Regular Polytopes of Full Rank

Peter McMullen

Department of Mathematics, University College London,
Gower Street, London WC1E 6BT, England
p.mcmullen@ucl.ac.uk

To the memory of Donald Coxeter

Abstract. The dimension of a faithful realization of a finite abstract regular polytope in some euclidean space is no smaller than its rank. Similarly, that of a discrete faithful realization of a regular apeirotope is at least one fewer than the rank. Realizations which attain the minimum are said to be of full rank. The regular polytopes and apeirotopes of full rank in two and three dimensions were classified in an earlier paper. In this paper these polytopes and apeirotopes are classified in all dimensions. Moreover, it is also shown that there are no chiral polytopes of full rank.

1. Introduction

In [6] (see also Sections 7E and 7F of [7]) we classified completely all the faithfully realized regular polytopes and discrete regular apeirotopes in dimensions up to three. In this paper we find all such polytopes and apeirotopes of maximal rank in each higher dimension.

We should initially remark that there are two quite different ways to approach realizations. The first, for which a fairly complete theory exists (at least, in the finite case), asks for a description of the space of all realizations of a given regular polytope or apeirotope, with rank playing only a minor rôle (see Sections 5B and 5C of [7] for further details). The second, about which much less is known in general terms, asks for a classification of the realizations of all these polytopes and apeirotopes in a euclidean space of given dimension (in this case, it is usual to impose conditions such as faithfulness and discreteness). This problem is solved in three dimensions. The finite regular polyhedra have long been known; Grünbaum [5] found all but one of the regular apeirohedra, while Dress [3], [4] found the missing example, and proved that the classification was then complete. A much quicker proof of this is in [6], where the regular apeirotopes were also described.

In the present context we classify by dimension, but with a restriction of rank suggested by the previous paragraph.

At the root of our classification lie two results, Theorems 3.1 and 3.5. The first, which is already known (see Theorems 5B20 and 5C3 of [7]), gives lower bounds for the dimensions of euclidean spaces in which regular polytopes or apeirotopes of given rank can be faithfully realized. The second places a strong restriction on the dimensions of the mirrors of the generating reflexions of the symmetry group, when this minimum is achieved.

The paper is organized as follows. In Section 2 we outline that part of the general theory of abstract regular polytopes we need to appeal to. In Section 3 we define the basic notions of realizations, and establish some general results which will subsequently be useful. In Section 4 we describe a construction for a particular class of apeirotopes.

The next six sections are devoted to the solution of the classification problem. In Sections 5 and 6 we look at those polytopes and apeirotopes which fall into families represented in each dimension. In Section 7 we briefly revisit the three-dimensional case. Perhaps the most interesting parts of the paper are Sections 8 and 9, because of the relatively large numbers of regular 4-polytopes and discrete 5-apeirotopes which occur. A striking rôle is played here by the three Petrie–Coxeter polyhedra (sponges) of [1] in ordinary space, all of which occur as 3-faces of apeirotopes. Section 10 completes the classification, showing that two putative examples in dimensions six and eight do not actually exist.

Chiral polytopes are nearly regular; we show in Section 11 that any putative chiral polytope of full rank is regular, so that there are, in fact, no chiral polytopes of full rank.

In Sections 12 and 13 we make some remarks about the non-discrete and hyperbolic cases, pointing out that, without imposition of extra conditions, the classification problems here appear to be hopelessly difficult. Finally, in Section 14, we provide a complete listing of the regular polytopes and apeirotopes of full rank.

2. Regular Polytopes

For the general background on abstract regular polytopes, we refer the reader to the recently published monograph [7]; for the most part, we do not cite original papers directly. In this paper we largely concentrate on the geometric aspects of the theory, that is, on realizations of regular polytopes. Hence, we can adopt (see Theorem 2E11 of [7]) the viewpoint that an abstract regular polytope $\mathcal{P}$ is to be identified with its automorphism group Γ . The latter is a *string C-group*: if $\mathcal{P}$ has rank n , then Γ is a group generated by n involutions ρ_j (the *distinguished generators*) with $j \in \mathbf{N} := \{0, \dots, n-1\}$, such that ρ_j and ρ_k commute if $0 \leq j \leq k-2 \leq n-3$, and

$$\langle \rho_i \mid i \in \mathbf{J} \rangle \cap \langle \rho_i \mid i \in \mathbf{K} \rangle = \langle \rho_i \mid i \in \mathbf{J} \cap \mathbf{K} \rangle \quad (2.1)$$

for each $\mathbf{J}, \mathbf{K} \subseteq \mathbf{N}$; the last is the *intersection property*. The j -faces of $\mathcal{P}$ are then the right cosets $\Gamma_j \sigma$ of the *distinguished subgroup*

$$\Gamma_j := \langle \rho_i \mid i \neq j \rangle$$

for each $j \in \mathbb{N}$, and two faces are incident just when they intersect (as cosets). In fact, incidence actually induces an order relation:

$$\Gamma_j \sigma \leq \Gamma_k \tau \iff \Gamma_j \sigma \cap \Gamma_k \tau \neq \emptyset \quad \text{and} \quad j \leq k.$$

Formally, we also adjoin two copies of Γ itself, as the (unique) (-1) - and n -faces of $\mathcal{P}$.

The maximal chains (with respect to this ordering) are the *flags* of $\mathcal{P}$; the group Γ is then simply transitive on the flags of $\mathcal{P}$. (Note that we do not usually mention the (-1) - and n -faces in flags.) With appropriate conditions on the poset of faces of an abstract polytope $\mathcal{P}$ (see Section 2A of [7]), this latter condition can be used as a definition of regularity of $\mathcal{P}$. Calling two flags *adjacent* if they differ by one face, the distinguished generator ρ_j takes the *base flag* $\Phi := \{\Gamma_0, \Gamma_1, \dots, \Gamma_{n-1}\}$ into the flag Φ^j which differs from it in Γ_j .

Note that the distinguished subgroups $\Gamma_{n-1} = \langle \rho_0, \dots, \rho_{n-2} \rangle$ and $\Gamma_0 = \langle \rho_1, \dots, \rho_{n-1} \rangle$ are themselves string C-groups; the corresponding polytopes are the *facet* and *vertex-figure* of $\mathcal{P}$, respectively.

To avoid cases which, in our context, turn out to be trivial, we always assume that adjacent generators ρ_{j-1} and ρ_j of Γ do not commute (this is justified in Section 3). In other words, the period p_j of the product $\rho_{j-1}\rho_j$ satisfies $p_j > 2$ (possibly, $p_j = \infty$). We call $\{p_1, \dots, p_{n-1}\}$ the *Schläfli type* of the polytope. If the polytope is determined just by the p_j , then we have the *universal* regular polytope (of that Schläfli type), for which we use the same symbol $\{p_1, \dots, p_{n-1}\}$ (but without qualification); we write $[p_1, \dots, p_{n-1}]$ for the corresponding *Coxeter* group. Generally, however, the group Γ will satisfy additional relations as well, for some of which we introduce special notation later.

