with a nondegenerate linear map B' has the first property. The influence of B on the volume is most easily studied by use of parameterizations.

A parameterized d -surface in $\mathbb{R}^{d+1}$ is a differentiable injective map $\Phi : G \subset \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ such that for every $x \in G$ the Jacobi-matrix $(\partial\Phi/\partial x)$ is nondegenerate. For $M \subset \Phi(G)$ the volume of M is given by

$$V(M) = \int_{\Phi^{-1}(M)} \det \Phi \, dx,$$

where $\det \Phi$ denotes the volume of the d -dimensional parallelepiped spanned by $\partial\Phi/\partial x_1, \dots, \partial\Phi/\partial x_d$. For a differentiable map $\psi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ with appropriate properties, $\Psi \circ \Phi$ is a parameterized surface and consequently

$$V(\Psi(M)) = \int_{\Phi^{-1}(M)} \det(\Psi \circ \Phi) \, dx$$

is the volume of $\Psi(M)$. Thus $(\det(\Psi \circ \Phi) / \det \Phi)(\Phi^{-1}(x))$ is the local change of volume in $x \in \Phi(G)$ under the map $\Psi : \Phi(G) \rightarrow \mathbb{R}^{d+1}$.

Here we are interested in the case

$$\Phi : B^d \rightarrow \mathbb{R}^{d+1}, \quad \Phi(x_1, \dots, x_d) = \left(x_1, \dots, x_d, \sqrt{1 - \sum_{i=1}^d x_i^2} \right)^T, \quad (13)$$

$$\Psi_\alpha : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1},$$

$$\Psi_\alpha(x_1, \dots, x_{d+1}) = (\alpha x_1, x_2, \dots, x_{d+1})^T / \sqrt{1 + (\alpha^2 - 1)x_1^2}, \quad \alpha > 0. \quad (14)$$

The restriction of Ψ_α on S^d is the spherical transformation $(\alpha x_1, x_2, \dots, x_{d+1})^T / \|(\alpha x_1, x_2, \dots, x_{d+1})^T\|$. We have chosen Ψ_α rather than the other transformation, as its derivatives can be computed more easily. In particular we consider the point $\bar{x} = (\beta, 0, \dots, 0, \sqrt{1 - \beta^2})^T$ with $0 \leq \beta < 1$. Computation of the Jacobi matrices and of the determinants with the help of Gram determinants yields

$$\det \Phi(\bar{x}) = \frac{1}{(1 - \beta^2)^{1/2}}, \quad \det(\Psi_\alpha \circ \Phi)(\bar{x}) = \frac{\alpha}{(1 - \beta^2)^{1/2} (1 + \beta^2(\alpha^2 - 1))^{(d+1)/2}}.$$

Thus the transformation ψ_α gives for all $(x_1, \dots, x_{d+1})^T \in S^d$ with $x_1 = \beta$ the local change of volume

$$f(\beta) = \frac{\alpha}{(1 + \beta^2(\alpha^2 - 1))^{(d+1)/2}}.$$

Consequently we obtain for $M \subset S^d$ with $x_1 \leq \beta$ for all $(x_1, \dots, x_{d+1})^T \in M$ and $\alpha > 0$ the estimate

$$V(\Psi_\alpha(M)) \geq \frac{\alpha}{(1 + \beta^2(\alpha^2 - 1))^{(d+1)/2}} V(M). \quad (15)$$

We may summarize the previous considerations in

Lemma 3.6. *Let $M \subset S^d$ such that for all $(x_1, \dots, x_{d+1})^T \in M$ it holds that $x_1 \leq \beta$. Further, let $\Psi_\alpha: S^d \rightarrow S^d$, $\alpha > 0$, be given by (14). Then the volume of $\Psi_\alpha(M)$ can be estimated by (15).*

Evidently we may generalize the transformation Ψ_α to the case of a transformation orthogonal to an arbitrary equator $a^T x = 0$ instead of the equator $x_1 = 0$.

