Approximation of the Normal Vector Field and the Area of a Smooth Surface*

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Abstract. This paper deals with the comparison of the normal vector field of a smooth surface S with the normal vector field of another surface differentiable almost everywhere. The main result gives an upper bound on angles between the normals of S and the normals of a triangulation T close to S . This upper bound is expressed in terms of the geometry of T , the curvature of S and the Hausdorff distance between both surfaces. This kind of result is really useful: in particular, results of the approximation of the normal vector field of a smooth surface S can induce results of the approximation of the area; indeed, in a very general case (T is only supposed to be locally the graph of a lipschitz function), if we know the angle between the normals of both surfaces, then we can explicitly express the area of S in terms of geometrical invariants of T , the curvature of S and of the Hausdorff distance between both surfaces. We also apply our results in surface reconstruction: we obtain convergence results when T is the restricted Delaunay triangulation of an ε -sample of S ; using Chew's algorithm, we also build sequences of triangulations inscribed in S whose curvature measures tend to the curvatures measures of S .

1. Introduction

In this paper we are interested in the relationship between the geometrical properties of a smooth surface S and the geometrical properties of another surface “close” to S .

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We wonder whether the geometrical properties of S can be well approximated by the “approximating” surface. The normal vector field is an important property: indeed, we show that if the angle between the normal vector field of a smooth surface and the normal vector field of a locally lipschitz surface (i.e. a surface that is locally the graph of a lipschitz function) M is known, then we can explicitly express the area of S in terms of geometrical invariants of M , the curvature of S and the Hausdorff distance between both surfaces.

Our main result gives an upper bound on the maximal angle $\alpha_{\max}$ between the normals of a smooth surface S and the normals of a triangulation T close to it that depends on the geometry of T , the curvature of S and the Hausdorff distance between both surfaces. It is worth noting that this upper bound can be small even if some triangle’s angles are very small. As corollaries, we bound $\alpha_{\max}$ from above with the largest circumradius of the triangles of T (up to a constant that only depends on S) and we get the convergence’s results.

These results are very general and must be compared with [14]. In the context of surface reconstruction from scattered sample points, Amenta et al. study the particular case of ε -samples (see [2] and [3]) and get a similar approximation: they build a triangulation (based on an ε -sample of S) whose triangle’s circumradii are small and they deduce an approximation of the normals of S . In [10] Fu has results of convergence of the curvature measures of a sequence of triangulations to the curvature measure of a smooth surface. Our results only deal with the normal vector field and with the area, but they are quantitative and more precise: we compare explicitly two surfaces and we get the convergence’s results as corollaries.

We give applications of our results in the theory of surface reconstruction from scattered sample points: in particular, if S is an ε -sample of a (closed) surface S (in the sense of [1]), then the limit of the area $\mathcal{A}(T_\varepsilon)$ of the Delaunay triangulation T_ε restricted to the ε -sample S satisfies

$$\lim_{\varepsilon \rightarrow 0} \mathcal{A}(T_\varepsilon) = \mathcal{A}(S).$$

This paper is organised as follows. Section 3 states the results of the approximation of the normals of a smooth surface with the normals of a triangulation close to it. In Section 4 we first give the results of the approximation of the area of a smooth surface with the area of a locally lipschitz surface. As a corollary (and by using the results of Section 3) we deduce the results of the approximation when the approximating surface is a triangulation. Section 5 deals with applications in surface reconstruction: we prove a result of convergence concerning restricted Delaunay triangulations; using Chew’s algorithm, we produce a sequence of triangulations which are close to S , and whose area tends to the area of S . Sections 6 and 7 sketch the proofs.

The background on differential geometry can be found in [4], [8], [12], and [15].

2. Definitions

We recall here some classical definitions which concern smooth surfaces, triangulations and the relative position of two surfaces. For more details on triangulations, one may refer to [10] and [6].

2.1. Smooth Surfaces

In the following, a smooth surface means a $\mathcal{C}^2$ surface which is regular, oriented, with or without boundary. Let S be a smooth surface of the (oriented) Euclidean space E^3 . Let ∂S denote the boundary of S . S is endowed with the Riemannian structure induced by the standard scalar product of E^3 . We denote by da the area form of S and by ds the canonical orientation of ∂S . Let ν be the unitary normal vector field (compatible with the orientation of S) and let h be the second fundamental form of S associated with ν . Its determinant at a point p of S is the Gauss curvature $G(p)$, its trace is the mean curvature $H(p)$. The maximal curvature of S at p is $\rho_p = \max(|\lambda^1(p)|, |\lambda^2(p)|)$, where $\lambda_1(p)$ and $\lambda_2(p)$ are the eigenvalues of the second fundamental form at p . Finally, the maximal curvature of S is

$$\rho_S = \sup_{p \in S} \rho_p.$$

We need the following:

Proposition 1. *Let S be a smooth compact (oriented) surface of E^3 . Then there exists an open set U_S of E^3 containing S and a continuous map ξ from U_S onto S satisfying the following property: if p belongs to U_S , then there exists a unique point $\xi(p)$ realising the distance from p to S (ξ is nothing but the orthogonal projection onto S).*

A proof of this proposition can be found in [9].

The open set U_S depends locally and globally on the smooth surface S . Locally, the reach is less than the radius of curvature. Globally, U_S depends on points which can be far from one another on the surface, but close in E^3 .