It is, however, useful also to be able to refer to the more combinatorial picture. Here, an (abstract) n -polytope is a poset $\mathcal{P}$ with a *rank* function, taking values in $\{-1, 0, \dots, n\}$ (the maximal and minimal faces are unique), such that each *flag*, or maximal chain, contains $n + 2$ faces (the elements of $\mathcal{P}$), one of each rank; for $j = 0, \dots, n - 1$ each flag Φ is *adjacent* to exactly one other flag Φ^j which differs from it in its j -face; last, $\mathcal{P}$ is *strongly flag connected*, in that, given two flags Φ, Ψ of $\mathcal{P}$, there is a sequence of successively adjacent flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$, with $\Phi \cap \Psi \subseteq \Phi_i$ for each $i = 1, \dots, k - 1$.

We then call an n -polytope $\mathcal{P}$ *regular* if its automorphism group $\Gamma(\mathcal{P})$ (preserving the partial ordering) is (simply) transitive on its flags; the generators ρ_j of $\Gamma(\mathcal{P})$ are then such that $\Phi^j = \Phi\rho_j$ for $j = 0, \dots, n - 1$. As we said earlier, Theorem 2E11 of [7] shows that this definition and the previous one are equivalent.

A central question in the abstract theory is the following. If a regular $(n + 1)$ -polytope has facets (of type) the n -polytope $\mathcal{P}$ and vertex-figures the n -polytope $\mathcal{Q}$, then the facets of $\mathcal{Q}$ must be isomorphic to the vertex-figures of $\mathcal{P}$. Conversely, if $\mathcal{P}$ and $\mathcal{Q}$ satisfy this latter criterion, then we write $\langle \mathcal{P}, \mathcal{Q} \rangle$ for the class of all regular $(n + 1)$ -polytopes with facet $\mathcal{P}$ and vertex-figure $\mathcal{Q}$. The question has two parts. First, is $\langle \mathcal{P}, \mathcal{Q} \rangle \neq \emptyset$; in other words, does there exist any such regular $(n + 1)$ -polytope at all? If there does, it then follows that there is a *universal* member $\{\mathcal{P}, \mathcal{Q}\}$ in the family $\langle \mathcal{P}, \mathcal{Q} \rangle$, of which every other one is a quotient (in the sense that its group is an appropriate quotient). Second, given that it exists, we ask what $\{\mathcal{P}, \mathcal{Q}\}$ is. (See Section 4B of [7] for further details.)

In the present context, we often pose this question in the form: is a given regular polytope, whose facet and vertex-figure are known, actually universal of its kind?

A regular polytope is called (*combinatorially*) *flat* if each of its vertices is incident with each of its facets (see Section 4E of [7]). Flatness is a hereditary property: a polytope $\mathcal{P}$ whose vertex-figure $\mathcal{Q}$ is flat is itself flat, and $\mathcal{P}$ has the same number of facets as $\mathcal{Q}$. (There is an obvious dual property for facets and their vertices, but we shall not need it.) We do not bother to introduce examples at this stage, since we meet several later.

Two different regular polytopes may be related by what is called a *mixing operation*; the distinguished generators of the second group are certain products of those of the first (see Chapter 7 of [7]). Apart from *duality* δ , which just reverses the order of the distinguished generators (and the order relation on the faces), there is one of these which plays an important rôle in our enumeration. Suppose that $\Gamma = \langle \rho_i \mid i \in \mathbb{N} \rangle$ is a string C-group. Let $j \neq k$, and consider the operation

$$(\rho_0, \dots, \rho_{n-1}) \mapsto (\rho_0, \dots, \rho_{j-1}, \rho_j \rho_k, \rho_{j+1}, \dots, \rho_{n-1}) =: (\sigma_0, \dots, \sigma_{n-1}).$$

Since, by the assumption we made earlier, adjacent generators of Γ do not commute, we easily see that $\Delta := \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ cannot possibly be a string C-group unless $(j, k) = (2, 0)$ or $(n-3, n-1)$. The former will rule itself out later for geometric reasons (see Section 3); the latter, namely,

$$\pi: (\rho_0, \dots, \rho_{n-1}) \mapsto (\rho_0, \dots, \rho_{n-4}, \rho_{n-3} \rho_{n-1}, \rho_{n-2}, \rho_{n-1}) =: (\sigma_0, \dots, \sigma_{n-1}), \quad (2.2)$$

which we denote by $\Gamma \mapsto \Gamma^\pi$, we call the *Petrie operation*, since it generalizes the operation with the same name when $n = 3$. (Implicitly, we take the Petrie operation to apply to the corresponding polytope as well.) Even when $n = 3$, the Petrie operation π does not always yield a C-group (though such cases are rather exceptional), but, for higher rank, each application has to be checked directly. One general case can be settled easily.

Proposition 2.1. *If $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ is a string C-group with $n \geq 4$ for which p_{n-3} is odd, then the Petrial Γ^π is not a C-group.*

Proof. Indeed, with the notation of (2.2), we have

$$(\sigma_{n-4} \sigma_{n-3})^{p_{n-3}} = (\rho_{n-4} \rho_{n-3} \cdot \rho_{n-1})^{p_{n-3}} = \varepsilon \cdot \rho_{n-1}^{p_{n-3}} = \rho_{n-1},$$

since p_{n-3} is odd. This says that $\sigma_{n-1} = \rho_{n-1} \in \langle \sigma_{n-4}, \sigma_{n-3} \rangle$, which violates the intersection property. $\square$

In spite of Proposition 2.1, we must, however, note that an important class of regular polyhedra or apeirohedra consists of those which are determined by their Schläfli type and Petrie polygons. The geometric picture of a Petrie polygon is one which shares two successive edges of each 2-face which it meets, but not a third. We write $\{p, q\}_r$ for the polyhedron (possibly infinite) of Schläfli type $\{p, q\}$, whose Petrie polygons of length r determine it. Its group is the Coxeter group $\langle \rho_0, \rho_1, \rho_2 \rangle = [p, q]$, with the imposition of the single extra relation

$$(\rho_0 \rho_1 \rho_2)^r = \varepsilon. \quad (2.3)$$

We note that, if it is a genuine polyhedron, then the Petrial of $\{p, q\}_r$ is $\{r, q\}_p$.

