Tchebyshev Posets*

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Abstract. We construct for each n an Eulerian partially ordered set T_n of rank $n + 1$ whose ce -index provides a non-commutative generalization of the n th Tchebyshev polynomial. We show that the order complex of each T_n is shellable, homeomorphic to a sphere, and that its face numbers minimize the expression $\max_{|x| \leq 1} |\sum_{j=0}^n (f_{j-1}/f_{n-1}) \cdot 2^{-j} \cdot (x-1)^j|$ among the f -vectors of all $(n-1)$ -dimensional simplicial complexes. The duals of the posets constructed have a recursive structure similar to face lattices of simplices or cubes, offering the study of a new special class of Eulerian partially ordered sets to test the validity of Stanley's conjecture on the non-negativity of the cd -index of all Gorenstein* posets.

Introduction

The object of this paper is the study of a sequence $T_0, T_1, T_2, \dots$ of Eulerian partially ordered sets constructed in such a way that for each n , the ce -index of T_n is a non-commutative generalization of the Tchebyshev polynomial $T_n(x)$ of the first kind. We call them Tchebyshev posets. One remarkable property of this sequence of posets is “self-similarity”: for each atom a of T_{n+1} the partially ordered set of elements above a is isomorphic to a copy of T_n . Since each Tchebyshev poset may be represented as the face poset of a CW -complex that is homeomorphic to a ball, at the level of CW -complexes we may state that any face of a “dual Tchebyshev cell” is a “dual Tchebyshev cell”, in analogy to the statements “every face of a simplex is a simplex” and “every face of a cube is a cube”. Therefore the same way one studies simplicial complexes and cubical complexes, it is possible to focus on CW -complexes whose faces are dual Tchebyshev cells.

The construction that yields our Tchebyshev posets may be defined in greater generality. This possibility is explored in Section 2. Here we define an operator T that assigns

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a partially ordered set $T(Q)$ to any given locally finite partially ordered set Q , by introducing a partial order on its non-singleton intervals. An interval of Q is considered smaller than another in $T(Q)$ if either this holds elementwise, or the smaller interval is an initial segment of the larger one. If Q has a rank function and a unique minimum element x_0 covered by a unique atom then $T(Q)$ has a rank function and a unique minimum element. Under these circumstances every interval of $T(Q)$ is Eulerian if and only if the same holds for $Q \setminus \{x_0\}$. Hence we may obtain a new, more complex Eulerian poset from any given one by adding a new minimum element below its minimum element, and applying the operator T to it. The partially ordered sets T_n we call Tchebyshev posets are obtained by applying this procedure to the simplest possible Eulerian posets, i.e., the “ladder” posets which have exactly two elements at any given rank between the maximum and the minimum element. A worthy subject of future research is what the effect of the operator T on other, more complicated (and not even necessarily Eulerian), posets? Part of this work (studying matrices of formal power series associated to Tchebyshev transforms of binomial posets) is done in a subsequent paper [22].

In Section 2 we describe the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ as a triangulation of the boundary of the n -dimensional cross-polytope. This description implies that the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ is homeomorphic to a sphere. The triangulation may be realized with “straight” faces, which raises the open question of whether the order complex has a line shelling induced by a geometric picture. It is also worth noting that the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ provides a balanced triangulation of the boundary complex of the cross-polytope that needs fewer faces than barycentric subdivision.

In Section 3 we take a closer look at the partially ordered set T_n and describe an important labeling of its cover relations, using only four symbols. This labeling is used to present a shelling of the order complex of T_n in Section 5, which allows us to describe the cd -index of T_n in Section 7 in terms of an “ascent–descent” statistic, in a way that is analogous to Purtill’s work [27] on the cd -index of the Boolean algebra. The major difference is that while in the case of the Boolean algebra one needs to sum over permutations, here we sum essentially over all strings of given length formed by four letters. The computation of the cd -index is followed by formulas for the ce -index and the flag f -vector in Section 8. Finally, in Section 9 we use a well-known extremal property of the Tchebyshev polynomials to show an extremal property of the f -vector of the order complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$ among f -vectors of all simplicial complexes of the same dimension. Since we transform a classic inequality that holds for all normalized polynomials of a given degree, the surprising fact is not that all with simplicial complexes satisfy the inequality in question, but the fact that this transformation yields a sharp inequality.

Gorenstein* posets built from Tchebyshev posets may serve as a testing ground to verify Stanley’s conjecture on the non-negativity of the cd -index of Gorenstein* posets in a new special setting that is genuinely different from the (partially) known simplicial and cubical cases. Section 10 provides remarks about some open questions and potential future applications.

It is beyond the scope of this paper to even mention the many uses of Tchebyshev polynomials, from classical results in numerical methods and continued fractions, to recent developments in enumerative combinatorics. (As examples of enumerative results, many papers by T. Mansour and his co-authors on the closely related Tchebyshev polynomials

of the second kind come to mind. For a detailed bibliography see [26].) Considering the very symmetric nature of the Tchebyshev posets presented in this paper, one can hope that they may help in many ways to illustrate the properties and understanding of the nature of this extremely important sequence of polynomials. At least, in analogy to Purtill's results [27], new links between flag-enumeration in Eulerian posets and various permutation statistics could perhaps be found.

1. Preliminaries

1.1. Graded Partially Ordered Sets

A partially ordered set P is *locally finite* if every interval $[x, y] \subseteq P$ contains a finite number of elements. An element $y \in P$ *covers* $x \in P$ if $x < y$ and there is no element strictly between x and y . We use the notation $x < y$. A function $\rho: P \rightarrow \mathbb{Z}$ is a rank function for P if $\rho(y) = \rho(x) + 1$ is satisfied whenever $x < y$. The restriction of any rank function to any interval $[x, y] \subseteq P$ is unique up to a constant shift, and the *rank* $\rho(x, y) = \rho(y) - \rho(x)$ of an interval $[x, y]$ is always the same number: the common length of all maximal chains in $[x, y]$. A finite poset is *graded* when it has a unique minimum element $\hat{0}$, a unique maximum element $\hat{1}$, and a rank function ρ . In this situation we require $\rho(\hat{0}) = 0$ and call $\rho(\hat{1})$ the rank of the graded poset. Given a graded poset P of rank $n + 1$ and a set $S \subseteq \{1, \dots, n\}$ we define the *S-rank selected subposet* of P to be the poset $P_S = \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. We denote by $f_S(P)$ the number of maximal chains of P_S . Equivalently, $f_S(P)$ is the number of chains $x_1 < \dots < x_{|S|}$ in P such that $\{\rho(x_1), \dots, \rho(x_{|S|})\} = S$. The vector $(f_S(P) : S \subseteq \{1, \dots, n\})$ is called the *flag f-vector* of P .

Two equivalent encodings of the flag f -vector are the *flag h-vector* $(h_S(P) : S \subseteq \{1, \dots, n\})$ and the *flag ℓ -vector* $(\ell_S(P) : S \subseteq \{1, \dots, n\})$. They are defined by $h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P)$ and $\ell_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1, n] \setminus S} (-1)^{|T|} \cdot f_T(P)$.

1.2. Eulerian Posets

A graded poset is *Eulerian* if every interval $[x, y]$ of positive rank in the poset satisfies $\sum_{x \leq z \leq y} (-1)^{\rho(z)} = 0$. All linear relations holding for the flag f -vector of an arbitrary Eulerian poset of rank n were determined by Bayer and Billera in [3]. These linear relations were rephrased by J. Fine as follows (see paper [6] by Bayer and Klapper). For any $S \subseteq \{1, \dots, n\}$ define the non-commutative monomial $u_S = u_1 \cdots u_n$ by setting

$$u_i = \begin{cases} b & \text{if } i \in S, \\ a & \text{if } i \notin S. \end{cases}$$

Then the polynomial $\Psi_{ab}(P) = \sum_S h_S u_S$ in non-commuting variables a and b , called the *ab-index* of P , is a polynomial of $c = a + b$ and $d = ab + ba$. This form of $\Psi_{ab}(P)$ is called the *cd-index* of P . Further proofs of the existence of the *cd-index* may be found in [5], [15], and [30]. As noted by Stanley in [30], the existence of the *cd-index*

is equivalent to saying that the ab -index rewritten as a polynomial of $c = a + b$ and $e = a - b$ involves only even powers of e . It was observed by Bayer and Hetyei in [4] that the coefficients of the resulting ce -index may be computed using a formula that is analogous to the definition of the ℓ -vector. In fact, given a ce -word $u_1 \cdots u_n$, let S be the set of positions i satisfying $u_i = e$. Then the coefficient $L_S(P)$ of the ce -word is given by $L_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} \left(-\frac{1}{2}\right)^{|T|} f_T(P)$. The fact that the ce -index is a polynomial of c and e^2 is equivalent to stating that $L_S(P) = 0$ unless S is an *even* set, that is, a union of disjoint intervals of even cardinality.

1.3. Simplicial Complexes

A simplicial complex Δ is a family of subsets (*faces*) of a finite vertex set V , containing all singletons and closed under taking subsets. The *dimension* of a face $\sigma \in \Delta$ is $\dim(\sigma) = |\sigma| - 1$. Maximal faces are called *facets*. The number of i -dimensional faces is denoted f_i , and the vector $(f_{-1}, f_0, \dots, f_d)$ is called the f -vector.

In this paper we are only concerned with *order complexes* of posets. The vertices of the order complex $\Delta(P)$ of a poset P are the elements of P ; its faces are the increasing chains. When P is graded, $\Delta(P)$ is a *cone* over its maximum element $\hat{1}$ and its minimum element $\hat{0}$. Because of this, many authors define the order complex of a graded poset P as the order complex of $P \setminus \{\hat{0}, \hat{1}\}$. In a few cases we will have to diverge from this convention for geometric reasons. To avoid confusion, for any poset P , *by default we include all elements (even the “special” ones) as vertices of $\Delta(P)$* and rather use the operation “ $\setminus$ ” if needed.

A simplicial complex is *pure*, if every facet has the same dimension. The order complex $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ of a graded poset P of rank $n + 1$ is pure of dimension $n - 1$. A pure simplicial complex Δ is *shellable* if there is an ordering $F_0, F_1, \dots, F_t$ of its facets such that for every $k \in \{2, \dots, t\}$ the collection of faces of F_k contained in some earlier F_i is a pure simplicial complex of dimension $\dim \Delta - 1$. Equivalently, there exists a face $R(F_k)$ of F_k , not contained in any earlier facet such that every face $\sigma \subseteq F_k$ not contained in any earlier F_i contains $R(F_k)$. In many cases special *chain-edge labelings* are used to obtain a shelling of the order complex of a graded poset.