We shall also need the notion of the *reach of a surface*, introduced in [9].

Definition 1. The reach of a surface S is the largest $r > 0$ for which ξ is defined on the (open) tubular neighbourhood $U_r(S)$ of radius r of S .

This implies that the reach r_S of S is smaller than the minimal radius of curvature of S . Thus, we have

$$\rho_S r_S \leq 1.$$

2.2. Triangulations

A *triangulation* T is a topological surface which is a (finite and connected) union of triangles of E^3 , such that the intersection of two triangles is either empty, or equal to a vertex, or equal to an edge.

We denote by T_T the set of triangles of T and by Δ a generic triangle of T .

- $\eta(\Delta)$ denotes the length of the longest edge of Δ , and $\mathcal{A}(\Delta)$ its area.
- The *fatness* of Δ is the real number

$$\theta(\Delta) = \frac{\mathcal{A}(\Delta)}{\eta(\Delta)^2}.$$

Fig. 1. Rightness of a triangle Δ .

- We introduce the notion of *rightness* of a triangle Δ (Fig. 1). It is the real number

$$\text{rig}(\Delta) = \sup_{p \text{ vertex of } \Delta} |\sin(\theta_p)|,$$

where θ_p is the angle at p of Δ .

Globally:

- The *area* $\mathcal{A}(T)$ is the sum of the areas of all the triangles of T .
- The *height* of T is

$$\eta(T) = \sup_{\Delta \in T_T} \eta(\Delta).$$

- The *fatness* of T is

$$\theta(T) = \min_{\Delta \in T_T} \theta(\Delta).$$

- The *rightness* of T is

$$\text{rig}(T) = \min_{\Delta \in T_T} \text{rig}(\Delta).$$

Note that the rightness is a weaker condition than the fatness. The rightness of a triangle is large if there exists at least one angle whose sinus is large. The fatness of a triangle is large if the sinus of every angle is large.

2.3. Surface Closely Near a Smooth Surface

We introduce the following definitions:

- A subset M of E^3 is *closely near* a smooth surface S if it lies in $U_r(S)$, where r is the reach of S and if the restriction of ξ to M is one-to-one.
- Let M be a subset closely near a smooth surface S . The *relative curvature of M with respect to S* is the real number defined by

$$\omega_S(M) = \sup_{m \in M \setminus \partial M} \|\xi(m) - m\| \rho_{\xi(m)}.$$

- If M is a surface closely near S and differentiable almost everywhere, one can define at almost every point m of M the angle $\alpha_m \in [0, \pi/2]$ between the tangent planes $T_m M$ and $T_{\xi(m)} S$. We put

$$\begin{aligned} \alpha_{\max} &= \max_{m \in M} \alpha_m, \\ \alpha_{\min} &= \min_{m \in M} \alpha_m. \end{aligned}$$

$\alpha_{\max}$ is called *the maximal angle between the normals of S and M* .

Remark. If M lies in $U_r(S)$, then $\omega_S(M) \leq 1$ (this is due to the fact that for every $m \in M$, the point $m - 2\overrightarrow{m\xi(m)}$ —which is the orthogonal symmetric of m with respect to the tangent plane of S at $\xi(m)$ —belongs to $U_r(S)$).

- We say that a triangulation of E^3 is *inscribed* in a smooth surface S if all its vertices belong to S .
- A triangulation T is *closely inscribed* in a smooth surface S if:
 1. T is closely near S ,
 2. all vertices of T belong to S ,
 3. all vertices of ∂T belong to ∂S .
- The *relative height* of T with respect to S is the real number defined by

$$\pi_S(T) = \sup_{\Delta \in T} \sup_{m \in \Delta} \eta(\Delta) \rho_{\xi(m)}.$$

Remark. Let T be a triangulation closely inscribed in a smooth surface S . We have

$$\omega_S(T) \leq \pi_S(T).$$

3. An Approximation on the Normal Vector Field

In this section we compare the behaviour of the normal vector field of a smooth surface S with the normal of each face of a triangulation which is closely inscribed in S . We express the approximation in terms of the geometric invariants that we introduced in the previous section.

Theorem 1. *Let S be a smooth surface and let T be a triangulation closely inscribed in S . Then the maximal angle $\alpha_{\max}$ between the normals of S and T satisfies*

$$\sin \alpha_{\max} \leq \left(\frac{\sqrt{10}}{2 \operatorname{rig}(T)(1 - \omega_S(T))} + \frac{1}{1 - \omega_S(T)} \right) \pi_S(T).$$

Corollary 1. *Let S be a smooth surface and let T be a triangulation closely inscribed in S . If*

$$\pi_S(T) \leq \frac{1}{2},$$

then the maximal angle $\alpha_{\max}$ between the normals of S and T satisfies

$$\sin \alpha_{\max} \leq \left(\frac{4}{\operatorname{rig}(T)} + 2 \right) \pi_S(T).$$

Corollary 2. *Let S be a (compact orientable) C^2 surface in E^3 . Let T_n be a sequence of triangulations closely inscribed in S . If*

1. *the length of the edges of T_n tends to zero when n goes to infinity,*
2. *the rightness of T_n is (uniformly) bounded from below by a positive constant,*

then the normals of T_n tend to the normals of S .