While we are discussing polyhedra, another class is also of great importance. A *hole* of a regular polyhedron is an edge-circuit which exits from the *second* edge (in some local orientation) emanating from a vertex from the edge by which it entered. The designation of a (possibly infinite) regular polyhedron of Schläfli type $\{p, q\}$, which is determined by its holes of length h , is $\{p, q | h\}$. The corresponding relation to be imposed on the Coxeter group $\langle \rho_0, \rho_1, \rho_2 \rangle = [p, q]$ is

$$(\rho_0 \rho_1 \rho_2 \rho_1)^h = \varepsilon. \quad (2.4)$$

Various examples of such polyhedra occur later; for now, let us observe that the great dodecahedron $\{5, \frac{5}{2}\}$ is, as an abstract regular polyhedron, $\{5, 5 | 3\}$.

One further abstract construction, which is a special case of mixing, is the following. Again, suppose that $\Gamma = \langle \rho_i \mid i \in \mathbb{N} \rangle$ is a string C-group. Let τ be an involution which commutes with all ρ_j , and consider the operation

$$(\rho_0, \dots, \rho_{n-1}, \tau) \mapsto (\rho_0 \tau, \rho_1, \dots, \rho_{n-1}) =: (\sigma_0, \dots, \sigma_{n-1}). \quad (2.5)$$

This is called *mixing with a segment*, because τ can be regarded as the generating involution of the group of the segment (1-polytope) $\{\}$ (see Section 3 for the notation). We have (see Theorem 7A8 of [7])

Theorem 2.2. *Mixing a string C-group Γ with a segment always yields another C-group. This is isomorphic to Γ if all edge-circuits in the associated regular polytope $\mathcal{P}$ have even length; otherwise, it is isomorphic to the direct product $\Gamma \times C_2$ of Γ with a cyclic group C_2 of order 2.*

We denote the resulting regular polytope (which again we say is obtained from $\mathcal{P}$ by mixing with a segment) by $\mathcal{P} \diamond \{\}$. This has twice as many vertices as $\mathcal{P}$ precisely when some edge-circuit of $\mathcal{P}$ has odd length.

A technique which we have less use for here is *twisting* (see Chapter 8 of [7]). In this, a given group (usually itself a C-group) is augmented by means of one or more automorphisms. In our only application (in Section 10), it turns out that the automorphisms are, in fact, inner.

3. Realizations

The usual (and generally most useful) context of realizations is of those in euclidean spaces, because it is in these that we obtain the richest structure. However, initially at least, it is appropriate for us to broaden the definition. Thus, for the time being, E is a k -dimensional spherical space $\mathbb{S}^k$, euclidean space $\mathbb{E}^k$ or hyperbolic space $\mathbb{H}^k$, for some k .

For a *faithful realization* of a regular polytope (or apeirotope – for the moment, we use the generic term, not distinguishing between the finite and infinite cases), we have two ingredients (see Chapter 5 of [7] for the general background here). First, we need a suitable space E which admits a group $\mathcal{G}$ of isometries isomorphic to Γ ; this is the *symmetry group* of the realization. (If Γ is a finite group, then E will be spherical; if Γ

is infinite, then, since we are generally interested only in discrete realizations, E will be euclidean or hyperbolic.) We identify the *reflexion* R_j corresponding to the involution ρ_j with its *mirror*

$$\{x \in E \mid xR_j = x\}$$

of fixed points; we thus use the same symbol E for the ambient space to denote the identity. The intersection

$$W := R_1 \cap \cdots \cap R_{n-1}$$

is called the *Wythoff space* of the realization. The realization then arises from some choice of *initial vertex* $v \in W$. (When E is hyperbolic, we also assume that v is a finite point, or lies on the absolute.) The vertex-set of the realization is $V := v\mathcal{G}$, and we always assume that E is spanned by V (as a subspace of the appropriate kind), so that E is thought of as the *ambient space* of the realization, namely, the space (of one of the three kinds) of smallest dimension which contains it. Then $\dim E$ is called the *dimension* of the realization.

We remark here that, if the realization were to be such that $R_j = E$, the identity, then $R_k = E$ for all $k > j$ as well. In particular, this will happen if $p_j = 2$, which is why we excluded this possibility in Section 2.

The induced geometric structure of the realization is defined as follows. Write $F_0 := v$, and, for $j \geq 1$, let

$$F_j := F_{j-1}\langle R_0, \dots, R_{j-1} \rangle;$$

then the j -faces are the F_jG with $G \in \mathcal{G}$, with the order relation given by iterated membership. Thus *edges* are composed of the two vertices which belong to them (we also think of an edge as the line-segment between its vertices – there will be no ambiguity, even in the spherical case, because antipodal points of the sphere will never determine an edge), 2-faces of the edges which belong to them, and so on up to the *ridges* or $(n-2)$ -faces and *facets* or $(n-1)$ -faces. We use a symbol such as P to denote a geometric polytope in this sense, and define its *dimension* by $\dim P := \dim E$; further, we denote by $V(P) := V$ its vertex-set. For the realization to be faithful, we demand that, for each $j = 1, \dots, n-1$, a j -face be uniquely determined by the $(j-1)$ -faces which belong to it. It is important to observe that some regular polytopes do not admit faithful realizations, because this latter condition implies a corresponding purely combinatorial condition on $\mathcal{P}$.

It is clear that F_{n-1} (and its induced structure, with the same initial vertex v) gives a realization of the facet of $\mathcal{P}$; its symmetry group is the image $\mathcal{G}_{n-1}$ of Γ_{n-1} . If we write w for the mid-point of the edge between v and vR_0 , then w is the initial vertex of a realization of the vertex-figure of $\mathcal{P}$, with symmetry group the image $\mathcal{G}_0$ of Γ_0 . (In the hyperbolic case of a polytope with vertices on the absolute, then w is well defined as the intersection of R_0 with the line between v and vR_0 ; in any event, w will always lie in this intersection.) If the original realization of $\mathcal{P}$ is faithful, then the realizations of the facet and vertex-figure of $\mathcal{P}$ are also faithful. In a similar way, $\langle R_0, \dots, R_{j-1} \rangle$ is the symmetry group of the basic j -face F_j of P , while $\langle R_{j+1}, \dots, R_{n-1} \rangle$ is that of the basic *co- j -face* P/F_j , which is the $(j+1)$ -fold iterated vertex-figure. Thus the vertex-figure

itself is P/F_0 . Even more generally, $\langle R_{j+1}, \dots, R_{k-1} \rangle$ is the symmetry group of the section F_k/F_j (for $j \leq k-2$).

Note, by the way, that we often use vR_0 rather than w as the initial vertex of the vertex-figure; for most purposes, this makes little difference, since the combinatorics are not altered.