1.4. Chain-Edge Labelings

Given a graded poset P of rank $n + 1$, let $\mathcal{M}(P)$ be the set of its maximal chains, and let $\mathcal{E}(P) = \{(x, y) : x, y \in P, x < y\}$ be the set of its cover relations. An *edge labeling* of P is a map $\lambda: \mathcal{E}(P) \rightarrow \Lambda$, into some partially ordered set of labels Λ . Introducing $\mathcal{ME}(P) = \{(C, e) : C \in \mathcal{M}(P), e \in \mathcal{E}\}$, a *chain-edge labeling* of P is a map $\lambda: \mathcal{ME}(P) \rightarrow \Lambda$ satisfying the following axiom:

- (CE) If two maximal chains $C: \hat{0} = x_0 < x_1 < \cdots < x_n < x_{n+1} = \hat{1}$ and $C': \hat{0} = x'_0 < x'_1 < \cdots < x'_n < x'_{n+1} = \hat{1}$ satisfy $x_i = x'_i$ for $i = 1, 2, \dots, k$, then $\lambda(C, (x_{i-1}, x_i)) = \lambda(C', (x'_{i-1}, x'_i))$ holds for $i = 1, 2, \dots, k$.

In other words, $\lambda(C, (x, y))$ depends only on x, y , and the subchain $C(\rho(x))$: $\widehat{0} = x_0 < x_1 < \cdots < x_{\rho(x)-1} < x_{\rho(x)} = x$. (For any i , $C(i)$ denotes the subchain $C(i)$: $\widehat{0} = x_0 < x_1 < \cdots < x_i$ of C .) $C(\rho(x))$ is a maximal chain of the graded poset $[\widehat{0}, x]$. An interval $[x, y]$ enriched by a maximal chain r of $[\widehat{0}, x]$ is a *rooted interval*, denoted by $[x, y]_r$. Given a rooted interval $[x, y]_r$, and a maximal chain C of $[x, y]$, for any $u, v \in C$ satisfying $\rho(x) \leq u < \rho(y)$ and $u < v$ we define $\lambda_r(C, (u, v)) = \lambda(C', (u, v))$, where $C' \in \mathcal{M}(P)$ is an arbitrary maximal chain containing $C \cup r$. (By axiom (CE), $\lambda(C', (u, v))$ depends only on C and r , not on the particular choice of C' .) Thus λ and r induce a chain-edge labeling of $[x, y]$.

A chain-edge labeling $\lambda: \mathcal{ME}(P) \rightarrow \Lambda$ has the *first atom property* (or it is an *FA-labeling*) if for every rooted interval $[x, y]_r$ there is a unique atom $a \in [x, y]$ satisfying $\lambda_r(C, (x, a)) < \lambda_r(C', (x, a'))$ for every other atom $a' \in [x, y]$ and every pair (C, C') of maximal chains of $[x, y]_r$, containing a and a' , respectively. We call the atom a the *first atom* of $[x, y]_r$ with respect to λ . FA-labelings were introduced by Billera and Hetyei in [8] as a flag-enumeration tool in arbitrary graded posets (generalizing the proof of Theorem 2.1 of [7]). Every graded poset has at least one FA-labeling, obtained by drawing the poset in the plane and numbering all cover relations $x < y$ for a fixed x from left to right with $1, 2, \dots$ (thus obtaining an edge-labeling). Given an FA-labeling, let φ be the operation assigning to each rooted interval $[x, y]_r$ its first atom. For each maximal chain C : $\widehat{0} = x_0 < x_1 < \cdots < x_n < x_{n+1} = \widehat{1}$ define the function $\psi_C: [1, n] \rightarrow [1, n]$ by $\psi_C(i) = \max\{j : x_i = \varphi([x_{i-1}, x_j]_{C(i-1)})\}$, and associate the family of intervals $\mathcal{I}_C = \{[i, \psi_C(i)] : i \in [1, n], \psi_C(i) \neq n+1\}$ to C . Let $C_1, \dots, C_m$ be an ordering of $\mathcal{M}(P)$ that extends the lexicographic order on their labels. Then a subset $\{x_s : s \in S\}$ occurs first in $C \in \mathcal{M}(P)$ if and only if $S \subseteq [1, n]$ blocks $\mathcal{I}_C$, that is $I \cap S \neq \emptyset$ for all $I \in \mathcal{I}_C$.

Intuitively speaking, $[i, j] \in \mathcal{I}_C$ means that there exists a chain $C' = \widehat{0} < x_1 < \cdots < x_{i-1} < x'_i < \cdots < x'_j < x_{j+1} < \cdots < x_{n+1} = \widehat{1}$ preceding C in the ordering that provides an “earlier” path from x_{i-1} to x_{j+1} in the Hasse diagram of P . Hence every subchain, not included in the union of the maximal chains preceding C , must contain at least one element whose rank is between i and j .

Lemma 1.1. *If an FA-labeling of a graded poset P satisfies that for every maximal chain C the minimal elements of the interval system $\mathcal{I}_C$ are singletons, then any ordering of the maximal chains of P that extends the lexicographic order on their labels yields a shelling of $\Delta(P \setminus \{\widehat{0}, \widehat{1}\})$.*

In fact, for every facet $C \in \Delta(P \setminus \{\widehat{0}, \widehat{1}\})$, $R(C)$ is the set of those elements whose rank s satisfies $\{s\} \in \mathcal{I}_C$. Following the terminology used in [2], [21], and [23], we call the elements of $R(C)$ *topological descents*.

This approach to shellings induced by chain-edge labelings is the most comfortable for the present author, but others must be credited for the discovery of the underlying results. Lemma 1.1 is applicable precisely to Kozlov’s *CC-labelings* defined in [25]. The same labelings were rediscovered by Hersh and Kleinberg in [23] (see the Introduction of [2]). All authors were generalizing *CL-labelings* introduced by Björner and Wachs in [9]. A *CR-labeling* is a chain-edge labeling for which every rooted interval has a unique maximal chain with rising labels. If this chain is always the first in the lexicographic

order, then it is called a *CL*-labeling. A *CR*- or *CL*-labeling that comes from an edge-labeling is called *ER*- or *EL*-labeling, respectively. Björner and Wachs have shown that *CL*-labelings induce a shelling of the order complex, and our language concerning chain-edge labelings (except where noted otherwise) is adopted from the paper of Björner and Wachs [11]. *ER*-labelings are called *R*-labelings in Stanley's book [32, Section 3.13].

Finally, it is worth noting that the entry h_S in the flag h -vector is precisely the number of maximal chains whose set of topological descents is S .

2. Elementary Properties of General Tchebyshev Posets

The elementary properties of Tchebyshev posets are most easily proven by introducing a definition that is more general than what is needed for most of this paper. The results of this section are also used in the subsequent paper [22] studying matrices of formal power series arising as generating functions associated to a larger class of general Tchebyshev posets (induced by binomial posets).

Definition 2.1. We define the *Tchebyshev poset* $T(Q)$ of a locally finite poset Q as follows. Its elements are all ordered pairs $(x, y) \in Q \times Q$ satisfying $x < y$, and we set $(x_1, y_1) \leq (x_2, y_2)$ when $y_1 \leq x_2$ or $x_1 = x_2$ and $y_1 \leq y_2$.

In other words, the elements of $T(Q)$ are identifiable with all intervals of Q that are not singletons. We consider an interval larger than the other if either every element of the larger interval is larger than every element of the smaller interval or the smaller interval is an “initial segment” of the larger interval.

Proposition 2.2. $T(Q)$ is a partially ordered set.

Proof. The relation is obviously reflexive. Antisymmetry and transitivity are easily shown using the implication $(x_1, y_1) \leq (x_2, y_2) \Rightarrow y_1 \leq y_2$. $\square$

Proposition 2.3. If $\rho: Q \rightarrow \mathbb{Z}$ is a rank function for Q then setting $\rho(x, y) = \rho(y)$ provides a rank function for $T(Q)$. In fact, the set of elements covering $(x, y) \in T(Q)$ is

$$\{(x, \dot{y}) : y < \dot{y}\} \cup \{(y, \dot{y}) : y < \dot{y}\}.$$

(Here $\dot{y}$ denotes an arbitrary element covering y in Q .)

Proof. If $(x, y) < (x_1, y_1)$ then either $x_1 \geq y$ or $x_1 = x$. If $x_1 > y$ then $(x, y) < (y, x_1) < (x_1, y_1)$, so either $x_1 = x$ or $x_1 = y$. In either case $y_1 > y$ is easy to verify. The other implication is obvious. $\square$

Proposition 2.4. Assume that every element of Q is comparable with at least one other element of Q . Then $T(Q)$ has a unique minimum element if and only if Q has a unique minimum element x_0 covered by a unique atom y_0 . In that case the unique minimum element of $T(Q)$ is (x_0, y_0) .

Proof. The proof of the “if” part is straightforward. Assume (x_0, y_0) is the unique minimum element of $T(Q)$. We claim that $x_0 \leq x$ holds for all $x \in Q$. If x is not a maximum element of Q then there exists an $(x, y) \in T(Q)$ for some $y \in Q$, and then $x_0 \leq x$ follows from $(x_0, y_0) \leq (x, y)$. If $x \in Q$ is a maximal element of Q then any comparable $z \in Q$ satisfies $x_0 \leq z < x$. Any y strictly between x_0 and y_0 would give rise to $(x_0, y) < (x_0, y_0)$, so $x_0 < y_0$. Any other atom z would give rise to an (x_0, z) that is incomparable with (x_0, y_0) . $\square$

Lemma 2.5. *Given $x_1 < y_1 \leq y_2 \in Q$, the interval $[(x_1, y_1), (x_1, y_2)] \subseteq T(Q)$ is isomorphic to $[y_1, y_2] \subseteq Q$.*

In fact, an element $(x, y) \in T(Q)$ belongs to $[(x_1, y_1), (x_1, y_2)]$ if and only if $x = x_1$ and $y \in [y_1, y_2]$.

Proposition 2.6. *Assume that every element of Q is comparable with some other element and that $T(Q)$ has a unique minimum element (x_0, y_0) . Then every interval of $T(Q)$ is an Eulerian poset if and only if the same holds for every interval of $Q \setminus \{x_0\}$.*

Proof. The necessity is a consequence of Lemma 2.5, since for any $[y_1, y_2] \subseteq Q \setminus \{x_0\}$ we have $x_0 < y_1 \leq y_2$, thus (x_0, y_1) and (x_0, y_2) are the endpoints of an interval in $T(Q)$ that is isomorphic to $[y_1, y_2]$.