As we work with the homogenized problem, we have to study the influence of (3) on the volume. Using the same technique as before we find for $M \subset \mathbb{R}^d$ for the volume of the homogenized set

$$V(\Phi(M)) = \int_M \frac{1}{(\|x\|^2 + 1)^{(d+1)/2}} dx.$$

Thus we have

Lemma 3.7. *Let $M \subset \mathbb{R}^d$ with $\|x\| \leq R$ for all $x \in M$. Then $V(\Phi(M)) \geq (R + 1)^{-(d+1)} V(M)$.*

The following considerations concerning polynomial algorithms could be done in the context of oracle-polynomial time algorithms for convex bodies in [5]. The presentation becomes somewhat simpler if we follow the presentation in [9] for algorithms for the feasibility problem of linear systems.

According to the corollary to Lemma 8.7 in [9] we can solve F in polynomial time, if we can solve the following problem in polynomial time: For the set

$$M = \{x \in \mathbb{R}^d \mid a_i^T x > b_i, i = 1, \dots, n\} \quad (16)$$

of size L find a point in M or show that there exists no such point in time polynomial in L . Here size L of the system measures the amount of space needed to write down the inequalities. For a precise definition of the size see [5] or [9].

We have that either M is empty or the volume of M is not too small:

Lemma 3.8. *If the set M from (16) is nonempty, then*

$$V(M \cap \{x \mid \|x\| \leq d 2^L\}) \geq 2^{-(d+2)L}.$$

Proof. Lemma 8.14 of [9]. □

For the a_i, b_i from (16) let $a'_i = (a_i^T, -b_i)^T$. Then we have for the homogenization of M ,

$$\Phi(M) = \{x \in S^d \mid a'_i{}^T x > 0, i = 1, \dots, n, x_{d+1} > 0\}.$$

We may combine Lemmas 3.7 and 3.8 to obtain

Lemma 3.9. *Let M be the set from (16) with size L . Then*

$$V(\Phi(M)) \geq 2^{-3(d+2)L}$$

or $\Phi(M)$ is empty.

Now we have collected the parts to show that the following modification of Algorithm 2.8 is polynomial.

Algorithm 3.10.

Input: A linear feasibility problem (1), such that the set of feasible points is either empty or contains interior points.

Output: A feasible point x or the assertion that such a point does not exist.

begin

const $\beta_d = \sqrt{((4/3)^{2/(d+2)} - 1)/3}$, $max = 6(d + 2)L$;

Homogenize the problem and denote the vectors determining the constraints by $a_1, \dots, a_n$;

$p := 0$;

$feasible := TRUE$;

while $p \leq max$ **do**

begin

 Let $j \in \{1, \dots, n\}$ be arbitrary;

$k := 1$;

$x^1 := a_j$;

$Q_1 := \{a_j\}$;

$finished := FALSE$;

while $finished = FALSE$ **do** (* beginning of a stage *)

begin

if $x^k / \|x^k\|$ is feasible **then**

begin

$finished := TRUE$;

stop;

end

else

begin

 Let m be the index of a constraint, which is violated for $x^k / \|x^k\|$;

if $Q_k \cup \{a_m\}$ is positively spanning **then**

begin

$feasible := FALSE$;

$finished := TRUE$;

stop;

end

else

```

begin
  Let  $y$  be the point on  $\overline{x^k a_m}$  closest to the origin;
  Let  $C$  be the center of the touching sphere of  $Q_k \cup \{a_m\}$ ;
  while  $C \notin \text{conv}(Q_k \cup \{a_m\})$  do
    begin
      Let  $y$  be the point, where the line segment  $\overline{yC}$  intersects
      the relative boundary of  $\text{conv}(Q_k \cup \{a_m\})$ ;
      Let  $F$  be a facet of  $\text{conv}(Q_k \cup \{a_m\})$ , which contains  $y$ ;
      Let  $a_j$  be the point of  $Q_k$ , which is not contained in  $F$ ;
       $Q_k := Q_k \setminus \{a_j\}$ ;
    end;
     $k := k + 1$ ;
     $x^k := C$ ;
     $Q_k := Q_{k-1} \cup \{a_m\}$ ;
    if  $\text{def } Q_k < \beta_d / (d + 1)$  then
      begin
         $\text{finished} := \text{TRUE}$ ;
         $p := p + 1$ ;
        Transform;
      end;
    end;
  end;
end;
end;
end;

```

procedure *Transform*

```

begin
  Determine the vertex diameter of  $\text{co } Q_k$  and let  $a_i$  be the vertex which gives
  the vertex diameter;
  Apply the transformation  $\Psi_2$  with respect to the equator  $a_i^T x = 0$ .
end.