As very basic examples, in $\mathbb{E}^0$ we just have the point, the finite regular 1-polytope is a segment $\{\}$, which is naturally realized in the 0-sphere $\mathbb{S}^0$ (the term is thus a slight misnomer, but we usually think of $\{\}$ as a genuine line-segment), while the regular apeirogon is $\{\infty\}$ is naturally realized discretely in $\mathbb{E}^1 = \mathbb{R}$. In the unit circle $\mathbb{S}^1$, there is an infinite family of (finite) regular polygons. The mirrors R_0 and R_1 are lines through its centre at a *rational* angle π/p , meaning that $p > 2$ is a rational number (always in its lowest terms); the resulting regular polygon is denoted $\{p\}$. In addition, $\{\infty\}$ has non-discrete faithful realizations in $\mathbb{S}^1$.

There are important restrictions on faithful realizations.

Theorem 3.1. *Let P be a faithful realization of an abstract regular polytope $\mathcal{P}$, whose ambient space E is a spherical, euclidean or hyperbolic space. Then $\dim P \geq \text{rank } \mathcal{P} - 1$.*

Proof. The result trivially holds (with equality) when $\text{rank } \mathcal{P} = 1$, because the only 1-polytope is the segment $\{\}$, which is realized in $\mathbb{S}^0$. So, suppose that $\text{rank } \mathcal{P} \geq 2$, and let Q denote the vertex-figure of $\mathcal{P}$; we make the obvious inductive assumption that a faithful realization Q of Q satisfies $\dim Q \geq \text{rank } Q - 1 = \text{rank } \mathcal{P} - 2$. In particular, if Q is the (geometric) vertex-figure of P , then $V(Q)$ lies on a sphere Ω in E with centre the initial vertex v (in the hyperbolic case, Ω may be a horosphere). Remembering that we must think of $V(Q)$ as spanning an appropriate subspace of Ω (with the corresponding dimension), we thus see at once that

$$\dim P = \dim E = \dim \Omega + 1 \geq \dim Q + 1 \geq (\text{rank } \mathcal{P} - 2) + 1 = \text{rank } \mathcal{P} - 1, \quad (3.1)$$

as required. $\square$

If P is a realization of a regular polytope $\mathcal{P}$ for which equality holds in Theorem 3.1, then we say that P is *of full rank*. The emphasis is placed this way round, because our aim (as explained in Section 1) is to classify such polytopes by dimension.

Recall that a realization of an abstract regular polytope $\mathcal{P}$ is *vertex-faithful* if it is one-to-one on the vertices of $\mathcal{P}$; we can similarly define terms such as *edge-faithful*. We make the following comment, which partially answers a question posed by one of the referees of the paper.

Theorem 3.2. *Let $\mathcal{G}$ be a faithful representation in E of the group Γ of the abstract regular polytope $\mathcal{P}$ with Wythoff space W , and let $v \in W$ be such that $V := v\mathcal{G}$ is a vertex-faithful realization of $\mathcal{P}$ of full rank. Then the realization is faithful.*

Proof. We can proceed by induction on the dimension. Suppose, to the contrary, that the realization is not edge-faithful. Then the initial vertex v is joined by multiple edges to

the adjacent vertex $w := vR_0$. However, this immediately implies that the corresponding representation $\mathcal{G}_0$ of the group Γ_0 of the vertex-figure of $\mathcal{P}$ is not faithful, so that there are non-identity elements of Γ_0 whose images under the induced representation are the identity in $\mathcal{G}_0$. Since v and w together span E , there are thus non-identity elements of Γ whose images act as the identity on E , contradicting the assumption that the representation was faithful. $\square$

Remark 3.3. The recursive argument employed here can be made to show that the groups of realizations of full rank are actually irreducible (affinely, in the case of euclidean realizations). We do not bother to prove this formally, because it emerges in any case from the classification.

Remark 3.4. Our main concern is with discrete realizations, which for a finite polytope obviously means in some spherical space, while for an apeirotope means in a euclidean or hyperbolic space. The non-discrete case is, as we said in Section 1, much more difficult; we briefly comment on this in Section 12. For now, we just observe that (as we noted earlier) there exist bounded realizations of regular apeirotopes (for example, an apeirogon with vertices on a circle, whose edges subtend a fixed irrational angle at the centre).

We can go further than Theorem 3.1, and place restrictions on the dimensions of the mirrors of the generating reflexions of the realizations when equality prevails.

Theorem 3.5. *Let P be a faithful realization of full rank of a regular n -polytope $\mathcal{P}$ in the ambient space E , with symmetry group $\mathcal{G} = \langle R_0, \dots, R_{n-1} \rangle$. Then $\dim R_j = j$ or $n - 2$ for $j = 0, \dots, n - 3$, and $\dim R_{n-2} = \dim R_{n-1} = n - 2$.*

Proof. We first observe that equality in (3.1) in the proof of Theorem 3.1 forces the vertex-figure Q of P (realizing the vertex-figure $\mathcal{Q}$ of $\mathcal{P}$) to be of full rank also; this will then hold for all face-figures (iterated vertex-figures) as well. The core case to be established is that for $j = 0$; the remaining cases will then follow easily. The proof is by induction on n , beginning with the segment $\{ \}$, whose symmetry group is generated by the point-reflexion in the centre of a 0-sphere.

Suppose, therefore, that P is an n -polytope (or apeirotope) with $n \geq 2$, and that the result holds for polytopes of rank $n - 1$. Let the initial vertex of P be v . Then the subset of the ambient space E fixed pointwise by the group $\mathcal{G}_0$ of Q consists of the point v (and the antipodal point of E , if E is a sphere), but contains no higher-dimensional subspace through v ; the vertices of Q lie on a sphere (possibly a horosphere) Ω in E centred at v . Let w be the initial vertex of Q , so that w is the mid-point of the segment joining v and vR_0 , and the initial edge F_1 spans the line L joining v and w . Then the group $\mathcal{G}_{01}$ (say) of the co-edge P/F_1 has fixed-point set exactly L (by the same token, since P/F_1 is the vertex-figure of Q), and acts irreducibly on the hyperplane H perpendicular to L through w . Hence, because R_0 commutes with $\mathcal{G}_{01}$, it follows that, if R_0 fixes any point of H distinct from w , then it must fix the whole of H . We conclude at once that either $R_0 = \{w\}$ or $R_0 = H$. This is the required result.