To prove sufficiency, again by Lemma 2.5, we may restrict our attention to those $[(x_1, y_1), (x_2, y_2)] \subseteq T(Q)$ which satisfy $x_1 \neq x_2$, and verify that for such intervals

$$\sum_{(x,y) \in [(x_1, y_1), (x_2, y_2)]} (-1)^{\rho(x,y)} = 0$$

holds. Recall that, by Proposition 2.3, every (x, y) in the above sum satisfies $\rho(x, y) = \rho(y)$ where $\rho(y)$ is the rank of y in Q , hence we need to show

$$\sum_{(x,y) \in [(x_1, y_1), (x_2, y_2)]} (-1)^{\rho(y)} = 0.$$

Every $(x, y) \in [(x_1, y_1), (x_2, y_2)]$ satisfies exactly one of $x = x_1$, $x_1 < x < x_2$, or $x = x_2$. Depending on this condition, we may partition our summands $(-1)^{\rho(y)}$ into three partial sums as follows:

- (i) If $x = x_1$ then (x, y) belongs to $[(x_1, y_1), (x_2, y_2)]$ if and only if $y_1 \leq y \leq x_2$. The contribution of the summands in this first subset is $\sum_{y_1 \leq y \leq x_2} (-1)^{\rho(y)}$. Since every interval of Q is Eulerian, we obtain a contribution of $(-1)^{\rho(x_2)} \cdot \delta_{y_1, x_2}$, where δ is the Kronecker delta function.
- (ii) If $x_1 < x < x_2$ then x must satisfy $y_1 \leq x < x_2$ and y must satisfy $x < y \leq x_2$. The contribution of the summands falling into this category is

$$\sum_{y_1 \leq x < x_2} \sum_{x < y \leq x_2} (-1)^{\rho(y)} = \sum_{y_1 \leq x < x_2} (-1)^{\rho(x)+1} = (-1)^{\rho(x_2)} \cdot (1 - \delta_{y_1, x_2}).$$

- (iii) If $x = x_2$ then y must satisfy $x_2 < y \leq y_2$. The contribution of the summands in this last subset is $\sum_{x_2 < y \leq y_2} (-1)^{\rho(y)} = (-1)^{\rho(x_2)+1}$.

Summing up the three partial sums yields 0. $\square$

3. The Tchebyshev Posets T_n

Consider the locally finite poset $\mathbb{P}^\pm$: $-1 < 1 < -2, 2 < -3, 3 < -4, 4 < \dots$. Pairs of elements separated with a comma are considered incomparable. By Proposition 2.4, $T(\mathbb{P}^\pm)$ has a unique minimum element $(-1, 1)$. It is easy to verify that

$$\rho(x) = \begin{cases} |x| - 1 & \text{if } x \neq -1, \\ -1 & \text{if } x = -1 \end{cases}$$

is a rank function for this poset, so Proposition 2.3 implies that $\rho(x, y) = |y| - 1$ is a rank function for $T(\mathbb{P}^\pm)$. Every interval $[x, y] \subset \mathbb{P}^\pm \setminus \{-1\}$ has exactly two elements at each rank strictly between $\rho(x)$ and $\rho(y)$, and is thus Eulerian. Therefore, by Proposition 2.6, every interval of $T(\mathbb{P}^\pm)$ is Eulerian.

Definition 3.1. The Tchebyshev poset T_n is the interval $[(-1, 1), (-(n+1), -(n+2))]$ in $T(\mathbb{P}^\pm)$.

Figure 1 represents the Tchebyshev poset T_3 . The meaning of the labels on the edges is explained in Section 5. Let us summarize what we already know about the Tchebyshev poset T_n using its definition and the results of Section 2. It has a unique minimum element $\hat{0} = (-1, 1)$, a unique maximum element $\hat{1} = (-(n+1), -(n+2))$, it is graded and Eulerian. The elements strictly between $\hat{0} = (-1, 1)$ and $\hat{1} = (-(n+1), -(n+2))$ are

Fig. 1. The Tchebyshev poset T_3 .

pairs of non-zero integers (x, y) such that $|x| < |y|$, $|x|, |y| \in \{1, 2, \dots, n\}$, with the restriction that for $|y| = n + 1$ we must have $y = -(n + 1)$. The relation $(x_1, y_1) < (x_2, y_2)$ holds if $|y_1| < |y_2|$, and either $|y_1| < |x_2|$, or $y_1 = x_2$, or $x_1 = x_2$. The rank of the element (x, y) is $|y| - 1$. In particular, the rank of T_n is $n + 1$. Using symbols ε and η to denote signs from the set $\{-, +\}$, the cover relations of T_n are the following:

- (i) $\widehat{0} = (-1, 1) < (\varepsilon 1, \eta 2)$,
- (ii) $(x, y) < (x, \varepsilon(|y| + 1))$ for $|x| < |y| \leq n$ (where ε is $-$ for $|y| = n$),
- (iii) $(x, y) < (y, \eta(|y| + 1))$ for $|x| < |y| \leq n$ (where η is $-$ for $|y| = n$), and
- (iv) $(\varepsilon x, -(n + 1)) < (-(n + 1), -(n + 2)) = \widehat{1}$.

In particular, every element of rank less than $n - 1$ is covered by exactly four elements. The cover relations may be naturally labeled by the symbols L^+ , L^- , R^+ , R^- as follows. We think of each label as an operator assigning the covering element to the covered element.

Definition 3.2. Given $\varepsilon, \eta \in \{+, -\}$ and (x, y) with $1 \leq |x| < |y| \leq n$ we set

$$R^\varepsilon(x, y) = (x, \varepsilon(|y| + 1)) \quad \text{and} \quad L^\eta(x, y) = (y, \eta(|y| + 1)).$$

We extend this definition of R^ε and L^η to $\widehat{0} = (-1, 1)$. We extend only L^- and R^- to elements of the form $(x, \pm n)$. We extend only L^- to the coatoms $(x, -(n + 1))$ which are sent into $\widehat{1} = (-(n + 1), -(n + 2))$ by L^- .

In other words, the operators R^ε remove the right component, and the operators L^η remove the left component of (x, y) . The sign ε , resp. η , indicates the sign of the new letter $(|y| + 1)$ introduced. In proving the results of Section 5, the following lemmata will be instrumental.

Lemma 3.3. *The range of the operators, L^+ , L^- , R^+ , and R^- is pairwise disjoint. In fact, $(x, y) \neq \widehat{0}$ is the image of some L^η exactly when $|y| = |x| + 1$, and it is the image of some R^ε exactly when $|y| > |x| + 1$. Moreover, the sign of any operator whose image is (x, y) is the same as the sign of y .*

This lemma is straightforward.

Lemma 3.4. *For any $x, y \in \mathbb{P}^\pm$ satisfying $|x| < |y|$, we have $(x', y') < (x, y)$ if and only if $(x', y') < (x, -y)$.*

In fact, if (x, y) is the image of some element under R^+ then $(x, -y)$ is the image of the same element under R^- and vice versa. The analogous claim also holds for the operators L^+ and L^- .

Lemma 3.5. *If (x, y) belongs to the image of $L^{\text{sign}(y)}$, then $(x, y) = L^{\text{sign}(y)} L^{\text{sign}(x)}(x', y')$ is satisfied for any (x', y') of rank $\rho(x, y) - 2$.*

In fact, the given assumptions imply $|x| = |y| - 1$, and $L^{\text{sign}(y)} L^{\text{sign}(x)}(x', y') = L^{\text{sign}(y)}(y', x) = (x, y)$.

Lemma 3.6. *For all $(x, y) \in T_n$ of rank at most $(n - 1)$ the identities $L^n R^\varepsilon(x, y) = L^\eta L^\varepsilon(x, y)$, $R^\varepsilon L^+(x, y) = R^\varepsilon L^-(x, y)$, and $R^\varepsilon R^+(x, y) = R^\varepsilon R^-(x, y)$ hold.*

This lemma is again straightforward. In analogy to Boolean algebras and the face lattice to the cube the duals of the Tchebyshev posets are “self-similar” in the following sense.

Proposition 3.7. *For any $n \geq 1$, and any atom $(\varepsilon 1, \eta 2) \in T_n$ the interval $[(\varepsilon 1, \eta 2), \widehat{1}]$ is isomorphic to T_{n-1} .*

Proof. Without loss of generality we may assume that the atom in question is $(-1, -2)$. Exactly those pairs $(\varepsilon i, \eta j)$ are above $(-1, -2)$ for which the letters 1 or 2 occur only with negative sign. Hence the interval $[(-1, -2), \widehat{1}] = [(-1, -2), (-n, -(n+1))]$ is precisely

$$T(-1 < -2 < -3, 3 < -4, 4 < \cdots < -n, n < -(n+1) < -(n+2)).$$

Consider the map φ from the partially ordered set

$$-1 < -2 < -3, 3 < -4, 4 < \cdots < -n, n < -(n+1) < -(n+2)$$

onto the partially ordered set

$$-1 < 1 < -2, 2 < -3, 3 < \cdots < -(n-1), (n-1) < -n < -(n+1)$$

given by the formula

$$\varphi(\varepsilon i) = \begin{cases} \varepsilon(i-1) & \text{if } i > 2, \\ 1 & \text{if } i = 2, \\ -1 & \text{if } i = 1. \end{cases}$$

It is easy to verify that φ is a poset-isomorphism, inducing an isomorphism of the associated Tchebyshev posets. $\square$

4. The Geometric Significance of the Tchebyshev Poset T_n

In this section we show that $\Delta(T_n \setminus \{\widehat{0}, \widehat{1}\})$ induces a triangulation of the boundary of the n -dimensional cross-polytope.

Let $e_1, \dots, e_n$ be the basis vectors of an n -dimensional Euclidean space. The n -dimensional cross-polytope is the convex hull of the vectors $\pm e_1, \dots, \pm e_n$. It has $2n$ vertices and 2^n facets, each facet being the convex hull of $\varepsilon_1 \cdot e_1, \dots, \varepsilon_n \cdot e_n$ for some $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$. Label the vertex $\varepsilon_i \cdot e_i$ with $(\varepsilon_i \cdot i, -(n+1)) \in T_n$. Assuming $i < j$, label the midpoint of the edge connecting $\varepsilon_i \cdot e_i$ and $\varepsilon_j \cdot e_j$ with $(\varepsilon_i \cdot i, \varepsilon_j \cdot j)$. We define the convex hull of a set of labeled points to be a face in our triangulation if and only if the corresponding set of labels forms an increasing chain in the Tchebyshev poset T_n . Thus we obtain a geometric realization of the $\Delta(T_n \setminus \{(-1, 1), (-(n+1), -(n+2))\})$.

Fig. 2. The triangulation of the octahedron induced by T_3 .