Since S is compact, the second condition may be weakened by asking that $\pi_S(T_n)$ tends to zero when n goes to infinity (*in some sense, the length of the edges may be “large” when the curvature is “small”...*).

Remark 1. The geometrical quantities $\pi_S(T)$ and $\text{rig}(T)$ are linked to the circumradii of the triangles of T . If Δ is a triangle of T , and if R_Δ denotes the circumradius of Δ , then we have

$$\rho_{\xi(\Delta)} R_\Delta = \frac{\rho_{\xi(\Delta)} \eta_\Delta}{2 \text{rig}(\Delta)} = \frac{\pi_\Delta(\xi(\Delta))}{2 \text{rig}(\Delta)}.$$

This implies the following:

Corollary 3. *Let S be a smooth surface and let T be a triangulation closely inscribed in S . We get*

$$\alpha_{\max} \leq K(S) \left(\sup_{\Delta \text{ triangle of } T} R_\Delta \right),$$

where $K(S)$ is a constant depending on S and R_Δ is the circumradius of Δ .

4. Approximation of the Area

This section deals with the approximation of the area of a smooth surface S with the area of a surface M close to it.

1. In Section 4.1 we show that if a surface M differentiable almost everywhere is close enough to a smooth surface S , and if the normals of M are close to the normals of S , then the area of M is close to the area of S . In other words, this means that the approximation of the normals induces results of the approximation of the area.
2. In Section 4.2 we deduce from Section 4.1 and from Section 3 results of the approximation of the area when M is a triangulation.

4.1. Locally Lipschitz Surfaces

The following result shows that if M is closely near S and has enough regularity to have a tangent plane almost everywhere, then the area of S is bounded from above and from below by quantities depending on *the area of M , the Hausdorff distance between M and S , the curvature of S and the angle between the corresponding tangent planes*.

A surface is said to be *locally lipschitz* if it is locally the graph of a lipschitz function. The chosen framework is the class of locally lipschitz surfaces (*these surfaces are differentiable almost everywhere* [13]; *this class contains all the triangulations*).

Theorem 2. *Let S be a (compact orientable) C^2 surface in E^3 . Let M be a locally lipschitz surface closely near S . Then the area satisfies*

$$\mathcal{A}(S) = \int_M \frac{\cos \alpha_m}{1 + \delta_m \varepsilon_m H_{\xi(m)} + \delta_m^2 G_{\xi(m)}} da_M(m),$$

where $\delta_m = \|\xi(m) - m\|$ and $\varepsilon_m \in \{-1, +1\}$ is the scalar product between $v_{\xi(m)}$ and $(\xi(m) - m)/\|\xi(m) - m\|$.

The Hausdorff distance δ_{Hauss} between two subsets A and B of E^3 is defined by

$$\delta_{\text{Hauss}}(A, B) = \max \left(\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right).$$

Corollary 4. *Let M be a locally lipschitz surface closely near S .*

1. *Then*

$$\frac{\cos \alpha_{\max}}{(1 + \omega_S(M))^2} \mathcal{A}(M) \leq \mathcal{A}(S) \leq \frac{\cos \alpha_{\min}}{(1 - \omega_S(M))^2} \mathcal{A}(M).$$

2. *In particular,*

$$|\mathcal{A}(S) - \mathcal{A}(M)| \leq K(S)(\alpha_{\max}^2 + \omega_S(M))$$

and

$$|\mathcal{A}(S) - \mathcal{A}(M)| \leq K(S)(\alpha_{\max}^2 + \delta_{\text{Hauss}}(S, M)),$$

where $K(S)$ is a constant depending on S .

From the previous corollary we deduce the following:

Corollary 5. *Let S be a (compact orientable) C^2 surface in E^3 . Let M_n be a sequence of locally lipschitz surfaces closely near S . If*

1. *the Hausdorff limit of M_n is S , when n goes to infinity,*
2. *the angle between tangent planes $T_m M_n$ and $T_{\xi(m)} S$ tends to zero almost everywhere, when n goes to infinity,*

then

$$\lim_{n \rightarrow \infty} \mathcal{A}(M_n) = \mathcal{A}(S).$$

We now present two examples showing the relevance of the introduced geometric quantities:

1. First, consider two spheres S and M with same centres but different radii, 1 and R (Fig. 2). The angle between the normals of S and M is 0, but the relative curvature of M with respect to S is equal to $1 - R$. If R tends to 0, then $\omega_S(M)$ tends to 1 and the area of M to 0.
2. Consider now a flat surface S , and a sequence $(M_n)_{n \geq 1}$ of locally lipschitz surfaces such that $\omega_S(M_n)$ is equal to 0 and $\alpha_{\max}$ is equal to $\pi/4$ (Fig. 3). The area of M_n is equal to $\sqrt{2}$ times the area of S .

Fig. 2. $\text{Cos}\alpha_{\max} = 1$ and $\omega_S(M)$ is large.