For the remaining assertions, if the group of Q is $\mathcal{G}_0 = \langle S_0, \dots, S_{n-2} \rangle$, where the S_j are mirrors in Ω , then R_j is the subspace spanned by S_{j-1} and v for $j = 1, \dots, n-1$, and the obvious inductive assumption yields the result. $\square$

We pay little heed to the hyperbolic case; see Section 13 for the reasons. For the finite case, we now find it convenient to revert to the former definition of realization. In other words, henceforth we regard a sphere which carries the vertices of a realization P of a finite regular polytope as sitting in the euclidean space of one larger dimension with centre the origin o . The mirrors R_j of its group $\mathcal{G}$ are then thought of as linear subspaces, also of one larger dimension than before; in particular, in the minimal case, R_0 is either a line or a hyperplane.

Remark 3.6. In the case of a faithful realization of full rank of a finite regular n -polytope with centre o , if the mirror R_0 is a line, then $-R_0 = (-I)R_0$, the product of R_0 with the central inversion, is a hyperplane reflexion. (In the euclidean situation, it is convenient to depart from our previous convention, and write I instead of E ; this lets us use $-I$ for the central inversion.) If we replace R_0 by $-R_0$, then at worst we have replaced the symmetry group $\mathcal{G}$ by $\mathcal{G} \times \mathcal{C}_2$, with $\mathcal{C}_2 = \{\pm I\}$; in any event, we always have another finite group. We are thus already quite close to proving Theorem 5.1.

Remark 3.6 presents us with the opportunity to introduce some important geometric operations on polytopes of full rank. In that situation, since o is the sole fixed point of the ambient space E under the group $\mathcal{G}$, it follows that $K_0 := R_0 \cap \dots \cap R_{n-1} = \{o\}$. Further, the central reflexion $-I$, identified with its mirror $\{o\}$, is thus K_0 , so the replacement of the remark is $R_0 \mapsto R_0 K_0$. It is extremely useful to have variant operations, which act on the co- $(j-1)$ -face P/F_{j-1} . With

$$K_k := R_k \cap \dots \cap R_{n-1},$$

we see that (the reflexion in) K_k induces the central inversion on $\text{aff}(P/F_{k-1})$; recall our general assumption of full rank. For $0 \leq j \leq k \leq n-1$, we then define the operation κ_{jk} by

$$\kappa_{jk}: (R_0, \dots, R_{n-1}) \mapsto (R_0, \dots, R_{j-1}, R_j K_k, R_{j+1}, \dots, R_{n-1}) =: (S_0, \dots, S_{n-1}). \quad (3.2)$$

We abbreviate κ_{jj} to κ_j , because this is the most important case (and here usually only with $j = 0, 1$), but we also make use of κ_{02} . Thus κ_j interchanges the two possibilities for R_j which can occur in Theorem 3.5. Just as with the Petrie operation, though, it must be emphasized that it is by no means generally the case that κ_{jk} will yield a C-group when it is applied to another – note that, for S_j to be an involution, we need $j = k$ or $j \leq k-2$. Observe also, by the way, that $K_{n-1} = R_{n-1}$, so that the Petrie operation can be written as $\pi = \kappa_{n-3, n-1}$. It is also worth noting that, in general, duals of faithfully realizable regular polytopes are not necessarily realizable at all (Petrials are particular examples), let alone in the same space.

However, we do have one result, for which again we do not have a general proof.

Theorem 3.7. *Let P be a finite regular polytope of full rank. Then P^{κ_0} is also a polytope.*

One problem in proving this result in general terms is that applying κ_0 may do one of three things, even when the first group is a hyperplane reflexion group: it may double the order, leave it the same or even halve it. For simple examples to illustrate this, in $\mathbb{E}^3$ take, respectively, the tetrahedron, octahedron and cube (note that, in each case, whereas the old facets were of full rank, the new ones are skew polygons, and so are not). In the planar case we have $\{p\}^{\kappa_0} = \{q\}$, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

Remark 3.8. If $K_k \in \langle R_j, \dots, R_{n-1} \rangle$, then κ_{jk} results in a mixing operation.

When the abstract polytope $\mathcal{P}$ is finite, we may assume that the centroid of the vertex-set V of its realization is the origin o of E , so that $\mathcal{G}$ is an orthogonal group. If $\mathcal{P}$ is infinite, in which case we call it an *apeirotope*, we will additionally demand of its realization that it be discrete, so that the group $\mathcal{G}$ acts discretely on the ambient space E .

In order to avoid constant repetition of various fixed phrases in the subsequent discussion, we adopt the following conventions. Henceforth, *regular polytope* will mean “faithfully realized finite abstract regular polytope”, while *regular apeirotope* will mean “discrete faithfully realized abstract regular apeirotope”.

4. An Apeirotope Construction

We already have much of the apparatus we need to carry out our classification. However, there is one more general construction which it is appropriate to introduce.

Let X be a point-set in a euclidean space E . We call X *rational* if the points of X can be chosen to have rational coordinates with respect to some (linear or affine) coordinate system in E . The following remark is obvious.

Lemma 4.1. *Let E be a euclidean space, and let X be a finite point-set in E . Let $\mathcal{R}(X)$ be the group generated by the point-reflexions (inversions) in the points of X . Then $\mathcal{R}(X)$ is discrete if and only if X is rational.*

Proof. The product of the point-reflexions in $t, u \in X$ (in this order) is the translation by $2(u - t)$. The discreteness of $\mathcal{R}(X)$ is equivalent to these translations generating a discrete lattice, which, in turn, is equivalent to the condition of the lemma. $\square$

If P is a regular polytope with ambient space E , then we similarly call P *rational* if its vertex-set is rational. We have the following very general construction.

Theorem 4.2. *Let P be a rational regular n -polytope in the euclidean space E , with symmetry group $\mathcal{G}_0 = \langle R_1, \dots, R_n \rangle$ and initial vertex w , and suppose that $v \in R_1 \cap \dots \cap R_n$. Let $R_0 = \{w\}$ be the point-reflexion in the point w . Then $\mathcal{G} := \langle R_0, \dots, R_n \rangle$ is the group of a discrete regular $(n + 1)$ -apeirotope $\text{apeir } P$, with 2-faces apeirogons, and vertex-figure P at the initial vertex v .*

Proof. Observe that we do not insist that P be of full rank here. Moreover, we do not demand that $v \in \text{aff } V_0$, with V_0 the vertex-set of P .

We first observe that the 2-faces of apeir P are indeed apeirogons, which are zigzag unless $P = \{ \}$ (a segment) and $v \in \text{aff } P$.

Note that $xR_0 = 2w - x$. We see that $\mathcal{G}$ is the semidirect product $\mathcal{G} = \mathcal{R}(V_0) \ltimes \mathcal{G}_0$, and that the set $V := v\mathcal{G}$ is the vertex-set of apeir P . Moreover, the translation subgroup Λ of $\mathcal{R}(X)$ is

$$\Lambda = \langle 2(u - t) \mid t, u \in V_0 \rangle.$$

It then follows that a general element of $\mathcal{G}$ is uniquely expressible in the form $x \mapsto xG + y$ or $x \mapsto (2w - x)G + y$, with $G \in \mathcal{G}_0$ and $y \in \Lambda$.