Theorem 4.1. *The geometric realization of $\Delta(T_n \setminus \{(-1, 1), (-(n+1), -(n+2))\})$ described above induces a triangulation of the boundary of the n -dimensional cross-polytope.*

Figure 2 shows the triangulation of the octahedron induced by T_3 . Invisible edges are indicated with dotted lines, except for the invisible edge connecting $(1, 2)$ with $(3, -4)$, which is completely covered by visible edges. The fact that $\Delta(T_3 \setminus \{(-1, 1), (-4, -5)\})$ is realized as a triangulation of the boundary of the octahedron is obvious from Fig. 2. To prove Theorem 4.1 for an arbitrary n , observe first that every maximal chain of T_n is of the form

$$\begin{aligned}\widehat{0} &= (-1, 1) \prec (\varepsilon_1 1, \varepsilon_2 2) \prec (x_2, \varepsilon_3 3) \prec \cdots \prec (x_{n-1}, \varepsilon_n n) \prec (x_n, -(n+1)) \\ &\prec (-(n+1), -(n+2)) = \widehat{1},\end{aligned}$$

and no matter what the elements $x_2, \dots, x_n$ are, if $|x_i| = j$ then the sign of x_i must be ε_j . Therefore the correspondingly labeled vertices all belong to the convex hull of $\varepsilon_1 e_1, \dots, \varepsilon_n e_n$, that is, to the same facet of the boundary of the cross-polytope. Without loss of generality we may assume

$$\varepsilon_1 = \cdots = \varepsilon_n = 1, \quad (1)$$

Fig. 3. The unsigned Tchebyshev poset U_3 .

and we only need to show that the geometric realization of all maximal chains satisfying this condition induce a triangulation of the convex hull of $e_1, \dots, e_n$. Observe finally that the maximal chains satisfying condition (1) are exactly the maximal chains in the Tchebyshev poset of the chain $\hat{0} = 1 < 2 < \dots < n+1 < n+2 = \hat{1}$. This motivates us to introduce the following “unsigned version” of the Tchebyshev posets.

Definition 4.2. Consider the set of non-negative integers $\mathbb{N}$ ordered by the usual $<$ relation. We define the *unsigned Tchebyshev poset* U_n to be the interval $[(0, 1), (n+1, n+2)]$ in $T(\mathbb{N})$.

Figure 3 represents U_3 . To triangulate the faces of the octahedron, U_2 is used. We prefer to include an illustration of U_3 only because its picture reflects the structure of U_n better in general. As noted above, the maximal chains satisfying (1) are the ones of $T(1 < \dots < n+2)$ which is isomorphic to $T(0 < \dots < n+1) = U_{n-1}$. Thus we only need to show the following “unsigned version” of Theorem 4.1.

Proposition 4.3. Consider an $(n+1)$ -dimensional Euclidean space with basis $\{e_0, \dots, e_n\}$, and the unsigned Tchebyshev poset U_n . Let Δ_n be the convex hull of $e_0, e_1, \dots, e_n$. Label the vertex e_i of Δ_n with $(i, n+1)$, and label the midpoint of the edge connecting e_i and e_j with (i, j) , whenever $0 \leq i < j \leq n$ holds. Then the convex hulls of points whose labels are the increasing chains in $U_n \setminus \{(n+1, n+2)\}$ provide a triangulation of the n -simplex Δ_n .

Proof. We proceed by induction on n . For $n = 1$ the simplex Δ_1 is the line segment connecting e_0 and e_1 , and U_1 consist of $(0, 1) < (0, 2), (1, 2) < (2, 3)$. The label of e_0 is $(0, 2)$, the label of e_1 is $(1, 2)$, and the label of the midpoint $(e_0 + e_1)/2$ is $(0, 1)$. The two

maximal chains of $U_1 \setminus \{(2, 3)\}$ correspond to the line segments from e_0 to $(e_0 + e_1)/2$ and from $(e_0 + e_1)/2$ to e_1 . Thus we obtain a (one-dimensional) “triangulation”.

Assume that our proposition holds for some n and consider Δ_{n+1} and U_{n+1} . The minimum element $(0, 1)$ is covered by two atoms: $(0, 2)$ and $(1, 2)$ in U_{n+1} . It is easy to see that, in analogy to Proposition 3.7, the intervals $[(0, 2), (n+2, n+3)]$ and $[(1, 2), (n+2, n+3)]$ are isomorphic to U_n . Note that to obtain the first interval from U_n we must replace every positive i with $i+1$ (leaving 0 unchanged) while the second interval is obtained by increasing every i by 1. Hence, by our induction hypothesis, the increasing chains from $[(0, 2), (n+2, n+3)] \setminus \{(n+2, n+3)\}$ induce a triangulation of the convex hull of $\{e_0, e_2, \dots, e_{n+1}\}$ while the increasing chains from $[(1, 2), (n+2, n+3)] \setminus \{(n+2, n+3)\}$ induce a triangulation of the convex hull of $\{e_1, e_2, \dots, e_{n+1}\}$. By adding $(0, 1)$ to all of these chains we obtain a triangulation of the convex hull of $\{(e_0 + e_1)/2, e_0, e_2, \dots, e_{n+1}\}$ and of $\{(e_0 + e_1)/2, e_1, e_2, \dots, e_{n+1}\}$. The union of these two simplices is the convex hull of $\{e_0, e_1, e_2, \dots, e_{n+1}\}$. $\square$

An important corollary of Theorem 4.1 is the following:

Corollary 4.4. *The dual of the poset T_n is the face poset of a CW-complex that is homeomorphic to an n -ball.*

In fact, consider the triangulation of the boundary of the n -dimensional cross-polytope induced by $T_n \setminus \{(-1, 1), (-(n+1), -(n+2))\}$. By associating the center of the cross-polytope to $(-1, 1)$ we may easily convince ourselves that $T_n \setminus \{(-(n+1), -(n+2))\}$ induces a triangulation of the n -dimensional cross-polytope. Associate to each $(\varepsilon i, \eta j) \in T_n \setminus \{(-(n+1), -(n+2))\}$ the convex hull of all vertices represented by some $(\varepsilon' i', \eta' j') \geq (\varepsilon i, \eta j)$. Thus we associate the entire cross-polytope to $(-1, 1)$, the union of some 2^{n-2} facets of its boundary to each $(\varepsilon 1, \eta 2)$, a vertex to each $(\varepsilon i, -(n+1))$, and so on. It is easy to verify that the convex hulls associated to the elements of $T_n \setminus \{(-(n+1), -(n+2))\}$ form the non-empty faces of a CW-complex, ordered by reverse inclusion. Since they arose as unions of faces from the triangulation of a polytope, the resulting CW-complex is homeomorphic to a ball.

Definition 4.5. An n -dimensional *dual Tchebyshev cell* is a CW-complex whose face poset is isomorphic to the dual of T_n .

Repeated use of Proposition 3.7 yields the following:

Corollary 4.6. *Every face of a dual Tchebyshev cell is a dual Tchebyshev cell.*

To conclude this section, we note that for a graded poset P of rank $n+1$, the $(n-1)$ -dimensional complex $\Delta(P \setminus \{\widehat{0}, \widehat{1}\})$ is *balanced*, i.e., n colors suffice to color the vertices without repeating the same color within any face. (The rank function may be used to assign colors.) Hence T_n induces a balanced triangulation of the n -dimensional cross-polytope, using fewer simplices than barycentric subdivision, the “usual way” to create a balanced triangulation. On Fig. 2 we used white circles to denote those midpoints of edges whose color should be 2, while the remaining midpoints of edges should have

color 3 and the vertices of the octahedron should have color 1. For more information on balanced simplicial complexes see Stanley's book [31] or his paper [28].

5. A Shelling of the Order Complex of $T_n \setminus \{\widehat{0}, \widehat{1}\}$

The fact that $\Delta(T_n \setminus \{\widehat{0}, \widehat{1}\})$ may be geometrically realized as a triangulation of the boundary of a convex polyhedron suggests that this order complex may be shellable. After all, the first shelling was introduced as an ordering of the facets of convex polytope by Bruggeser and Mani in [13]. In this section we introduce an *FA*-labeling for T_n that satisfies the conditions of Lemma 1.1, i.e., a *CC*-labeling.

Using the operators introduced in Definition 3.2, every element covering $(x, y) \in T_n$ may be written in exactly one of the following four forms: $R^-(x, y)$, $R^+(x, y)$, $L^-(x, y)$, and $L^+(x, y)$.

Definition 5.1. We define $\lambda: \mathcal{E}(T_n) \longrightarrow \{1, 2, 3, 4\}$ as follows:

(i) If $\text{sign}(y)$ is positive, we set

$$\begin{aligned} \lambda((x, y), R^-(x, y)) &= 1, & \lambda((x, y), R^+(x, y)) &= 2, \\ \lambda((x, y), L^-(x, y)) &= 3, & \lambda((x, y), L^+(x, y)) &= 4. \end{aligned}$$

(ii) If $\text{sign}(y) = \text{sign}(x)$ is negative, we set

$$\begin{aligned} \lambda((x, y), L^-(x, y)) &= 1, & \lambda((x, y), R^-(x, y)) &= 2, \\ \lambda((x, y), L^+(x, y)) &= 3, & \lambda((x, y), R^+(x, y)) &= 4. \end{aligned}$$

(iii) If $\text{sign}(y)$ is negative but $\text{sign}(x)$ is positive, we set

$$\begin{aligned} \lambda((x, y), L^-(x, y)) &= 1, & \lambda((x, y), L^+(x, y)) &= 2, \\ \lambda((x, y), R^-(x, y)) &= 3, & \lambda((x, y), R^+(x, y)) &= 4. \end{aligned}$$

Here we consider $\{1, 2, 3, 4\}$ as the linearly ordered set $1 < 2 < 3 < 4$.

If the rank of (x, y) is $n - 1$ then only $L^-(x, y)$ and $R^-(x, y)$ exist, if (x, y) is a coatom then only $L^-(x, y)$ exists. As noted in Section 1, every edge-labeling may be considered as a special chain-edge labeling.

Theorem 5.2. *The FA-labeling introduced in Definition 5.1 has the property that any linear extension of the lexicographic order on the maximal chains of T_n provides a shelling.*

The poset T_3 is represented in Fig. 1 in such a way that for most (x, y) the labeling λ numbers the edges covering (x, y) from left to right, in increasing order. The only three exceptions are: $(1, 2)$, for which the order of labels is 3, 4, 1, 2; $(2, 3)$ and $(1, 3)$, both of which are covered by two edges, labeled from left to right by 2, 1, in this order. It is worth noting that there is no “rising chain” in the interval $[(-1, -2), (3, -4)]$, hence our labeling is *not a CR-labeling*, only a *CC-labeling*.

We prove Theorem 5.2 by establishing the validity of a sequence of lemmata. In each of these we consider the labeling λ given in Definition 5.1, an arbitrary maximal chain $C: \widehat{0} < (x_1, y_1) < \cdots < (x_n, y_n) < \widehat{1}$, and we assume that $[i, j]$ is an arbitrary minimal interval of $\mathcal{I}_C$. We will be done with the proof of Theorem 5.2 once we have shown that $i = j$ must hold. By definition, the fact that $[i, j]$ belongs to $\mathcal{I}_C$ implies that there is an “earlier” maximal chain

$$C': \widehat{0} < (x_1, y_1) < \cdots < (x_{i-1}, y_{i-1}) < (x'_i, y'_i) < \cdots < (x'_j, y'_j) \\ < (x_{j+1}, y_{j+1}) < \cdots < (x_n, y_n) < \widehat{1}$$

differing from C exactly at the ranks belonging to $[i, j]$.