```

Theorem 3.11. *Algorithm 3.10 solves F in polynomial time.*

Proof. We denote by M_p the set of feasible points before the p th transformation. First we show the correctness of the algorithm. If $x^k / \|x^k\|$ is feasible, then the algorithm trivially gives the correct result. If we find that $Q_k \cup \{a_m\}$ is positively spanning, then $M \subset P_{Q_k}$. P_{Q_k} is a lower-dimensional sphere and thus $V(P_{Q_k}) = 0$ and we have by our assumption that the instance is infeasible (see Lemma 3.9).

Now we consider the state of the algorithm at the call of procedure *Transform*. We have $\cos R'(Q_k) \leq \beta_d / (d + 1)$. By (12) we find that all points of M_p are contained in the zone $\{x \in S^d \mid 0 \leq a_i^T x \leq \beta_d\}$ for the a_i giving the vertex diameter. From this we obtain by Lemma 3.6 that $V(M_{p+1}) / V(M_p) \geq \frac{3}{2}$. Finally we obtain by Lemma 3.9 that after max steps we have $V(M_{\max}) \geq 2 > V(S^d)$. Consequently, an existing feasible point must be found before this step. This proves the correctness of the algorithm.

To calculate the number of computations we only have to estimate the number of iterations in a stage. By a power series expansion we see easily that $\beta \geq c d^{-1/2}$ for suitable $c > 0$. Thus by Lemma 2.10 there are $O(d^3)$ iterations.

It remains to show that it is sufficient to do all calculations with rational numbers whose size is bounded by a polynomial in L . We must show that by some rounding process the number of steps is not increased too much. While the necessary steps are not very difficult the calculations tend to become rather longish if one wants to work out the precise number of steps.

Thus we only indicate that there are no particular difficulties and skip all the details. We work with unnormalized constraints. In this way the homogenization is just to replace the inequality $a_i^T x \geq b_i$ by $(a_i^T, b_i)x \geq 0$. We denote the inner while-loop as a stage of the algorithm. During a stage we essentially have to compute the centers C of touching spheres. We have $C = \sum_{a_j \in Q_k} \lambda_j a_j$ and $C^T a_i = R \|a_i\|$, $a_i \in Q_k$, for the radius R of the touching sphere. Thus we have to solve a sequence of linear equations, which is a polynomial operation in the size of the encoding length of the input. As (5) which controls the progress of the algorithm is independent of the encoding length, we may round the quantities $R \|a_i\|$ in a suitable way such that a stage still works and (5) remains essentially valid.

In each stage the a_i are changed by the procedure *Transform*. Calculating with unnormalized integer vectors a_i and $\alpha = 2$ the transformation becomes

$$a_i := a_i + a_j a_j^T a_i, \quad i = 1, \dots, n, \quad (17)$$

for a suitable j . In this way the encoding length may increase exponentially. Using a simple rounding process we obtain from a_j a new vector $\bar{a}_j$ such that for a suitable constant γ we have

$$\max_k (\bar{a}_{jk}) \leq \gamma d$$

and

$$\angle(a_j, \bar{a}_j) < 1/(2d).$$

We replace (17) by

$$a_i := a_i + \bar{a}_j \bar{a}_j^T a_i, \quad i = 1, \dots, n,$$

for a suitable j . If we replace the stopping condition $\text{def } Q_k < \beta_d/(d+1)$ by $\text{def } Q_k < \beta_d/(2d+2)$ we find that the algorithm remains correct and the encoding length of the a_i grows in a polynomial way. $\square$

Some remarks concerning the algorithm:

1. If we compare Algorithms 2.8 and 3.10 we find that we have replaced the condition in Algorithm 2.8 that the new a_m is given by a most violated constraint by the weaker condition that the new a_m in Algorithm 3.10 is given by any violated constraint. We can easily check that it is sufficient to choose any $a \neq 0$, such that $a^T x^k < 0$, but $a^T x \geq 0$ for all $x \in P_A$, i.e. if $\{a^T x = 0\}$ is a separating great sphere for x^k and P_A . Moreover, we made no use of the fact that P_A is a polyhedron, but used only that the volume of P_A cannot

be too small. Now a separating hyperplane for the unhomogenized problem immediately yields a separating great sphere for the homogenized problem. Thus the algorithm is essentially a polynomial algorithm for the following convex feasibility problem: Given is a convex set $K \in \mathbb{R}^d$ which is either empty or well bounded in the language of [5] and an oracle which for any $x \in \mathbb{R}^d$ confirms that $x \in K$ or gives a separating hyperplane. Under these assumptions find a point in K or show that K is empty.

2. While the previous remark pointed out that from a theoretical point of view it is sufficient to choose any separating great sphere, from a practical point of view this is certainly not the case. This is shown by the following example, which shows that the inner loop in Algorithm 3.10 may need many iterations:

In this example the sets Q_k always have three elements which are for simplicity unnormalized. Let

$$Q_k = \{(1, \varepsilon, 0, \dots, 0)^T, (-1, \varepsilon, 0, \dots, 0)^T, (0, -\delta_k, 1, 0, \dots, 0)^T\},$$

where ε and δ_k are positive numbers of the kind that δ_k is small with respect to 1, and ε is small with respect to δ_k . The points x^k are of the form $x^k = (0, 1, \eta_k, 0, \dots, 0)^T$. $\{(0, -\delta_{k+1}, 1, 0, \dots, 0)^T x = 0\}$ is a separating great sphere if $\delta_{k+1} > \eta_k$. The computation of x^{k+1} gives

$$x^{k+1} = (0, 1, \eta_{k+1}, 0, \dots, 0)^T, \quad \eta_{k+1} = \delta_{k+1} + \varepsilon \frac{\sqrt{1 + \delta_{k+1}^2}}{\sqrt{1 + \varepsilon^2}}. \quad (18)$$

Thus if δ_{k+1} is close to η_k we have that the distance $\angle(x^k, x^{k+1})$ is of order ε . Now the iteration may be repeated with a δ_{k+2} which is of the same order as δ_{k+1} . This behavior does not affect the polynomiality as ε is bounded below by a function depending on the dimension, but demonstrates that the convergence may be rather slow.

3. If we choose the most violated constraint, then this phenomenon cannot occur, as is shown by the discussion after Lemma 2.10.

4. While the choice of x^k as the circumcenter of the sets Q_k is fundamental for our work, this is certainly not the case for the construction of the transformation. Although it did secure the polynomiality of the algorithm, there are some apparent disadvantages. We have to work with three different quantities (circumradius, vertex diameter, volume) and these quantities are rather weakly related. Moreover, the volume does not behave very nicely for transformations of the sphere. The apparent choice for a transformation would be to transform after each iteration $\text{co } Q_k$ into a regular simplex of the same circumradius such that the circumcenter of $\text{co } Q_k$ is transformed to the circumcenter of the resulting regular simplex. Unfortunately we do not know of any quantity which behaves nicely under this transformation and thus do not even know whether the resulting algorithm is convergent.

5. Still there are probably other transformations which are easier to handle than the one sketched above and lead to polynomial algorithms. One approach might be the following: We want to construct in polynomial time a positively spanning set for infeasible problems. For our heuristic considerations we assume that this positively spanning set is actually a positive basis. As we want to construct a positive basis, we look at the violation. Considering the fact that for a regular simplex the quotient of the