Now consider a subgroup $\langle R_0, \dots, R_k \rangle$ of $\mathcal{G}$ for some k . An element of this which also belongs to $\mathcal{G}_0$ must be of the form $x \mapsto xG$ with $G \in \mathcal{G}_0$, and therefore actually lies in $\langle R_1, \dots, R_k \rangle$. By Proposition 2E16 of [7], this shows that $\mathcal{G}$ is a C-group, so that apeir P is genuinely polytopal. $\square$

We call apeir P the *free abelian apeirotope on P* , or *with vertex-figure P* , and *base vertex v* . The construction is not quite the dual of the free extension of Theorem 4D4 of [7], because its geometric nature imposes commutativity conditions on it; additionally, apeir P depends on the particular realization P , rather than on the original abstract regular polytope $\mathcal{P}$. In fact, there is nothing in the construction which depends on P being rational; however, if it is not, then the resulting apeirotope will not be discrete, which is contrary to the ground rules established at the end of Section 3.

Remark 4.3. If E is the ambient space of apeir P , let $H := \text{aff } V_0$ (in the notation of Theorem 4.2). Then $H = E$ means that v is the centroid of V_0 . Otherwise, H is a hyperplane in E , and the vertex-set V of apeir P lies in two hyperplanes parallel to, and equidistant from, H .

Remark 4.4. It may happen that P is centrally symmetric about v . In this case the point-reflexion in v is in $\mathcal{G}_0$, and Λ is a subgroup of index 2 in the full group of translations, which is generated by the translations $2u$ with $u \in V_0 - v$.

5. Polytopes in All Dimensions

To avoid repetition of the same arguments in subsequent sections, we first describe the families of regular polytopes and apeirotopes of full rank which occur in all dimensions. The exceptional examples are to be found in dimensions two, three and four only. For dimensions greater than four, we therefore make a forward appeal to the completeness of the enumeration in Section 8, and the fact that two potential examples in dimensions six and eight are shown in Section 10 to be non-polytopal.

A core result about finite regular polytopes of full rank, which at present we cannot prove except on a case-by-case basis, is the following.

Theorem 5.1. *Let P be a finite faithfully realized regular polytope of full rank. Then its symmetry group is either a Coxeter group (generated by hyperplane reflexions) or the direct product of a Coxeter group with a cyclic group of order two (generated by the reflexion in a point). Moreover, if P is not a polygon or a pentagonal polytope, then it is obtained from a string Coxeter group by operations π and κ_0 alone.*

In the following we assume that Theorem 5.1 holds in each lower dimension than the one under consideration. As before, we write $\mathcal{G}(P) = \langle S_0, \dots, S_{n-1} \rangle$ for the symmetry group of a putative such polytope P of rank and dimension n ; the general method we apply is to consider the vertex-figure Q of P . As usual, we take o to be the centre of P . Then $\mathcal{G}(Q)$ contains a subgroup $\langle R_1, \dots, R_{n-1} \rangle$ of index at most 2, with $R_1, \dots, R_{n-1}$ hyperplane reflexions in E whose mirrors contain o . The distinguished generator S_0 of $\mathcal{G} := \mathcal{G}(P)$ is either a hyperplane or a line. In the former case set $R_0 := S_0$, and in the latter $R_0 := -S_0$. Then $\mathcal{K} := \langle R_0, \dots, R_{n-1} \rangle$ is a finite Coxeter group, although not necessarily with the standard generating reflexions. We therefore look among such groups $\mathcal{K}$ for ones to which operations such as π and κ_j can be applied (for some j , although only $j = 0$ and, initially at least, $j = 1$ are allowed); in this context, we also bear in mind Proposition 2.1, which eliminates most potential applications of π .

Thus, for general $n \geq 5$, with the two exceptions just mentioned, we begin with a string group $\mathcal{K} := \langle R_0, \dots, R_{n-1} \rangle$ generated by hyperplane reflexions, and attempt to apply one or more of π , κ_0 and κ_1 to it. However, we only have the possibilities $\mathcal{K} = [3^{n-1}]$, $[3^{n-2}, 4]$ and $[4, 3^{n-2}]$, with the latter two the same group in dual form. Proposition 2.1 excludes any application of π . The only cases, therefore, are the three universal polytopes

$$\{3^{n-1}\}, \quad \{3^{n-2}, 4\}, \quad \{4, 3^{n-2}\}, \quad (5.1)$$

namely, the n -simplex, n -cross-polytope and n -cube, respectively, and the results of applying κ_0 or κ_1 to them (if a polytope results). We treat them in turn.

Theorem 5.2. *Let $n \geq 3$. Then $\{3^{n-1}\}^{\kappa_0} = \{3^{n-1}\} \diamond \{ \}$ exists (as a polytope); it has group $[3^{n-1}] \times C_2$, and $2(n+1)$ vertices (namely, those of the simplex and its opposite). Further, $\{3^{n-1}\}^{\kappa_1}$ is not a polytope.*

Proof. The assertion for κ_0 is straightforward. We remark that we should verify the intersection property (2.1) in this case and those following, but, because we have explicit geometric descriptions of the groups, this is largely unnecessary. Note, by the way, that (in terms of the symmetry group $\mathcal{G} = \langle S_0, \dots, S_{n-1} \rangle$) we have $K_0 = (S_0 S_1)^3$.

For κ_1 , with the indices shifted by 1, we have $K_1 = (S_1 S_2)^3 \in \mathcal{G}$. Now K_1 is the half-turn whose mirror is an altitude of the regular n -simplex. We have all the conjugates of this half-turn; the angle between any two of their mirrors is $\arccos(1/n)$, and so their product is an ordinary rotation through $2 \arccos(1/n)$, which is an irrational multiple of π for each $n \geq 3$. Hence the resulting group is not finite, and so no polytope results. $\square$

For κ_0 , we can go down to $n = 2$ with the hexagon $\{6\}$, while for $n = 3$ we obtain the Petrial $\{6, 3\}_4$ of the cube $\{4, 3\}$.