4.2. Approximations with Triangulations

In Section 4.1 we got an approximation of the area of a smooth surface by the area of an almost-differentiable surface close to it, in terms of the angles between the corresponding normals. Using this result and the approximation of the normals (Corollary 1), we get the following:

Corollary 6. *Let S be a (compact orientable) C^2 -surface in E^3 and let T be a triangulation closely inscribed in S . If*

$$\left(\frac{4}{\text{rig}(T)} + 2 \right) \pi_S(T) \leq 1,$$

then the area of S satisfies

$$\frac{\sqrt{1 - (4/\text{rig}(T) + 2)^2 \pi_S(T)^2}}{(1 + \omega_S(T))^2} \mathcal{A}(T) \leq \mathcal{A}(S) \leq \frac{1}{(1 - \omega_S(T))^2} \mathcal{A}(T).$$

As an obvious consequence, we get the following convergence result:

Corollary 7. *Let S be a (compact orientable) C^2 surface in E^3 . Let T_n be a sequence of triangulations closely inscribed in S . If*

1. *the length of the edges of T_n tends to zero when n goes to infinity,*

Fig. 3. $\text{Cos}\alpha_m = \sqrt{2}/2$ and $\omega_S(M_n) = 0$.

2. the rightness of T_n is (uniformly) bounded from below by a positive constant, then

$$\lim_{n \rightarrow \infty} \mathcal{A}(T_n) = \mathcal{A}(S).$$

Using Remark 1, we also have:

Corollary 8. *Let S be a smooth surface and let T be a triangulation closely inscribed in S . Then*

$$|\mathcal{A}(T) - \mathcal{A}(S)| \leq K(S) \left(\sup_{\Delta \text{ triangle of } T} R_{\Delta} \right),$$

where $K(S)$ is a constant depending on S and R_{Δ} is the circumradius of Δ .

5. Applications in Surface Reconstruction

Our results are general. However, they can be applied to particular algorithms of surface reconstruction.

5.1. Restricted Delaunay Triangulations

Let $\mathcal{S}$ be a finite set of points inscribed in a (compact without boundary) surface S . We recall the following [1]:

Definition 2. The set T of Delaunay simplices of $\mathcal{S}$ whose dual intersects S is called the **restricted Delaunay triangulation** of $\mathcal{S}$ with respect to S .

If the sample has suitable properties, then T has interesting topological and geometrical properties. We use the concept of an ε -sample, introduced in [1]:

Definition 3. A set of points $\mathcal{S}$ on S is an ε -sample of S if and only if, for every point m of S , the ball $B(m, \varepsilon \text{f}_S(m))$ encloses at least one point of $\mathcal{S}$.

Let us summarise the main properties of the restricted Delaunay triangulation associated to an ε -sample. We denote by $T_{\mathcal{S}_\varepsilon}$ the restricted Delaunay triangulation of an ε -sample $\mathcal{S}_\varepsilon$ with respect to S . Amenta et al. [3] showed that if $\varepsilon < 0.08$, then $T_{\mathcal{S}_\varepsilon}$ is closely inscribed in S . Furthermore, Amenta and Bern proved the following property [1]:

Proposition 2. *The maximal angle $\alpha_{\max}$ between the corresponding tangent planes of $T_{\mathcal{S}_\varepsilon}$ and S (via ξ) satisfies*

$$\alpha_{\max} = O(\varepsilon).$$

As an immediate consequence of Proposition 2 and Theorem 2, we get the following:

Corollary 9. *Let S be a compact without boundary surface of E^3 . Let T_{S_ε} be the restricted Delaunay triangulation of an ε -sample S_ε with respect to S . Then*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{A}(T_{S_\varepsilon}) = \mathcal{A}(S).$$

5.2. An Application Using Chew's Algorithm

Let S be a compact surface, let $\mathcal{S}$ be a $\tilde{\varepsilon}$ -sample of S ($\tilde{\varepsilon} < 0.1$) and let $\text{Del}_S(\mathcal{S})$ be the restricted Delaunay triangulation associated to the sample $\mathcal{S}$ of S . From $\text{Del}_S(\mathcal{S})$, Chew's algorithm builds a triangulation which is finer than $\text{Del}_S(\mathcal{S})$, and whose fatness is $> 30^\circ$ if S is closed. One may control the size of the triangles too (see [7] or [5]).

To get a triangulation which is close to S (for the Hausdorff distance), we work as follows (in the particular case where S has no boundary, otherwise it is a little more complicated): We consider a (good) $\tilde{\varepsilon}$ -sample (with $\tilde{\varepsilon} < 0.1$). We fix $\varepsilon > 0$. To each face Δ of $\text{Del}(\mathcal{S})$, we associate the (unique) sphere whose centre c is on S and which contains the vertices of σ (we denote its radius by r_Δ). We consider all the faces Δ of $\text{Del}(\mathcal{S})$ whose radius r_Δ is larger than ε or for which one angle is too small. We add the corresponding centres c'_S to S and we build $\text{Del}(S \cup \{c'_S\})$ and $\text{Del}_S(S \cup \{c'_S\})$. Iterating this process, we obtain a triangulation $T(\varepsilon)$ such that

- the length of every edge is $< 2\varepsilon$,
- the fatness is bounded from below by a *universal* constant (independent of ε).

This algorithm terminates (see [7] or [5]). Furthermore, if $\varepsilon < 0.08$ and if S has no boundary, then $T(\varepsilon)$ is closely inscribed in S [3]. In that case it is easy to see that the Hausdorff distance between S and the final triangulation is less than ε .