Lemma 5.3. *Under our assumptions either $i = j$ or $(x_{j+1}, y_{j+1}) = L^{\text{sign}(y_{j+1})}(x_j, y_j)$ holds. In the latter case we must also have $(x_{k+1}, y_{k+1}) = R^{\text{sign}(y_{k+1})}(x_k, y_k)$ for $i + 1 \leq k < j$.*

Proof. Assume first that $(x_{j+1}, y_{j+1}) = R^{\text{sign}(y_{j+1})}(x_j, y_j)$ holds. Then, by Lemma 3.3, we also have $(x_{j+1}, y_{j+1}) = R^{\text{sign}(y_{j+1})}(x'_j, y'_j)$. Since the operators R^+ and R^- do not change the first coordinate, $x_j = x'_j = x_{j+1}$ follows. This also implies that the only way for (x_j, y_j) to be different from (x'_j, y'_j) is $y'_j = -y_j$. Now we may use Lemma 3.4 to conclude from $(x'_{j-1}, y'_{j-1}) < (x'_j, y'_j)$ that $(x'_{j-1}, y'_{j-1}) < (x_j, y_j)$. However, then, unless $i = j$, the maximal chain

$$C_1: \widehat{0} < (x_1, y_1) < \cdots < (x_{i-1}, y_{i-1}) < (x'_i, y'_i) < \cdots < (x'_{j-1}, y'_{j-1}) \\ < (x_j, y_j) < \cdots < (x_n, y_n) < \widehat{1},$$

which is “earlier” than C , demonstrates the fact that $[i, j - 1]$ belongs to $\mathcal{I}_C$, in contradiction with the minimality of $[i, j]$. This contradiction is avoided only if $i = j$.

From now on we may assume $(x_{j+1}, y_{j+1}) = L^{\text{sign}(y_{j+1})}(x_j, y_j)$. Let k be the smallest integer satisfying $k \geq i + 1$ and let $(x_{k+1}, y_{k+1}) = L^{\text{sign}(y_{k+1})}(x_k, y_k)$. For this k we have

$$(x_{k+1}, y_{k+1}) = L^{\text{sign}(y_{k+1})} L^{\text{sign}(x_{k+1})}(x'_{k-1}, y'_{k-1})$$

by Lemma 3.5. Hence the maximal chain

$$C_2: \widehat{0} < (x_1, y_1) < \cdots < (x_{i-1}, y_{i-1}) < (x'_i, y'_i) < \cdots < (x'_{k-1}, y'_{k-1}) \\ < L^{\text{sign}(x_{k+1})}(x'_{k-1}, y'_{k-1}) < (x_{k+1}, y_{k+1}) < \cdots < (x_n, y_n) < \widehat{1},$$

which is “earlier” than C demonstrates $[i, k] \in \mathcal{I}_C$. By the minimality of $[i, j]$ we must have $k = j$, therefore every (x_{k+1}, y_{k+1}) for $i + 1 \leq k < j$ must be in the image of some R -operator. $\square$

Lemma 5.4. *Under our assumptions $j \leq i + 1$ holds.*

Proof. Assume, by way of contradiction, $j \geq i + 2$. In this case, by Lemma 5.3, we have $(x_{j+1}, y_{j+1}) = L^{\text{sign}(y_{j+1})}(x_j, y_j)$ and $(x_{k+1}, y_{k+1}) = R^{\text{sign}(y_{k+1})}(x_k, y_k)$ for $i + 1 \leq k < j$. As a consequence of $(x_{j+1}, y_{j+1}) = L^{\text{sign}(y_{j+1})}(x_j, y_j)$ we must have

$$x_{j+1} = y_j. \quad (2)$$

We claim that in this situation $y_{i+1}, y_{i+2}, \dots, y_{j-1}$ must all have a negative sign. In the contrary event, let m be the smallest integer satisfying $\text{sign}(y_m) = +$ and $i + 2 \leq m \leq j - 1$. Consider the maximal chain

$$C_3: \quad \widehat{0} < (x_1, y_1) < \dots < (x_{m-1}, y_{m-1}) < (x_m, -y_m) \\ < (x_{m+1}, y_{m+1}) < \dots < (x_n, y_n) < \widehat{1}.$$

This is a chain by Lemma 3.4 and $(x_{m+1}, y_{m+1}) = R^{\text{sign}(y_{m+1})}(x_m, y_m) = R^{\text{sign}(y_{m+1})}(x_m, -y_m)$. Since y_m is positive, either $(x_m, y_m) = R^+(x_{m-1}, y_{m-1})$ or $(x_m, y_m) = L^+(x_{m-1}, y_{m-1})$. In the first case $(x_m, -y_m) = R^-(x_{m-1}, y_{m-1})$, in the second $(x_m, y_m) = L^-(x_{m-1}, y_{m-1})$. Since the label of $((x, y), R^-(x, y))$ is always less than the label of $((x, y), R^+(x, y))$ and the analogous statement holds also for the operators L^- and L^+ , the chain C_3 defined above is “earlier” than C , and demonstrates $[m, m] \in \mathcal{I}_C$, in contradiction with the minimality of $[i, j]$. This contradiction proves that $y_{i+1}, y_{i+2}, \dots, y_{j-1}$ must have a negative sign. In particular, the sign of y_{j-1} is negative. Consider now the maximal chain

$$C_4: \quad \widehat{0} < (x_1, y_1) < \dots < (x_{j-1}, y_{j-1}) < (y_{j-1}, y_j) \\ < (x_{j+1}, y_{j+1}) < \dots < (x_n, y_n) < \widehat{1}.$$

This is a chain since $(y_{j-1}, y_j) = L^{\text{sign}(y_j)}(x_{j-1}, y_{j-1})$ is obvious, and (2) implies $(x_{j+1}, y_{j+1}) = L^{\text{sign}(y_{j+1})}(y_{j-1}, y_j)$. Since y_{j-1} is negative, the label of

$$((x_{j-1}, y_{j-1}), (y_{j-1}, y_j)) = ((x_{j-1}, y_{j-1}), L^{\text{sign}(y_j)}(x_{j-1}, y_{j-1}))$$

is less than the label of

$$((x_{j-1}, y_{j-1}), (x_j, y_j)) = ((x_{j-1}, y_{j-1}), R^{\text{sign}(y_j)}(x_{j-1}, y_{j-1})).$$

Therefore C_4 is “earlier” than C , and $[j, j]$ belongs to $\mathcal{I}_C$, in contradiction with the minimality of $[i, j]$. This contradiction may only be avoided if $j \leq i + 1$. $\square$

The last part of the proof of Lemma 5.4 also allows us to observe the following.

Lemma 5.5. *If $j > i$ then $j = i + 1$ and y_i has a positive sign.*

Proof. The fact that $j = i + 1$ is a direct consequence of Lemma 5.4. Assume that y_i has a negative sign. If $(x_{i+1}, y_{i+1}) = R^{\text{sign}(y_{i+1})}(x_i, y_i)$ then we may reach a contradiction similar to the one in the last part of the proof of Lemma 5.4 by considering the maximal chain

$$C_5: \quad \widehat{0} < (x_1, y_1) < \dots < (x_i, y_i) < (y_i, y_{i+1}) < (x_{i+2}, y_{i+2}) < \dots < (x_n, y_n) < \widehat{1},$$

which is earlier than C and demonstrates $[i + 1, i + 1] \in \mathcal{I}_C$, a contradiction with the minimality of $[i, j] = [i, i + 1]$. Hence we may assume $(x_{i+1}, y_{i+1}) = L^{\text{sign}(y_{i+1})}(x_i, y_i)$. Since y_i is negative, either $(x_i, y_i) = R^-(x_{i-1}, y_{i-1})$ or $(x_i, y_i) = L^-(x_{i-1}, y_{i-1})$ holds.

Let (x_i^*, y_i^*) be that element of $\{R^-(x_{i-1}, y_{i-1}), L^-(x_{i-1}, y_{i-1})\}$, which is different from (x_i, y_i) . Consider the maximal chain

$$C_6: \quad \widehat{0} < (x_1, y_1) < \cdots < (x_{i-1}, y_{i-1}) < (x_i^*, y_i^*) \\ < (x_{i+1}, y_{i+1}) < \cdots < (x_n, y_n) < \widehat{1}.$$

The fact that (x_i^*, y_i^*) is less than (x_{i+1}, y_{i+1}) follows from the fact that $L^{\text{sign}(y_{i+1})}(x_i^*, y_i^*)$ belongs to $\{L^{\text{sign}(y_{i+1})}R^-(x_{i-1}, y_{i-1}), L^{\text{sign}(y_{i+1})}L^-(x_{i-1}, y_{i-1})\}$, a singleton by Lemma 3.6, and so it is also equal to $(x_{i+1}, y_{i+1}) = L^{\text{sign}(y_{i+1})}(x_i, y_i)$ which belongs to the same singleton. No matter what the signs of x_{i-1} and y_{i-1} are, one of $((x_{i-1}, y_{i-1}), L^-(x_{i-1}, y_{i-1}))$ and $((x_{i-1}, y_{i-1}), R^-(x_{i-1}, y_{i-1}))$ has label 1. That cover relation cannot be $((x_{i-1}, y_{i-1}), (x_i, y_i))$ since otherwise $[i, j] \in \mathcal{I}_C$ is not possible. Hence C_6 is an “earlier” maximal chain than C , demonstrating $[i, i] \in \mathcal{I}_C$. We have reached again a contradiction with the minimality of $[i, j]$. $\square$

Lemma 5.6. *Under our assumptions only $j = i$ is possible.*

Proof. Assume by way of contradiction that $j > i$ holds. Then, by Lemma 5.5, $j = i + 1$ and the sign of y_i is positive. Moreover, by Lemma 5.3, $(x_{i+2}, y_{i+2}) = L^{\text{sign}(y_{i+2})}(x_{i+1}, y_{i+1})$. From here on we distinguish two cases, depending on whether (x_{i+1}, y_{i+1}) is in the image of some L -operator or some R -operator.

Assume first that (x_{i+1}, y_{i+1}) is in the image of an L -operator. Then, by Lemma 3.3, $(x_{i+1}, y_{i+1}) = L^{\text{sign}(y_{i+1})}(x_i, y_i)$. Consider the maximal chain

$$C_7: \quad \widehat{0} < (x_1, y_1) < \cdots < (x_i, y_i) < R^{\text{sign}(y_{i+1})}(x_i, y_i) \\ < (x_{i+2}, y_{i+2}) < \cdots < (x_n, y_n) < \widehat{1}.$$

This is a chain since $L^{\text{sign}(y_{i+2})}R^{\text{sign}(y_{i+1})}(x_i, y_i) = L^{\text{sign}(y_{i+2})}R^{\text{sign}(y_{i+1})} = (x_{i+2}, y_{i+2})$. The label of $((x_i, y_i), R^{\text{sign}(y_{i+1})}(x_i, y_i))$ is smaller than the label of $((x_i, y_i), L^{\text{sign}(y_{i+1})}(x_i, y_i))$, since y_i is positive. Hence C_7 is an “earlier” chain, exhibiting the fact that $[i + 1, i + 1] \in \mathcal{I}_C$, in contradiction with the assumed minimality of $[i, j]$.