circumradius and the inradius is d we find that $v(x^k) \geq 1/(d+1)$ if the points of the positive basis form a regular simplex. Now, let us assume that we have found a point x^k such that $v(x^k)$ is small compared with $1/(d+1)$. Let Q be an arbitrary positive basis spanned by the a_i . Then it is easy to see that there must be a face F of $\text{conv } Q$, such that F is nearly parallel to $\{(x^k)^T x = 0\}$ and has a small distance from the origin. We apply for suitable $\alpha > 1$ a transformation Ψ_α with respect to the equator $\{(x^k)^T x = 0\}$. Locally we increase the violation. Globally it may be seen that the distance to the origin of all faces of $\text{conv } Q$, which have a dimension less than or equal to the dimension of F , is increased. One may hope that the minimal rate of increase is given by a function, which depends only on the dimension. Thus one should obtain fast, i.e. polynomial, convergence.

We have not tried to work out the theoretical details of this algorithm. Thus we do not know whether it works and have no information on its theoretical quality. We may say that this algorithm would avoid the problems of our algorithm which we pointed out in the previous remark, as in our context it is certainly more natural to deal with distances than with volume. Moreover, the transformation with a pole x^k is more natural than the transformation with a pole a_i and it is easy to implement this algorithm. This was in fact done with a transformation taking place whenever $v(x^k) \leq 1/\sqrt{d}$. The surprisingly good results of this algorithm are presented in the next section.

Altogether we may say that solely as a consequence of the weak estimate in Lemma 2.10 it is possible to construct a sequence of transformations which transform the combinatorial Algorithm 2.8 into a polynomial one. This is a remarkable contrast to the combinatorial simplex algorithm, as for the simplex algorithm it is even unknown whether an optimal choice of pivot steps would lead to a polynomial algorithm (Hirsch conjecture, for an account see [13]).

Moreover, there is some hope that Algorithm 2.6 is polynomial. That would be of particular interest, as an algorithm which is combinatorial as well as polynomial might be a candidate for a strongly polynomial algorithm for F. It is still an open question whether such an algorithm exists. For details about this problem see [5].

4. Implementation of the Algorithms and Numerical Experiments

Up to now we have only studied the geometry of the algorithms and we still have the problem of finding a practical way to perform the necessary calculations. We do not want to go into details or find the optimal way to do this but just show that there are no particular problems and that the complexity of a single step is on average comparable with the simplex algorithm, i.e. $O(dn)$.

The only steps in Algorithm 2.6 where the computations are not completely obvious are where we compute the center C of the touching sphere and check whether it is in the convex hull of the a_i . We observe that by our construction Q_k is linearly independent and $Q_k \cup \{a_m\}$ is affinely independent. We compute a least squares representation

$$a_m = a + \sum_{a_i \in Q_k} \lambda_i a_i, \quad a \text{ orthogonal to all } a_i \in Q_k, \quad (19)$$

and distinguish two cases:

1. $a = 0$. Then $Q_k \cup \{a_m\}$ is linearly dependent, $C = 0$ and using (19) we easily find a representation of the origin as an affine combination of $Q_k \cup \{a_m\}$.
2. $a \neq 0$. We observe that C is characterized by the following three properties:
 - (a) $a_i^T C = a_j^T C > 0$ for all $a_i, a_j \in Q_k \cup \{a_m\}$.
 - (b) $C = \sum_{a_i \in Q_k \cup \{a_m\}} \mu_i a_i$ with $\mu_i \in \mathbb{R}$.
 - (c) $\sum \mu_i = 1$.

Thus if a $C' \neq 0$ has the first two properties, then C is just a multiple of C' .

We look at the points y constructed in the algorithm. We find inductively that we have

$$\begin{aligned} \angle(y, a_i) &= \angle(y, a_l) \leq \angle(y, a_m) \quad \text{for all } a_i, a_l \in Q_k, \\ y &= \sum_{a_i \in Q_k \cup \{a_m\}} \mu_i a_i, \quad \text{with all } \mu_i \geq 0. \end{aligned}$$

Thus using (19) we find that there is a C' of the form $C = y + \rho a$ for suitable $\rho > 0$. From this we determine C as an affine combination of $Q_k \cup \{a_m\}$.