In particular, using Corollary 1 we get the following:

Corollary 10. *Let S be a smooth (compact without boundary) surface and let S be a (good) ε -sample of S . Let $\varepsilon > 0$. Then the triangulation $T(\varepsilon)$ obtained by the previous algorithm is such that the angle $\alpha_{\max}$ between the normals of S and $T(\varepsilon)$ satisfies*

$$\sin \alpha_{\max} \leq C(\rho_S)\varepsilon,$$

where $C(\rho_S)$ denotes a constant depending only on the maximal curvature ρ_S of S .

Consider now the sequence $\varepsilon_n = 1/n$. By the previous process, we associate a triangulation T_n satisfying the following properties:

- each T_n is closely inscribed in S ;
- the Hausdorff distance between T_n and S tends to zero when n tends to infinity;
- the fatness of T_n is uniformly bounded from below by a positive constant.

This implies, in particular (by using Corollary 7),

$$\lim_{n \rightarrow \infty} \mathcal{A}(T_n) = \mathcal{A}(S).$$

Moreover, we can apply classical results on geometric measure theory, and deduce immediately that the curvature measures of T_n converge to the curvature measures of S for the flat topology, and in particular for the weak topology (see [9] and [10] for the background and details on *curvature measures*).

6. Proof of Theorem 1

We first need some technical lemmas.

6.1. A Purely Geometric Result

The following lemma is a purely geometric result in E^3 :

Lemma 1. *Let Δ be a triangle whose vertices are p , p_1 and p_2 . If $\alpha_p \in [0, \pi/2]$ denotes the angle between a normal to the triangle and the axis (O, z) , then*

1.

$$\cos^2(\alpha_p) = \frac{\cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \gamma + 2 \cos \gamma \sin \theta_1 \sin \theta_2}{\sin^2(\theta_2 - \theta_1) + 2 \sin \theta_1 \sin \theta_2 \cos(\theta_2 - \theta_1) + \cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \gamma},$$

where $\theta_i \in [-\pi/2, \pi/2]$ is the angle between $\vec{p}\vec{p}_i$ and the orthogonal projection of $\vec{p}\vec{p}_i$ onto the plane orthogonal to (O, z) which contains p ($\theta_i \geq 0$ iff the third component of $\vec{p}\vec{p}_i$ is positive) and $\gamma \in]0, \pi[$ is the angle of Δ at p .

2. In particular, if $|\sin \theta_1| \leq \varepsilon$ and $|\sin \theta_2| \leq \varepsilon$, then

$$\sin(\alpha_p) \leq \frac{\sqrt{10}\varepsilon}{\sin \gamma},$$

where $\gamma \in]0, \pi[$ is the angle of Δ at p .

Proof. 1. We put

$$\eta_1 = pp_1, \quad \eta_2 = pp_2 \quad \text{and} \quad N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can assume that $p = 0$. In spherical coordinates, we get

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_1 = \begin{pmatrix} \eta_1 \cos \varphi_1 \cos \theta_1 \\ \eta_1 \sin \varphi_1 \cos \theta_1 \\ \eta_1 \sin \theta_1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} \eta_2 \cos \varphi_2 \cos \theta_2 \\ \eta_2 \sin \varphi_2 \cos \theta_2 \\ \eta_2 \sin \theta_2 \end{pmatrix},$$

$$\text{with} \quad \begin{cases} \theta_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ \varphi_i \in [0, 2\pi]. \end{cases}$$

We put

$$p\vec{p}_1 \wedge p\vec{p}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Thus

$$\cos(\alpha_p) = \left| \left\langle N, \frac{p\vec{p}_1 \wedge p\vec{p}_2}{\|p\vec{p}_1 \wedge p\vec{p}_2\|} \right\rangle \right| = \sqrt{\frac{c^2}{a^2 + b^2 + c^2}}.$$

We have

$$\begin{aligned} a &= \eta_1 \eta_2 (\cos \theta_1 \sin \varphi_1 \sin \theta_2 - \cos \theta_2 \sin \varphi_2 \sin \theta_1), \\ b &= \eta_1 \eta_2 (\cos \theta_2 \cos \varphi_2 \sin \theta_1 - \cos \theta_1 \cos \varphi_1 \sin \theta_2), \\ c &= \eta_1 \eta_2 \cos \theta_1 \cos \theta_2 (\cos \varphi_1 \sin \varphi_2 - \sin \varphi_1 \cos \varphi_2) \\ &= \eta_1 \eta_2 \cos \theta_1 \cos \theta_2 \sin(\varphi_2 - \varphi_1). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{a^2 + b^2}{\eta_1^2 \eta_2^2} &= \cos^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_2 \sin^2 \theta_1 \\ &\quad - 2 \cos \theta_1 \cos \theta_2 \sin \theta_1 \sin \theta_2 \cos(\varphi_2 - \varphi_1) \\ &= \sin^2(\theta_2 - \theta_1) + 2 \cos \theta_1 \cos \theta_2 \sin \theta_1 \sin \theta_2 (1 - \cos(\varphi_2 - \varphi_1)). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} \cos \gamma &= \frac{\langle p\vec{p}_1, p\vec{p}_2 \rangle}{\eta_1 \eta_2} \\ &= \cos \theta_1 \cos \theta_2 (\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2) + \sin \theta_1 \sin \theta_2 \\ &= \cos \theta_1 \cos \theta_2 \cos(\varphi_2 - \varphi_1) + \sin \theta_1 \sin \theta_2. \end{aligned}$$