Thus from now on we may assume $(x_{i+1}, y_{i+1}) = R^{\text{sign}(y_{i+1})}(x_i, y_i)$. Since the sign of y_i is positive, either $(x_i, y_i) = L^+(x_{i-1}, y_{i-1})$ or $(x_i, y_i) = R^+(x_{i-1}, y_{i-1})$ holds. In the first case the maximal chain

$$C_8: \quad \widehat{0} < (x_1, y_1) < \cdots < (x_{i-1}, y_{i-1}) < L^-(x_{i-1}, y_{i-1}) \\ < (x_{i+1}, y_{i+1}) < \cdots < (x_n, y_n) < \widehat{1},$$

in the second case the maximal chain

$$C_9: \quad \widehat{0} < (x_1, y_1) < \cdots < (x_{i-1}, y_{i-1}) < R^-(x_{i-1}, y_{i-1}) \\ < (x_{i+1}, y_{i+1}) < \cdots < (x_n, y_n) < \widehat{1},$$

is “earlier” than C , therefore we reach the contradiction $[i, i] \in \mathcal{I}_C$ in either case. (The last part of Lemma 3.6 may be used to verify that C_8 and C_9 are chains.) $\square$

6. Topological Descents as a Function of the LR -Operators

In this section we identify the topological descents of each maximal chain C , i.e., the set $\{i \in [1, n] : [i, i] \in \mathcal{I}_C\}$. Since there is no possibility of confusion, we just use the term “descent” to stand for “topological descent” in our proofs.

We observe first that every maximal chain of T_n may be uniquely described as a word of length $n+1$ using letters from the alphabet $\{L^-, L^+, R^-, R^+\}$ subject to the restriction that the penultimate letter must be negative, and the last letter must be L^- . Here the i th letter in the code of the chain will stand for the operator that takes (x_{i-1}, y_{i-1}) into (x_i, y_i) . For example, the maximal chain

$$\widehat{0} = (-1, 1) < (1, -2) < (1, 3) < (1, -4) < (-4, -5) = \widehat{1} \quad (3)$$

in T_3 corresponds to the word $L^- R^+ R^- L^-$.

Definition 6.1. We call the word associated to a maximal chain of T_n in the above described way the LR -code of the maximal chain.

In particular, the number of maximal chains in T_n is $2 \cdot 4^{n-1}$. The descents are associated to the *gaps* between the letters and before the first letter. In fact, the element (x_i, y_i) of rank i in C : $\widehat{0} < (x_1, y_1) < \dots < (x_n, y_n) < \widehat{1}$ corresponds to the gap between the letter (at position i) of the operator taking (x_{i-1}, y_{i-1}) into (x_i, y_i) and the letter (at position $(i+1)$) of the operator taking (x_i, y_i) into (x_{i+1}, y_{i+1}) . We may mark the descents using vertical bars between the letters. For the example above we see that the only descent occurs at rank 2, thus we may write $L^- R^+ | R^- L^-$ to mark that descent.

Theorem 6.2. Consider the shelling of $\Delta(T_n \setminus \{\widehat{0}, \widehat{1}\})$ described in Section 5. An element i is a topological descent of a maximal chain C : $\widehat{0} < (x_1, y_1) < \dots < (x_n, y_n) < \widehat{1}$ if and only if one of the following holds for the LR -code of C :

- (i) the letter at position $(i-1)$ is negative, the letter at position i is an R -operator, and the letter at position $(i+1)$ is an L -operator (pattern $\dots X^- R^\varepsilon | L^\eta \dots$),
- (ii) the letter at position $(i-1)$ is positive, the letter at position i is an L -operator, and the letter at position $(i+1)$ is an L -operator (patterns $\dots X^+ L^\varepsilon | L^\eta \dots$ and $L^\varepsilon | L^\eta \dots$), or
- (iii) the letter at position i is positive, and the letter at position $(i+1)$ is an R -operator (pattern $\dots X^+ | R^\varepsilon \dots$).

Proof. The element i is a descent if and only if there exists an “earlier” maximal chain C' : $\widehat{0} < (x_1, y_1) < \dots < (x_{i-1}, y_{i-1}) < (x'_i, y'_i) < (x_{i+1}, y_{i+1}) < (x_n, y_n) < \widehat{1}$. Equivalently there must be an element (x'_i, y'_i) strictly between (x_{i-1}, y_{i-1}) and (x_{i+1}, y_{i+1}) such that the label of $((x_{i-1}, y_{i-1}), (x'_i, y'_i))$ is less than the label of $((x_{i-1}, y_{i-1}), (x_i, y_i))$.

Case 1: (x_{i+1}, y_{i+1}) is in the image of some L -operator. In this case we must have $(x_{i+1}, y_{i+1}) = L^{\text{sign}(y_{i+1})}(x_i, y_i) = L^{\text{sign}(y_{i+1})}(x'_i, y'_i)$, and $y_i = y'_i = x_{i+1}$. Since $y_i = y'_i$, both (x_i, y_i) and (x'_i, y'_i) are in the image of some operator with the same sign $\text{sign}(y_i)$. Hence the set $\{(x_i, y_i), (x'_i, y'_i)\}$ is equal to the set $\{R^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}), L^{\text{sign}(y_i)}(x_{i-1},$

$y_{i-1})\}$. There are two possible ways to match the elements in these two sets. The first possibility is

$$(x_i, y_i) = R^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}) \quad \text{and} \quad (x'_i, y'_i) = L^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}).$$

In this case the maximal chain C' is “earlier” if and only if the sign of y_{i-1} is negative. This is equivalent to saying that (x_{i-1}, y_{i-1}) is in the image of some operator X^- (where $X \in \{L, R\}$), and we recover the pattern $\dots X^- R^\varepsilon | L^\eta \dots$. The second possibility is

$$(x_i, y_i) = L^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}) \quad \text{and} \quad (x'_i, y'_i) = R^{\text{sign}(y_i)}(x_{i-1}, y_{i-1}).$$

In this case the maximal chain C' is “earlier” if and only if the sign of y_{i-1} is positive. This is equivalent to saying that (x_{i-1}, y_{i-1}) is in the image of some operator X^+ (where $X \in \{L, R\}$), or that $(x_{i-1}, y_{i-1}) = (-1, +1)$ and we recover the patterns $\dots X^+ L^\varepsilon | L^\eta \dots$ and $L^\varepsilon | L^\eta \dots$.

Case 2: (x_{i+1}, y_{i+1}) is in the image of some R -operator. In this case we must have $(x_{i+1}, y_{i+1}) = R^{\text{sign}(y_{i+1})}(x_i, y_i) = R^{\text{sign}(y_{i+1})}(x'_i, y'_i)$ and $x_i = x'_i = x_{i+1}$. The only way for (x_i, y_i) and (x'_i, y'_i) to be different is by y_i and y'_i having opposite signs. Hence the set $\{(x_i, y_i), (x'_i, y'_i)\}$ is either equal to $\{R^-(x_{i-1}, y_{i-1}), R^+(x_{i-1}, y_{i-1})\}$ or it is equal to $\{L^-(x_{i-1}, y_{i-1}), L^+(x_{i-1}, y_{i-1})\}$. Either way, C' is “earlier” than C exactly when (x_i, y_i) is in the image of the positive-signed operator. We recover the pattern $\dots X^+ | R^\varepsilon \dots$. $\square$

Using Theorem 6.2 it is relatively easy to recover the flag h -vector of T_n directly, but for the sake of calculating the cd -index we observe the following. The two patterns in case (ii) may be merged into one pattern using the fact that, by Proposition 3.7, the poset T_n isomorphic to both intervals $[(1, -2), \widehat{1}]$ and $[(1, 2), \widehat{1}]$ of T_{n+1} . Depending on which way we embed T_n into T_{n+1} , we may convert our codes for the maximal chains of T_n to words of length $n + 1$ by adding an initial letter L^- or L^+ in front of each word. This yields the *negative augmented LR-code*, resp. *positive augmented LR-code*, of the maximal chain. In either case, descents correspond to the gaps between letters, except for the gap between the first two letters, and the description of the descent-yielding patterns may be simplified to

$$\dots X^- R^\varepsilon | L^\eta \dots, \quad \dots X^+ L^\varepsilon | L^\eta \dots, \quad \text{or} \quad \dots X^+ | R^\varepsilon \dots \quad (4)$$

The positive augmented LR -code yields the same descent-association as the short LR -code, while the negative yields a different descent-assignment. For example, for the maximal chain (3) we obtain the negative augmented LR -code $L^- L^- R^+ | R^- L^-$ and the descent set $\{2\}$, as marked. However, if we sum up the descents associated to the negative augmented LR -code for all maximal chains of T_n , we obtain the same flag h -vector statistic, since we end up computing the flag h vector of the same poset. We use both codes and take the average.

Corollary 6.3. *The flag h -vector of T_n may be calculated as follows:*

1. Write down all words $u = u_0 u_1 \dots u_{n+1}$ of length $n + 2$ using the alphabet $\{L^-, L^+, R^-, R^+\}$ satisfying $u_0 \in \{L^-, L^+\}$, $u_n \in \{L^-, R^-\}$, and $u_{n+1} = L^-$.

2. Number the gaps between the letters left to right with the numbers $1, \dots, n$, omitting the first gap. (Gap number i follows the letter u_i .)
3. Mark the descents wherever any of the patterns listed in (4) occurs.
4. $h_S(T_n)$ is half of the number of words for which exactly the gaps associated to the set $S \subseteq [1, n]$ are marked as descents.

7. The cd -Index of T_n

The use of Corollary 6.3 provides an easy way to calculate the cd -index of T_n .

Theorem 7.1. *The coefficient of the cd -word $w = c^{k_1}dc^{k_2}d \cdots dc^{k_r+1}$ in the cd -index of T_n is*

$$2^{\deg_d(w)}(k_1 + 1) \cdots (k_r + 1).$$

Here $\deg_d(w)$ denotes the number of d 's occurring in the cd -word w , and any of the k_i 's is allowed to be zero.

Proof. Consider the set $W_n^\pm$ of all words described in Corollary 6.3. The ab -index of T_n , using its definition and Corollary 6.3, may be described as follows. For each word $u_0 \cdots u_{n+1} \in W_n^\pm$ write a b below the gaps marked as descents, and a letter a to all other gaps, except for the gap between u_0 and u_1 . One-half times the sum of the associated ab -words over all words from $W_n^\pm$ yields the ab -index.