Theorem 5.3. *Let $n \geq 3$. Then $\{3^{n-2}, 4\}^{\kappa_0}$ exists (as a polytope); it has the same group $[3^{n-2}, 4]$ and the same vertices as $\{3^{n-2}, 4\}$. Further, $\{3^{n-2}, 4\}^{\kappa_1}$ is not a polytope.*

Proof. The argument in this case is rather different. First, we have

$$K_0 = (R_0 \cdots R_{n-1})^n$$

in the original group $\langle R_0, \dots, R_{n-1} \rangle = [3^{n-2}, 4]$. However, once again, we have $K_0 = (S_0 S_1)^3$ in the group $\mathcal{G} = \langle S_0, \dots, S_{n-1} \rangle$. It follows that the operation which yields the group is invertible. The new group does give a polytope, whose k -faces (for $k = 2, \dots, n-1$) are mixes $\{3^{k-1}\} \diamond \{ \}$, though these are $(k+1)$ -dimensional (they have the vertices of the $(k+1)$ -dimensional cross-polytope $\{3^{k-1}, 4\}$), and so are not copies of $\{3^{k-1}\}^{\kappa_0}$. Since the mix $\{3^{n-2}, 4\} \diamond \{ \}$ (with $4n$ vertices) is the universal regular polytope with facet $\{3^{n-2}\} \diamond \{ \}$ and vertex-figure $\{3^{n-3}, 4\}$, we see that $\{3^{n-2}, 4\}^{\kappa_0}$ is not universal of its kind. Notice that $\{3^{n-2}, 4\}^{\kappa_0}$ is flat (meaning, we recall, that each vertex belongs to each facet), with 2^{n-1} facets.

However, we cannot apply κ_1 . Taking the generators of $[3^{n-2}, 4]$ in the standard form

$$\begin{aligned} R_j: & \quad x \mapsto (\xi_1, \dots, \xi_j, \xi_{j+2}, \xi_{j+1}, \xi_{j+3}, \dots, \xi_n) \quad \text{for } j = 0, \dots, n-2, \\ R_{n-1}: & \quad x \mapsto (\xi_1, \dots, \xi_{n-1}, -\xi_n), \end{aligned}$$

with $x = (\xi_1, \dots, \xi_n)$, so that

$$S_1: \quad x \mapsto (\xi_1, -\xi_3, -\xi_2, -\xi_4, \dots, -\xi_n),$$

with $S_j = R_j$ for other j , we readily see that $\langle S_0, \dots, S_{n-2} \rangle = B_n$ (see Table 3B1 of [7]), the group consisting of all permutations of the coordinates $\xi_1, \dots, \xi_n$, together with all even numbers of changes of sign. The intersection of this with $\langle S_1, \dots, S_{n-1} \rangle$ clearly contains more than $\langle R_1, \dots, R_{n-2} \rangle \times \langle K_1 \rangle$, which consists of all permutations, together with changing the signs of all coordinates simultaneously. Thus the intersection property (2.1) is violated, and we do not obtain a polytope. $\square$

Observe that $\{3, 4\}^{\kappa_0} = \{6, 4\}_3 = \{3, 4\}^\pi$. Indeed, we have the same polyhedron, but with different generators of its group.

Theorem 5.4. *Let $n \geq 3$. Then $\{4, 3^{n-2}\}^{\kappa_0}$ exists (as a polytope). Further, $\{4, 3^{n-2}\}^{\kappa_1}$ is not a polytope.*

Proof. The result of applying κ_0 to the cube depends on whether n is even or odd. First note that

$$K_0 = (R_0 \cdots R_{n-1})^n$$

in the original Coxeter group $[4, 3^{n-2}]$; thus $S_0 = R_0 K_0$. We also have

$$(S_0 \cdots S_{n-1})^n = (R_0 \cdots R_{n-1})^n K_0^n = K_0^{n+1} = \begin{cases} K_0, & \text{if } n \text{ is even,} \\ I, & \text{if } n \text{ is odd.} \end{cases}$$

Thus κ_0 is an involutory mixing operation if n is even.

When n is even, we obtain an isomorphic copy of the cube with the same vertices; we denote this by $\{4, 3^{n-2}\}^{\text{alt}}$, where the affix “alt” stands for “alternative” (and not “alternate”, as when n is odd). When n is odd, however, the result identifies opposite vertices of the cube, and produces $\{4, 3^{n-2}\}/2 = \{4, 3^{n-2}\}_n$, where the suffix in the last expression indicates that the group is obtained from $\langle R_0, \dots, R_{n-1} \rangle = [4, 3^{n-2}]$ by the imposition of the extra relation $(R_0 \cdots R_{n-1})^n = E$. The vertices are alternate vertices of those of the cube, say all $(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_j = \pm 1$ for $j = 1, \dots, n$ and $\varepsilon_1 \cdots \varepsilon_n = 1$, and the symmetry group is the group B_n mentioned before; in this case we are safe in using the same notation for the abstract and geometric polytopes. (Compare here the description of the realization space of the cube in Section 5B of [7].) Moreover, as abstract polytopes,

$$\{4, 3^{n-2}\} = \{4, 3^{n-2}\}_n \diamond \{ \},$$

because the segment $\{ \}$ is not a realization of $\{4, 3^{n-2}\}_n$ when n is odd.

The polytope $\{4, 3^{n-2}\}_n$, with n odd, does not just have the same vertices as the half-cube ($h\gamma_n$ in the notation of [2]), but the same symmetry group. This is what leads to the anomalous cases which we have to eliminate in Section 10.

Finally, observe that κ_1 cannot be applied to $[4, 3^{n-2}]$. The argument (for $n \neq 4$) is similar to that in Theorem 5.2. This time, $(S_1 S_2)^3$ is the half-turn about the axis joining (say) (1^n) to the antipodal vertex. We also have the half-turn about the axis through e_1 , and the product of these two is an ordinary rotation through $2 \arccos(1/\sqrt{n})$. Except when $n = 4$, this angle is irrational, and so the resulting group is infinite.

The case $n = 4$ can also be excluded, but we postpone the discussion of this until Section 8. $\square$

6. Apeirotopes in All Dimensions

We now move on to the apeirotopes. We first recall that the basic generator R_0 must be a point or a hyperplane. In the latter case, inspection shows readily that the only possible vertex-figures (among those considered in Section 5) are the cross-polytope $\{3^{n-2}, 4\}$ and its image $\{3^{n-2}, 4\}^{\kappa_0}$ under κ_0 . Indeed, the same arguments as in the proofs of Theorems 5.2 and 5.4 show that otherwise $(R_0 R_1)^2$ is a rotation through an irrational angle (for $n = 3$ or $n \geq 5$). We thus only obtain $\{4, 3^{n-2}, 4\}$ (the cubic tessellation) and its image $\{4, 3^{n-2}, 4\}^{\kappa_1}$ under κ_1 (the index of κ is, of course, shifted by 1).

The apeirotope $\{4, 3^{n-2}, 4\}^{\kappa_1}$ is unfamiliar, and so deserves a little more attention. A purely geometric approach will tell us much of what we need to know. In fact, we begin (to avoid repetition of essentially the same arguments later) by working in a rather more general context than the present one.