Suppose that $\cos \theta_1 \neq 0$ and $\cos \theta_2 \neq 0$. Then

$$\cos(\varphi_2 - \varphi_1) = \frac{\cos \gamma - \sin \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2}.$$

Therefore

$$\begin{aligned} \frac{a^2 + b^2}{\eta_1^2 \eta_2^2} &= \sin^2(\theta_2 - \theta_1) + 2 \cos \theta_1 \cos \theta_2 \sin \theta_1 \sin \theta_2 \\ &\quad \times \left(1 - \frac{\cos \gamma - \sin \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2} \right) \\ &= \sin^2(\theta_2 - \theta_1) + 2 \sin \theta_1 \sin \theta_2 \cos(\theta_2 - \theta_1) - 2 \cos \gamma \sin \theta_1 \sin \theta_2, \end{aligned}$$

$$\begin{aligned}
\frac{c^2}{\eta_1^2 \eta_2^2} &= \cos^2 \theta_1 \cos^2 \theta_2 \left(1 - \left(\frac{\cos \gamma - \sin \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2} \right)^2 \right) \\
&= \cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \gamma + 2 \cos \gamma \sin \theta_1 \sin \theta_2, \\
\frac{a^2 + b^2 + c^2}{\eta_1^2 \eta_2^2} &= \sin^2(\theta_2 - \theta_1) + 2 \sin \theta_1 \sin \theta_2 \cos(\theta_2 - \theta_1) \\
&\quad + \cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \gamma.
\end{aligned}$$

Suppose now that $\cos \theta_1 = 0$. Thus the angle α_p is equal to $\pi/2$ or $-\pi/2$ and $\cos^2 \alpha_p = 0$. Since $\cos \gamma = \sin \theta_2$, the result is still true.

2. We have

$$|\sin \theta_i| \leq \varepsilon \quad \text{and} \quad |\cos \theta_i| \geq \sqrt{1 - \varepsilon^2}.$$

Thus

$$\begin{aligned}
&\cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \gamma + 2 \cos \gamma \sin \theta_1 \sin \theta_2 \\
&\geq (1 - \varepsilon^2) - \varepsilon^4 - \cos^2 \gamma - 2\varepsilon^2 \\
&= \sin^2 \gamma - 4\varepsilon^2
\end{aligned}$$

and

$$\begin{aligned}
&\sin^2(\theta_2 - \theta_1) + 2 \sin \theta_1 \sin \theta_2 \cos(\theta_2 - \theta_1) + \cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_1 \sin^2 \theta_2 - \cos^2 \gamma \\
&\leq 4\varepsilon^2 + 2\varepsilon^2 + 1 - \cos^2 \gamma \\
&= \sin^2 \gamma + 6\varepsilon^2.
\end{aligned}$$

Then

$$\cos^2(\alpha_p) \geq \frac{\sin^2 \gamma - 4\varepsilon^2}{\sin^2 \gamma + 6\varepsilon^2}.$$

Suppose now that $6\varepsilon^2/\sin^2 \gamma < 1$. Then we get

$$\cos^2(\alpha_p) \geq \left(1 - \frac{4\varepsilon^2}{\sin^2 \gamma}\right) \left(1 - \frac{6\varepsilon^2}{\sin^2 \gamma}\right) \geq 1 - \frac{10\varepsilon^2}{\sin^2 \gamma}$$

and

$$\sin(\alpha_p) \leq \frac{\sqrt{10}\varepsilon}{\sin \gamma}.$$

If $6\varepsilon^2/\sin^2 \gamma \geq 1$, the result is still true. $\square$

6.2. Comparing the Length of a Geodesic and Its Chord

We need the following result that gives the main differential properties of the function ξ when it is defined. It has been studied in [11] in the case where S is the boundary of a convex set of E^3 and in [14].

Proposition 3. *Let S be a smooth surface of E^3 without boundary, let U_S be an open subset of E^3 where the map $\xi : U_S \rightarrow S$ is well defined. Then:*

1. *The map ξ is C^1 in U_S and satisfies, for every $m \in U_S$,*

$$\begin{aligned} D\xi(m)(Z_m) &= 0, & \forall Z_m \text{ orthogonal to } T_{\xi(m)}S, \\ D\xi(m)(X_m) &= (Id + \delta_m \varepsilon_m A_{\xi(m)})^{-1}(X_m), & \forall X_m \text{ parallel to } T_{\xi(m)}S, \end{aligned}$$

where $\varepsilon_m = \langle v_{\xi(m)}, (\xi(m) - m) / \|\xi(m) - m\| \rangle \in \{-1, +1\}$ and $A_{\xi(m)} = -Dv(\xi(m))$ is the Weingarten endomorphism of S at the point $\xi(m)$.

2. *In particular, the matrix of $D\xi(m) : E^3 \rightarrow T_{\xi(m)}S$ (in local orthonormal frames $(e_{\xi(m)}^1, e_{\xi(m)}^2, v_{\xi(m)}^S)$ and $(e_{\xi(m)}^1, e_{\xi(m)}^2)$) is given by*

$$\begin{pmatrix} \frac{1}{1 + \delta_m \varepsilon_m \lambda_{\xi(m)}^1} & 0 & 0 \\ 0 & \frac{1}{1 + \delta_m \varepsilon_m \lambda_{\xi(m)}^2} & 0 \end{pmatrix},$$

where $e_{\xi(m)}^1$ and $e_{\xi(m)}^2$ are unitary principal vectors and $v_{\xi(m)}^S$ is the oriented normal of S at $\xi(m)$.