Switching the sign of any letter except for the sign of u_n and u_{n+1} takes $W_n^\pm$ into itself. Allowing all such sign switches induces a $\mathbb{Z}_2^n$ -action on $W_n^\pm$. Each equivalence class under this action may be characterized by a word of length $n + 2$ on the (unsigned) alphabet $\{L, R\}$ starting and ending with the letter L . (The sign of the last two letters may be dropped, since they have to be negative; the sign of the other letters may be ignored due to the $\mathbb{Z}_2^n$ -action.) We claim that each equivalence class contributes the constant multiple of exactly one cd -monomial w to the cd -index. The coefficient of the contributed cd -monomial will be $2^{\deg_d(w)-1}$. (Different equivalence classes may contribute the same $2^{\deg_d(w)}w$.) To see this, consider an arbitrary (signed) LR -word $u = u_0 \cdots u_{n+1} \in W_n^\pm$. We determine the effect of choosing the sign of u_i on the descent set where $0 \leq i \leq n$. Observe that the choice of the sign of u_n will not change the position of the descent, hence we may focus on the remaining u_i 's, each of which is followed by at least two letters.

Case 1: u_{i+1} is of the form R^ε and u_{i+2} is of the form L^η . According to the list of descent patterns (4), depending on the choice of the sign of u_i we see either the descent pattern $X^-R^\varepsilon|L^\eta$ or the descent pattern $X^+|R^\varepsilon L^\eta$ at positions $i, i + 1, i + 2$, and the sign of u_i alone will determine which of the gaps numbered i and $i + 1$, respectively, is a descent. Note also that the case $i = 0$ is special, since the gap between u_0 and u_1 is ignored for the purposes of the descent statistics. Choosing the sign of u_i will have no effect whatsoever on the existence of a descent at any other gap. Hence summing over the sign choices of u_i will contribute a factor of $ab + ba = d$ covering the gaps numbered i and $i + 1$ if $i > 0$ and a factor of $a + b = c$ for $i = 0$.

Case 2: both $u_{i+1} = L^\varepsilon$ and $u_{i+2} = L^\eta$ are signed L -letters. For these the sign of u_i will determine whether the descent pattern $X^+L^\varepsilon|L^\eta$ occurs or not at positions $i, i+1, i+2$, and will have no effect on the existence of descents at any other positions. Hence summing over the sign choices of such a u_i will contribute a factor $a+b=c$ covering gap number $i+1$.

Case 3: u_i is followed by two signed R -letters, $u_{i+1} = R^\varepsilon$ and $u_{i+2} = R^\eta$. The sign of such u_i 's will determine alone whether there is a descent between u_i and u_{i+1} (according to the pattern $X^+|R^\varepsilon$) and will have no effect whatsoever on the existence of any other descents. Hence such u_i 's contribute a factor of $a+b=c$ covering gap number i . Again the case $i=0$ is special, since the gap between u_0 and u_1 is ignored in the descent statistics.

Case 4: u_i is followed by $u_{i+1} = L^\varepsilon$ and $u_{i+2} = R^\eta$. In this case the sign of u_i has no bearing whatsoever on the existence of any descent, summing over the sign choices of u_i yields a factor of 2.

Observe also that that in any of the patterns listed in (4), switching the sign of the letter denoted by X makes the descent in question disappear. In fact, there is no descent between R^ε and L^η in a pattern $\dots X^+R^\varepsilon L^\eta \dots$, no descent between L^ε and L^η in a pattern $\dots X^-L^\varepsilon L^\eta \dots$, and no descent before R^ε in a pattern X^-R^ε . Moreover, in all patterns the letter X is followed by two letters, except for the last pattern, where we know that X^+ cannot be u_n , since the last letter must always be L^- . This means that the existence of any descent is controlled by the sign of some u_i followed by at least two letters, and the four cases above cover the generation (and “disappearance”) of all descents.

What we obtained is that the equivalence class of u contributes a multiple of that cd -word w in which the letters d cover exactly the gaps before and after those $u_{i+1} = R^\varepsilon$ satisfying $1 \leq i \leq n-1$, which are followed by a $u_{i+2} = L^\eta$. The coefficient of the cd -word will be $\frac{1}{2}$ times 2 raised to the power of the number of those u_i 's whose sign has no influence on the descent structure. Observe that the sign of every u_i associated to a letter d determines the existence of descents at two positions, making the role of one u_i in determining the descent structure unnecessary, while the u_i 's associated to a letter c determine the existence of a descent at one position. Hence the number of u_i 's whose sign has no influence on the descent structure is the number of d 's, and the coefficient of w is exactly $2^{\deg_d(w)-1}$.

Finally we need to answer the question, how many equivalence classes yield the same cd -word? Each equivalence class may be uniquely represented by a word $v = v_0 \cdots v_{n+1}$ on the (unsigned) alphabet $\{L, R\}$ that satisfies $v_0 = v_{n+1} = L$. From what was said above, it is clear that an equivalence class contributes $2^{\deg_d(w)} \cdot w$ for some cd word $w = c^{k_1} d c^{k_2} d \cdots d c^{k_{r+1}}$ if and only if the gaps before and after the letters v_i satisfying $v_i = R, i > 2$, and $v_{i+1} = L$, mark exactly the beginnings of the d 's in w . For example, the term $2^{2-1} \cdot d c c d c$ is contributed by the following equivalence classes of maximal chains in T_7 : $LLRLLLRL$, $LLRLLRRL$, $LLRLRRRL$, $LRRLLLRL$, $LRRLLRRL$, and $LRRLRRRL$. Here we may observe that whenever there is a c^{k_j} between two d 's, then that corresponds to k_j letters between two RL patterns. These k_j letters may contain any number of R 's between 0 and k_j , but once we fix this number there is a unique way of filling in the letters since we are not allowed to create a new

Table 1. The cd -index of T_n for $n \leq 5$.

$\Psi_{cd}(T_0) = 1$
$\Psi_{cd}(T_1) = c$
$\Psi_{cd}(T_2) = c^2 + 2d$
$\Psi_{cd}(T_3) = c^3 + 4cd + 2dc$
$\Psi_{cd}(T_4) = c^4 + 6c^2d + 4cdc + 2dc^2 + 4d^2$
$\Psi_{cd}(T_5) = c^5 + 8c^3d + 6c^2dc + 4cdc^2 + 2dc^3 + 8cd^2 + 8dcd + 4d^2c$

RL pattern, and so the R 's must be right-adjusted. This gives us $(k_j + 1)$ independent options. Concerning the letters before the first RL pattern marking a d we may make almost the same observation, but we also have to note that we are always allowed to choose the letter v_1 to be either R or L , since $v_1 = R, v_2 = L$ is an RL pattern that does not induce a letter d . For example, the term ccd is contributed by the following equivalence classes of maximal chains in T_4 : $LLLLRL, LLLRRL, LLRRRL, LRRRRL; LRLRL, LRLRRL$. If there is a c^{k_1} before the first d , then the corresponding RL pattern is preceded by $v_0v_1 \cdots v_{k+2}$. Here $v_0 = L$, we have two choices to set v_1 , and $k_1 + 1$ choices to choose the number of (right-adjusted) R letters among $v_2 \cdots v_{k+2}$. This yields a factor of $2 \cdot (k_1 + 1)$. Note finally that all letters after the last RL pattern must be L 's.

Therefore there are $2(k_1 + 1) \cdots (k_r + 1)$ equivalence classes contributing to the same cd monomial $w = c^{k_1}dc^{k_2}d \cdots dc^{k_r+1}$, and each such class contributes $2^{\deg_d(w)-1}$ to its coefficient. $\square$

The cd -index of T_n for the first few values of n is provided in Table 1.

Remark 7.2. With some effort, it is also possible to deduce the cd -index formulas of this section from the ce -index formulas that are shown in Section 8, without using shelling. However, it is always an intriguing challenge to describe the cd -index of a class of “symmetric” posets using some sort of statistics on words. The earliest example of such a calculation may be found in Purtill’s paper [27] where he calculated the cd -index of the Boolean algebras using ascent–descent statistics studied by Foata, Schützenberger, and Strehl in [16]–[19]. The existence of such interactions still needs a better explanation.

8. The ce -Index and the Flag f -Vector of T_n

As a consequence of Proposition 3.7, we obtain an easy recursion formula for the ce -index of T_n .

Proposition 8.1. The ce -index $\Psi_{ce}(T_n)$ of T_n satisfies the recursion formula

$$\Psi_{ce}(T_n) = 2c\Psi_{ce}(T_{n-1}) - e^2\Psi_{ce}(T_{n-2}) \quad \text{for } n \geq 2.$$

Proof. As is known (see, e.g., Stanley’s paper [29, Formula (5)]), the ab -index of an Eulerian poset of rank $n + 1$ may also be written as $\sum_{S \subseteq [1, n]} f_S \cdot v_S$ where $v_S = v_1 \cdots v_n$

and

$$v_i = \begin{cases} b & \text{if } i \in S, \\ a - b & \text{if } i \notin S. \end{cases}$$

We apply this formula to T_n , and split the sum into two parts depending on whether $S \subseteq [1, n]$ contains 1 as an element.

Case 1: $1 \in S$. Since T_n has four atoms, and above each atom, according to Proposition 3.7, there is a copy of T_{n-1} , we obtain

$$\sum_{\substack{S \subseteq [1, n] \\ 1 \in S}} f_S(T_n) \cdot v_S = 4b\Psi_{ab}(T_{n-1}).$$

Case 2: $1 \notin S$. All elements of a partial chain not containing any element of rank 1 are either above $(-1, -2)$ or $(+1, +2)$. If a partial chain is above both of these elements, then it contains no element of rank 2 and it is a partial chain in the Tchebyshev poset of

$$Q: -3, 3 < -4, 4 < \dots < -n, n < -(n+1) < -(n+2)$$

that avoids the top element $(-(n+1), -(n+2))$. The Tchebyshev poset of $-3 < 3 < -4, 4 < \dots < -n, n < -(n+1) < -(n+2)$ differs from $T(Q)$ only in having an extra minimum element $(-3, 3)$ and this latter poset is isomorphic to T_{n-2} . Hence summing over all partial chains of $T(Q)$ avoiding its top element has the same effect as summing over all partial chains of T_{n-2} that avoid its top and bottom element. Therefore

$$\sum_{\substack{S \subseteq [1, n] \\ 1 \notin S}} f_S(T_n) \cdot v_S = 2(a-b)\Psi_{ab}(T_{n-1}) - (a-b)^2\Psi_{ab}(T_{n-2}).$$

Summing the contributions of all partial chains from both cases yields

$$\begin{aligned} \Psi_{ab}(T_n) &= (4b + 2a - 2b)\Psi_{ab}(T_{n-1}) - (a-b)^2\Psi_{ab}(T_{n-2}) \\ &= 2(a+b)\Psi_{ab}(T_{n-1}) - (a-b)^2\Psi_{ab}(T_{n-2}), \end{aligned}$$

and so we are done by $c = a + b$ and $e = a - b$. $\square$

Corollary 8.2. *The substitution $c \mapsto x$, $e \mapsto 1$ sends the ce -index of T_n into the Tchebyshev polynomial $T_n(x)$.*

In fact, under this substitution, the $\Psi_{ce}(T_0)$ goes into 1, the $\Psi_{ce}(T_1)$ goes into x , and the recursion formula of Proposition 8.1 goes into the well-known recursion

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

of Tchebyshev polynomials.