The next a_i to leave Q_k is then apparently found by a simple line search. It remains to solve (19). To do this we consider the Q_k as matrices with columns a_i . We observe that consecutive matrices Q_k, Q_{k+1} only differ in that a column is added or taken away. If we solve (19) by computing a QR-factorization of Q_k , then it is well known that the QR-factorizations of consecutive Q_k, Q_{k+1} can be obtained by an updating process which takes only $O(d^2)$ computations, see [4]. Thus taking into account that the determination of a_m in step 2 takes $O(dn)$ computations and we have on average to add one column to Q_k and take one column away, we find that a step of Algorithm 2.6 takes $O(dn)$ computations on average.

Clearly our results above give no information whether the algorithms could be of practical use. An obvious way to test this would be to solve the test problems as provided by <http://www.netlib.org/lp>. These problems are sparse optimization problems, i.e. to solve them we have to deal with many issues which are rather unrelated to the basic algorithm, but will strongly influence the results. Among these are solving the optimization problem by an algorithm for feasibility, dealing with equality constraints, dealing with the sparsity. Thus we have taken a much more simple approach. We generated three different sets of problems. The most basic way was done by generating in a random way n vectors of unit length a_i and n negative constants b_i (Ex1 in the following tables). Thus we obtain a system of inequalities $a_i^T x \geq b_i$ which has the origin in the interior of the feasible region. In the next set of problems we chose $b_i = 0$ for $i = 1, \dots, d+1$ and $a_{d+1} = -\sum_{i=1}^d a_i / \|\sum_{i=1}^d a_i\|$. Thus in this case we obtained with probability 1 problems which have exactly one feasible point (Ex2). The last set differed from the second only in that we chose $b_{d+1} = 10$ to construct infeasible problems (Ex3). In order to use the origin as a starting point all systems were translated by a random vector.

If we look at the problems then we see that after choosing the correct set of indices the second problem is equivalent to the solution of a system of linear equations. If we use the Gauss algorithm, and allow it to transform the complete problem in every step, then we see that a solution would involve d steps, each with $O(nd)$ operations. In particular, the number of steps would be independent of the number of inequalities. Certainly we

would consider such an algorithm as excellent from a practical point of view and we may ask how close we can get.

We used three different algorithms for the solution of the problems. First we used the following variant of the simplex algorithm to have a basis of comparison. We consider the problem as the dual of a problem in standard form. We perform d pivot steps to generate a basis. This corresponds to asking for equality in the inequalities corresponding to the basic columns. Then we choose a positive constant right side and use the simplex algorithm to solve the generated problem. If the algorithm stops with a finite solution, this shows the existence of a feasible point in the dual problem in the intersection of the constraints corresponding to the basic columns. If the algorithm stops with unboundedness this shows the infeasibility of the dual problem. This algorithm is denoted Alg 1 in the following tables.

The next algorithm was a straightforward implementation of Algorithm 2.6 (Alg 2). The last experimental algorithm (Alg 3) used an additional rescaling in Algorithm 2.6. Here it does not make much sense to use the polynomial Algorithm 3.10 as the number of steps before the first transformation takes place is not much smaller than the total number of steps for the complete solution by Algorithm 2.6. Instead we used the following procedure. We have by (5) that every step of the algorithm makes relative progress of at least $\sqrt{1 - v(x^k)^2}$. Thus when $v(x^k)$ becomes less than $1/\sqrt{d}$ then we do the following. Let $v(x^k)$ be determined by the constraint a_r , i.e. $v(x^k) = -a_r^T x^k / \|x^k\|$. We set $a'_i = (I + \lambda x^k x^{kT})a_i$, $i = 1, \dots, n$, and renormalize the a'_i to unit length. Here λ is chosen in a way such that $a'_r x^k / \|x^k\| = -\sqrt{2/d}$. We observe that the matrix Q_k is transformed by the formula $Q_k = \mu(Q_k + w x^{kT})$ for a scalar μ and a vector w which are easily determined. For such a transformation it is possible to update the QR-factorization in $O(d^2)$ computations, see [4]. Thus the transformation takes a total of $O(nd)$ computations.