Sketch of proof.

- For every $m \in U_S$, the point $\xi(m)$ is defined by the following relation:

$$\forall m \in U_S, \quad \forall X \in T_{\xi(m)}S, \quad \langle \xi(m) - m, X_{\xi(m)} \rangle = 0.$$

Consequently, for every $m \in T_m S$, the function ξ is constant on the orthogonal of $T_m S$.

- Consider now a vector $X_m \in T_m U_S$ which is parallel to $T_{\xi(m)}S$. We have

$$D\xi(m)(X) = X + \delta_m \varepsilon_m Dv(\xi(m)) \circ D\xi(m)(X).$$

The endomorphism $(I + \delta_m \varepsilon_m A_{\xi(m)})$ is clearly invertible, and, consequently,

$$D\xi(m)(X) = (I + \delta_m \varepsilon_m A_{\xi(m)})^{-1}(X).$$

- The rest of the proof is obvious. □

Corollary 11. *Let S be a smooth compact surface of E^3 , let U_S be a neighbourhood of S where the map $\xi : U_S \rightarrow S$ is well defined, and let p and q be two points on S such that $[p, q] \subset U_S$ and $\xi([p, q]) \subset S \setminus \partial S$. Then the distance l_{pq} between p and q on S satisfies*

$$l_{pq} \leq \frac{1}{1 - \omega} pq,$$

where $\omega = \sup_{m \in [p, q]} \|\xi(m) - m\| \rho_{\xi(m)}$.

Proof. Since $\xi([p, q])$ is a curve on S , its length is larger than the length l_{pq} of the geodesic whose ends are p and q . Therefore,

$$l_{pq} \leq l(\xi([p, q])) \leq \sup_{m \in]p, q[} |D\xi(m)| l_{pq}.$$

On the other hand, for every $m \in]p, q[$, we have (see the remarks above)

$$\|\xi(m) - m\|_{\rho_{\xi(m)}} < 1.$$

Since $\xi(p) = p$ and $\xi(q) = q$, $\omega < 1$. Therefore Proposition 3 implies that

$$|D\xi(m)| \leq \frac{1}{1 - \|\xi(m) - m\|_{\rho_{\xi(m)}}} \leq \frac{1}{1 - \omega},$$

and Corollary 11 is proved. $\square$

6.3. Comparing the Normals at a Vertex

We shall prove the following:

Proposition 4. *Let S be a smooth surface, let Δ be a triangle closely inscribed in S and let p be a vertex of Δ . Then the angle $\alpha_p \in [0, \pi/2]$ between the normals of S and Δ at p satisfies*

$$\sin(\alpha_p) \leq \frac{\sqrt{10}\pi_S(\Delta)}{2 \sin \gamma_p(1 - \omega_S(\Delta))},$$

where γ_p is the angle of Δ at p .

This proposition is a consequence of the following:

Lemma 2. *Let S be a smooth surface and let $p, q \in S$ such that $[p, q] \subset S$. Then the angle $\theta \in [0, \pi/2]$ between $\vec{pq}$ and the orthogonal projection of $\vec{pq}$ onto $T_p S$ satisfies*

$$\sin \theta \leq \frac{\rho_S l}{2},$$

where l is the distance on S between p and q .

Proof. Let $\mathcal{C}$ denote a geodesic of S linking p and q . $\mathcal{C}$ is parametrized by arc length by

$$\gamma : [0, l] \rightarrow S,$$

with $\gamma(0) = p$ and $\gamma(l) = q$. A simple calculation gives

$$\gamma(l) - \gamma(0) = l\gamma'(0) + \int_0^l (l-t)\gamma''(t) dt,$$

thus

$$\gamma'(0) = \frac{\vec{pq}}{l} - \frac{1}{l} \int_0^l (l-t) \gamma''(t) dt.$$

Let $\vec{v} = (1/l) \int_0^l (l-t) \gamma''(t) dt$ and $\vec{e} = \vec{pq}/pq$. We have

$$\gamma'(0) = \frac{pq}{l} \vec{e} - \vec{v} \quad \text{with} \quad \begin{cases} \|\vec{e}\| = 1, \\ \|\vec{v}\| \leq \frac{l\rho_{\xi_S}}{2}, \end{cases}$$

and

$$\sin \theta = \inf_{\substack{\vec{u} \in T_p S \\ \|\vec{u}\|=1}} \|\vec{u} \wedge \vec{e}\| \leq \frac{l\rho_S}{2}. \quad \square$$

Proof of Proposition 4. Denote by l_1 the distance on S between p and p_1 , and by l_2 the distance on S between p and p_2 . Since T is closely inscribed in S , thanks to Corollary 11 we get

$$l_1 \leq \frac{pp_1}{1 - \omega_S(\Delta)} \leq \frac{\eta_\Delta}{1 - \omega_\Delta} \quad \text{and} \quad l_2 \leq \frac{\eta_\Delta}{1 - \omega_\Delta}.$$