Remark 8.3. The substitution $c \mapsto x$, $e \mapsto 1$ sends d into $(x^2 - 1)/2$, and thus Theorem 7.1 yields the following formula for Tchebyshev polynomials:

$$T_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} x^{n-2m} \left(\frac{x^2 - 1}{2} \right)^m \cdot 2^m \sum_{\substack{S \subseteq [1, n] \\ |S|=2m \\ S \text{ even}}} \prod_{\substack{I \in \mathcal{I}([1, n] \setminus S) \\ n \notin I}} (|I| + 1).$$

Here $\mathcal{I}([1, n] \setminus S)$ stands for the collection of longest possible intervals whose disjoint union is $[1, n] \setminus S$. It is also well known that

$$T_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} x^{n-2m} \left(\frac{x^2 - 1}{2} \right)^m \cdot 2^m \binom{n}{2m}.$$

Comparing coefficients yields

$$\binom{n}{2m} = \sum_{\substack{S \subseteq [1, n] \\ |S|=2m \\ S \text{ even}}} \prod_{\substack{I \in \mathcal{I}([1, n] \setminus S) \\ n \notin I}} (|I| + 1).$$

Using Proposition 8.1 it is easy to obtain the following closed formula for the ce -index of T_n .

Theorem 8.4. *The coefficient of a ce -word u in $\Psi_{ce}(T_n)$ is $2^{\deg_c(u)-1}(-1)^{\deg_e(u)/2}$ if the last e in u is followed by at least one c , and $2^{\deg_c(u)}(-1)^{\deg_e(u)/2}$ otherwise. Equivalently*

$$L_S^{n+1}(T_n) = \begin{cases} 2^{n-|S|-1}(-1)^{|S|/2} & \text{if } S \text{ is even and } n \notin S, \\ 2^{n-|S|}(-1)^{|S|/2} & \text{if } S \text{ is even and } n \in S, \\ 0 & \text{if } S \text{ is not even.} \end{cases}$$

The proof is a straightforward induction on n and is left to the reader.

We conclude this section with calculating the flag f -vector of T_n .

Proposition 8.5. *Assume that the elements of $S \subseteq [1, n]$ are $1 \leq s_1 < s_2 < \dots < s_k \leq n$. Then*

$$f_S(T_n) = \begin{cases} 4^{|S|} s_1(s_2 - s_1)(s_3 - s_2) \cdots (s_k - s_{k-1}) & \text{if } n \notin S, \\ \frac{4^{|S|}}{2} s_1(s_2 - s_1)(s_3 - s_2) \cdots (s_k - s_{k-1}) & \text{if } n \in S. \end{cases}$$

Proof. The number $f_S(T_n)$ is the number of increasing chains of the form

$$(\varepsilon_1 j_1, \eta_1(s_1 + 1)) < (\varepsilon_2 j_2, \eta_2(s_2 + 1)) < \dots < (\varepsilon_k j_k, \eta_k(s_k + 1)),$$

where the ε_i 's and η_j 's are signs. Assume first that $n \notin S$, i.e., $s_k < n$, and count the number of such increasing chains by filling it from the bottom up. The possible values of j_1 are $1, 2, \dots, s_1$. There are four ways to choose the signs ε_1 and η_1 , so $(\varepsilon_1 j_1, \eta_1(s_1 + 1))$ may be chosen $4s_1$ ways. Suppose now we have already chosen $(\varepsilon_1 j_1, \eta_1(s_1 + 1)), (\varepsilon_2 j_2, \eta_2(s_2 + 1)), \dots, (\varepsilon_{i-1} j_{i-1}, \eta_{i-1}(s_{i-1} + 1))$. The possible values of j_i are $j_{i-1}, s_{i-1} + 1, s_{i-1} + 2, \dots, s_i$. If $j_i = j_{i-1}$, then we must have $\varepsilon_i = \varepsilon_{i-1}$ and if $j_i = s_{i-1}$ then we must have $\varepsilon_i = \eta_{i-1}$. In all other cases there are two ways to choose ε_i . There are always two ways to choose η_i , so choosing $(\varepsilon_i j_i, \eta_i(s_i + 1))$ contributes a factor of $2 + 2 + 4(s_i - s_{i-1} - 1) = 4(s_i - s_{i-1} - 1)$. Multiplying the numbers of choices we made we obtain exactly the stated formula.

The only difference arising when $n \in S$ is that the sign η_k of $s_k + 1 = n + 1$ must be $-$, and so we have only half as many options to choose the last element of our increasing chain. $\square$

9. Extremal Property of the Tchebyshev Posets

The main result of this section is the following extremal property of the Tchebyshev poset T_d .

Theorem 9.1. *For $d \geq 1$, the f -vector $(f_{-1}, f_0, \dots, f_{d-1})$ of an arbitrary $(d-1)$ -dimensional simplicial complex Δ satisfies the inequality*

$$\max_{-1 \leq x \leq 1} \left| \frac{2^d \sum_{j=0}^d f_{j-1} ((x-1)/2)^j}{f_{d-1}} \right| \geq \frac{1}{2^{d-1}}.$$

Equality holds for $\Delta(T_d \setminus \{\widehat{0}, \widehat{1}\})$.

We begin the proof of Theorem 9.1 by collecting the terms of the polynomial $\sum_{j=0}^d f_{j-1} ((x-1)/2)^j$ and denoting the coefficient of x^{d-i} by L_i . We call the vector $(L_0, \dots, L_d)$ obtained this way the L -vector of Δ .

Definition 9.2. The L -vector $(L_0, L_1, \dots, L_d)$ of Δ is given by $L_i = (-1)^{d-i} \sum_{j=d-i}^d f_{j-1} \binom{j}{d-i}$.

Rewriting $\sum_{i=0}^d L_i x^{d-i} = \sum_{j=0}^d f_{j-1} ((x-1)/2)^j$ in terms of the variable $y = (x-1)/2$, and comparing the coefficients of y^j yields

$$f_{j-1} = 2^j \sum_{i=0}^{d-j} \binom{d-i}{j} L_i.$$

Hence the L -vector is a linearly equivalent encoding of the f -vector. It is easy to verify that the order complex of an arbitrary graded poset satisfies

$$L_i = \sum_{|S|=i} L_S. \quad (5)$$

As a consequence of (5) and Corollary 8.2 we obtain that for $\Delta(T_d \setminus \{\widehat{0}, \widehat{1}\})$ the polynomial $\sum_{j=0}^d f_{j-1} ((x-1)/2)^j$ is the Tchebyshev polynomial $T_d(x)$, with leading coefficient 2^{d-1} . For a general simplicial complex of dimension $(d-1)$ the leading coefficient of $\sum_{j=0}^d f_{j-1} ((x-1)/2)^j$ is $L_0 = (\frac{1}{2})^d f_{d-1}$, and so the leading coefficient of

$$\frac{2^{2d-1} \sum_{j=0}^d f_{j-1} ((x-1)/2)^j}{f_{d-1}}$$

is 2^{d-1} . As is well known, polynomials $f(x)$ of degree d with leading coefficient 2^{d-1} satisfy $\max_{-1 \leq x \leq 1} |f(x)| \geq 1$ with equality when $f(x) = T_d(x)$. Therefore we have

$$\max_{-1 \leq x \leq 1} \left| \frac{2^{2d-1} \sum_{j=0}^d f_{j-1} ((x-1)/2)^j}{f_{d-1}} \right| \geq 1.$$

Dividing both sides by 2^{d-1} yields the statement of Theorem 9.1.

10. Concluding Remarks

Perhaps the most important future use of Tchebyshev posets could be the investigation of a new special instance of Stanley's conjecture on the non-negativity of the cd -index of Gorenstein* posets. In Conjecture 2.1 of [30] Stanley made the conjecture that every Eulerian poset whose order complex has the Cohen–Macaulay property, i.e., every Gorenstein* poset, has a non-negative cd -index. In the same paper Stanley proved his conjecture (using spherical shelling) for boundary complexes of polytopes and (by reducing the statement to known results about simplicial spheres) for simplicial posets. In [14] Ehrenborg and Hetyei gave an analogous description of the cd -index of Eulerian cubical posets. Using our description, the verification of Stanley's conjecture in the case of cubical posets is reduced to the question of non-negativity of Adin's h -vector (defined in [1]) for Cohen–Macaulay cubical complexes.

In analogy to the case of simplicial and cubical posets, one could restrict attention to those Eulerian posets P for which every interval $[0, x]$ is isomorphic to the dual of a Tchebyshev poset T_n . If P is the face poset of a CW -complex then this complex is given by the property that each of its cells is a dual Tchebyshev cell.

Conjecture 10.1. *In the case of CW -complexes of dual Tchebyshev cells, Stanley's conjecture may be reduced to a non-negativity statement on an (appropriately defined) h -vector. Just like the simplicial h -vector, and Adin's cubical h -vector, this new h -vector will be a linear recoding of the f -vector of the complex.*

Two key issues are likely to make the treatment of such complexes more difficult than that of its simplicial or cubical “cousins”. First, both simplicial and cubical h -vectors may be defined in relation to shellings. This is unlikely to happen to “complexes of dual Tchebyshev cells”, since dual Tchebyshev cells themselves are non-polytopal and have only four facets in any dimension, thus the number of types of shelling components is not likely to grow with the dimension, while the dimension of the vector space generated by the f -vectors probably will. Second, the *Dehn–Sommerville equations* for cubical and simplicial spheres were already known to Grünbaum [20]. In terms of the cubical and simplicial h vectors, these equations may be rewritten as $h_i = h_{d-i}$. It is not yet known whether the dimension of the vector space of f -vectors of d -dimensional “spheres of dual Tchebyshev cells” would be essentially $d/2$. It is easy to write up that many (linearly independent) relations for the f -vector, but constructing that many examples with linearly independent f -vectors seems to be a daunting task, again because of the “lack of polytopality”.

On the other hand “spheres of dual Tchebyshev cells” are still much more special than Gorenstein* posets in general. Proving Stanley's conjecture for them, together with the simplicial case, and with examples of other constructions of Gorenstein* posets, e.g., Hsiao's work [24], could perhaps inspire a long-awaited “common generalization”.

Finally, we wish to present a challenge to experts of shellings and of polyhedral geometry: is there a way to use the geometric realization of $\Delta(T_n \setminus \{\widehat{0}, \widehat{1}\})$ to find a line shelling of the order complex? Although our representation is not a simplicial polytope, looking at Fig. 2, it seems to be hard to decide whether there is a special way to pull a few vertices and transform the picture into a genuine simplicial polytope.

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