To test the dependence on the dimension we chose the dimensions $d = 10, 20, 40, 80, 160, 320$ and 640 and $8d$ inequalities for each dimension. To test the dependence on the number of inequalities we chose $d = 100$ and $n = 400, 800, 1600, 3200, 6400$. For each case five examples were generated and solved. The average numbers of steps are shown in Tables 1 and 2.

In the tables the second column for the third algorithm gives the number of rescalings. Moreover, the numbers for the simplex algorithm do not give the d steps necessary to obtain the starting tableau. We see that all algorithms show little dependence on the number of constraints, in particular for Alg 3 this dependence appears to be sublogarithmic.

For the dependence on d we may not expect spectacular results, as we know by the remarks at the beginning that we may not hope for less than d iterations but on the other hand the simplex algorithm solved all examples in less than $12d$ iterations. However, we see that for large dimensions the new algorithms need fewer iterations and in particular Alg 3, the algorithm with the rescaling, did very well.

To obtain some more insight into the dependence on the dimension we did a best fit of the form αd^β for all series and all algorithms. Here we computed two values for the simplex algorithm, for the first value we counted the d steps for the initialization while in the second value these steps were neglected. For the algorithm with the rescaling we did not take into account the numbers of rescalings, as this number appears to have little dependence on the dimension. The results are presented in Table 3.

Table 1. Number of steps in dependence of the dimension.

	Dim	Alg 1	Alg 2	Alg 3	
Ex1	10	13.4	20.6	16.8	5.8
	20	31.2	57.8	35.2	12.0
	40	93.4	146.0	69.2	18.8
	80	274.4	353.2	146.6	23.2
	160	769.2	926.0	294.8	27.4
	320	2114.6	2156.6	585.0	29.8
Ex2	640	6321.0	4756.4	1179.0	33.8
	10	17.0	28.8	25.8	9.2
	20	42.6	62.6	54.2	14.0
	40	122.0	154.8	108.8	20.8
	80	346.6	385.2	228.8	29.6
	160	925.0	923.8	528.2	35.0
Ex3	320	2528.0	2296.8	939.0	41.2
	640	7294.4	5388.0	1909.4	45.4
	10	15.8	27.4	24.0	7.4
	20	41.0	58.4	50.2	11.0
	40	112.4	141.2	101.2	17.0
	80	326.2	368.6	210.0	19.8
	160	887.0	857.2	422.0	21.0
	320	2565.2	2183.2	867.0	22.6
	640	7151.0	5125.2	1787.0	23.4

Table 2. Number of steps in dependence of the number of inequalities.

	n	Alg 1	Alg 2	Alg 3	
Ex1	400	222.0	336.4	158.2	20.8
	800	386.6	481.0	185.8	24.4
	1600	502.0	597.2	210.8	27.6
	3200	644.6	670.2	225.4	28.6
	6400	745.2	781.4	244.2	30.2
Ex2	400	302.4	345.0	247.2	27.6
	800	456.6	536.0	294.6	32.2
	1600	570.6	602.8	312.2	32.8
	3200	735.0	713.4	330.0	34.6
	6400	869.6	794.2	360.0	38.0
Ex3	400	297.4	334.0	224.6	17.8
	800	449.8	491.8	261.0	19.8
	1600	553.8	545.6	279.2	21.0
	3200	732.6	657.8	293.6	20.6
	6400	838.6	699.4	304.2	20.8

Table 3. Rate of growth.

	Alg 1				Alg 2		Alg 3	
	α	β	α	β	α	β	α	β
Ex1	0.8735	1.3783	0.3890	1.4990	1.1122	1.3093	1.6269	1.0214
Ex2	1.0709	1.3432	0.5605	1.4621	1.4575	1.2719	2.4228	1.0334
Ex3	0.9950	1.3793	0.5010	1.4779	1.3538	1.2747	2.2439	1.0334

We find that the simplex algorithm and Algorithm 2.6 are superlinear, where the exponent of Algorithm 2.6 is slightly smaller for all series, while the experimental algorithm is very close to being linear for our examples and thus for these examples very close to the hypothetical Gauss-type algorithm.

Altogether the examples show that the concepts developed here might be of practical interest.

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