Therefore, Lemma 2 implies that

$$\sin \theta_1 \leq \frac{\rho_{\xi(\Delta)} l_1}{2} \leq \frac{\pi_S(\Delta)}{2(1 - \omega_S(\Delta))} \quad \text{and} \quad \sin \theta_2 \leq \frac{\pi_S(\Delta)}{2(1 - \omega_S(\Delta))}.$$

Then Lemma 1 implies

$$\sin(\alpha_p) \leq \frac{\sqrt{10}}{\sin \gamma_p} \frac{\pi_S(\Delta)}{2(1 - \omega_S(\Delta))} = \frac{\sqrt{10} \pi_S(\Delta)}{2 \sin \gamma_p (1 - \omega_S(\Delta))}. \quad \square$$

6.4. Comparing the Normals of a Smooth Surface

Proposition 5. *Let S be a smooth compact oriented surface of E^3 , let Δ be a triangle closely inscribed in S and let p and s be two points on Δ . Then the angle $\alpha_{sp} \in [0, \pi/2]$ between two normals $v_{\xi(p)}^S$ and $v_{\xi(s)}^S$ at $\xi(p)$ and $\xi(s)$ satisfies*

$$\sin(\alpha_{sp}) \leq \frac{\pi_S(\Delta)}{1 - \omega_\Delta},$$

where η_Δ is the height of Δ and ω_Δ is the relative curvature of Δ with respect to $\xi(\Delta)$.

This proposition is the consequence of the following lemma, which is a direct application of the mean value theorem:

Lemma 3. *Let S be a smooth compact oriented surface of E^3 and let a and b be two points of S . The angle $\alpha_{ab} \in [0, \pi/2]$ between two normals v_a^S and v_b^S at a and b satisfies*

$$\sin(\alpha_{ab}) \leq \rho_S L_S(a, b),$$

where ρ_S is the maximal curvature of S and $L_S(a, b)$ is the distance on S between a and b .

The proof of Proposition 5 is an obvious consequence of Lemma 3 and Proposition 3. Theorem 1 can be immediately deduced from Propositions 4 and 5, since

$$\sin(\alpha_s) \leq \sin(\alpha_p) + \sin(\alpha_{sp}).$$

7. Proof of Theorem 2

Consider the surface M lying in U_S . Since M is compact and locally lipschitz, there exists a finite family of lipschitz graphs M_i , such that $M = \bigcup_i M_i$ and $M_i \cap M_j = \emptyset$ (if $i \neq j$). We put $S_i = \xi(M_i)$. By assumption, the restriction $\xi|_{M_i}$ of ξ to M_i is one-to-one. Furthermore, the function ξ is differentiable at every point of M . We apply the general area-coarea formula [13]:

$$\mathcal{A}(S_i) = \int_{S_i} da_{S_i}(s) = \int_{M_i} |Jac_2 D\xi(m)| da_{M_i}(m),$$

where $Jac_2 D\xi(m)$ is the two-dimensional Jacobian of ξ at m . Since M_i is a lipschitz surface, M_i admits a tangent plane at almost every point m . In such a point m , $Jac_2 D\xi(m)$ equals the determinant of the linear map

$$D\xi|_{M_i} : T_m M_i \rightarrow T_{\xi(m)} S.$$

We have

$$\mathcal{A}(S_i) = \int_{M_i} |D\xi|_{M_i}(m)| da_{M_i}(m) = \int_{M_i} |D\xi|_M(m)| da_M(m).$$

Therefore, by taking the sum over all the M_i 's, we get

$$\mathcal{A}(S) = \int_M |D\xi|_M(m)| da_M(m).$$

The following lemma evaluates the Jacobian of $\xi|_M$.

Lemma 4. *Let S be a smooth surface of E^3 without boundary and let U_S be an open subset of E^3 where the map $\xi : U_S \rightarrow S$ is well defined. Let $M \subset U_S$ be a smooth surface. Then the Jacobian of the differential of $\xi|_M$ is given by*

$$|D\xi|_M(m)| = \frac{\cos \alpha_m}{(1 + \delta_m \varepsilon_m \lambda_{\xi(m)}^1)(1 + \delta_m \varepsilon_m \lambda_{\xi(m)}^2)},$$

where $\varepsilon_m = \langle v_{\xi(m)}, (\xi(m) - m)/\|\xi(m) - m\| \rangle \in \{-1, +1\}$, and $\delta_m = \|\xi(m) - m\|$.

This lemma is obvious, by considering the restriction $\xi|_M$ of ξ to M . Note that the condition of the theorem on the angle between corresponding tangent planes implies that $\xi|_M$ is a diffeomorphism between M and S .

End of the Proof of Theorem 2. Since M is closely inscribed in S , $\xi|_M$ is one-to-one between M and S . Therefore,

$$\mathcal{A}(S) = \int_M |D\xi|_M(m)| \, da_M(m) = \int_M \frac{\cos \alpha_m}{(1 + \delta_m \varepsilon_m \lambda_{\xi(m)}^1)(1 + \delta_m \varepsilon_m \lambda_{\xi(m)}^2)} \, da_M(m). \quad \square$$

8. Conclusion

In this paper we give a very general result relating the normal vector field of a smooth surface and the normal vector field of a triangulation. This result induces a result of the approximation of the area of a smooth surface with the area of a triangulation. These results are applied in surface reconstruction. We hope that this kind of result can be useful to get a general theoretical approach to the approximation of the geometrical properties of a smooth surface and a triangulation close to it.

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