We thus suppose that $n \geq 3$, and that we have a universal $(n + 1)$ -apeirotope $\{p_1, \dots, p_n\}$ in $\mathbb{E}^n$, whose (Coxeter) group is $[p_1, \dots, p_n] = \langle R_0, \dots, R_n \rangle$ in the usual way; thus each R_j is a hyperplane reflexion. Write $K = K_1$ for the point-reflexion in the initial vertex, which as usual we take to be o . Then the group of $\{p_1, \dots, p_n\}^{\kappa_1}$ is given by

$$(R_0, \dots, R_n) \mapsto (R_0, R_1 K, R_2, \dots, R_n) =: (S_0, \dots, S_n).$$

By definition, the new vertex-figure is $\{p_2, \dots, p_n\}^{\kappa_0}$. If the new apeirohedron is of type $\{q_1, \dots, q_n\}$, then clearly $\{q_3, \dots, q_n\} = \{p_3, \dots, p_n\}$, because κ_1 does not affect any of the relevant group generators $R_2, \dots, R_n$.

Now K clearly preserves the affine (actually, linear) hull L_k of the base k -face, so that $\dim L_k = k$ for each $k \leq n$. In particular, for $k = 3$, write $T_j := S_j \cap L_3$ for $j = 0, 1, 2$. Then $T_1 = S_1 \subset L_3$, while each of T_0 and T_2 is a plane; hence the corresponding *dimension vector* is $\dim(T_0, T_1, T_2) := (\dim T_0, \dim T_1, \dim T_2) = (2, 1, 2)$. Reference to [1], [6] or Section 7E of [7] then suggests that the new 3-face is a Petrie–Coxeter polyhedron (or sponge). To verify this, observe, first, that the polygon $\{q_1\}$ is given by

$$\frac{1}{q_1} + \frac{1}{p_1} = \frac{1}{2},$$

because the edge $F_1 S_1$ through o in F_3 adjacent to the base edge F_1 goes along the same line as before, but in the opposite direction. Second, the section $\{q_2\}$ is given by

$$q_2 = \begin{cases} p_2, & \text{if } p_2 \text{ is even,} \\ 2p_2, & \text{if } p_2 \text{ is odd.} \end{cases}$$

In practice, because $p_2 = 3$ or 4 , we see that (exactly as for q_1), we have

$$\frac{1}{q_2} + \frac{1}{p_2} = \frac{1}{2}.$$

Now the 2-faces are planar, and the vertex-figures of 3-faces are skew, which is appropriate for some Petrie–Coxeter apeirohedra $\{q_1, q_2 | h\}$, with a suitable h . Indeed, h is given by the period of

$$S_0 S_1 S_2 S_1 = R_0 \cdot R_1 K \cdot R_2 \cdot R_1 K = R_0 R_1 R_2 R_1,$$

so that h is 3 if $\{p_1, p_2\} = \{3, 3\}$ or 4 if $\{p_1, p_2\} = \{3, 4\}$ or $\{4, 3\}$ (when $n \geq 3$, these are the only possibilities); this is just that h for the Petrie–Coxeter polyhedron.

We now return to the specific case $\{4, 3^{n-2}, 4\}$, with $n \geq 3$. The 3-faces are thus Petrie–Coxeter apeirohedra $\{4, 6 | 4\}$, whose vertex-sets form copies of the cubic lattice $\mathbb{Z}^3$. To see what the general face looks like, we give suitable generators for the symmetry group. In terms of the usual cartesian coordinates $x = (\xi_1, \dots, \xi_n)$, they are

$$\begin{aligned} S_0: & x \mapsto (1 - \xi_1, \xi_2, \dots, \xi_n), \\ S_1: & x \mapsto (-\xi_2, -\xi_1, -\xi_3, \dots, -\xi_n), \\ S_j: & x \mapsto (\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \xi_j, \xi_{j+2}, \dots, \xi_n) \quad \text{for } j = 2, \dots, n-1, \\ S_n: & x \mapsto (\xi_1, \dots, \xi_{n-1}, -\xi_n). \end{aligned}$$

It is easy to check similarly that, for $k = 4, \dots, n$, the k -faces are k -dimensional apeirotopes, whose vertices form cubic lattices $\mathbb{Z}^k$. It is tedious, but routine, to verify that, with $k = 3, \dots, n$ and $\varepsilon_j = (-1)^j$ for $j = 1, \dots, k$, the translation by $(\varepsilon_1, \dots, \varepsilon_k, 0^{n-k})$ (in L_k), combined with the reflexion about L_k , is

$$\begin{aligned} & (S_0 \cdots S_{k-1})^k, & \text{if } k \text{ is odd,} \\ & (S_0 \cdots S_{k-1})^k (S_1 S_2)^3, & \text{if } k \text{ is even,} \end{aligned}$$

bearing in mind that $(S_1 S_2)^3 = K$. Conjugation by elements of $\langle S_0, \dots, S_{k-1} \rangle$ shows that, as a k -dimensional k -apeirotope, the symmetry group of the base face F_k has a translation subgroup generated by all translations of the form $((\pm 1)^k, 0^{n-k})$, but that, when $k < n$, only the subgroup $2\mathbb{Z}^k$ of index 2 generated by the $2e_j$ with $j = 1, \dots, k$ corresponds to translations in the whole group $\mathcal{G}$. Observe that F_k is built up from copies of F_{k-1} in k different orientations, corresponding to the $(k-1)$ -dimensional coordinate subspaces in L_k . The case $k = n$ shows that the facets are all 2^{n-1} distinct translates of the base facet F_n , corresponding to the index of the translation subgroup of F_n in $\mathbb{Z}^n$; this accords with the fact that the vertex-figure $\{3^{n-2}, 4\}^{k_0}$ is flat, also with 2^{n-1} facets.

Summarizing this discussion, we have

Theorem 6.1. *In each dimension $n \geq 3$, there are two discrete regular $(n+1)$ -apeirotopes $\{4, 3^{n-2}, 4\}$ and $\{4, 3^{n-2}, 4\}^{k_1}$ with square 2-faces; for $n = 3$ and $n \geq 5$, these are the only such apeirotopes with finite 2-faces. The 3-faces of $\{4, 3^{n-2}, 4\}^{k_1}$ are Petrie–Coxeter apeirohedra $\{4, 6|4\}$.*

The geometry of the facet F_n of $\{4, 3^{n-2}, 4\}^{k_1}$ can be explained as follows. Its vertex-figure is the mix $\{3^{n-2}\} \diamond \{ \}$ of an $(n-1)$ -simplex and a segment. It has the vertices and some of the edges of the cross-polytope $\{3^{n-2}, 4\}$; however, the edges of two opposite facets do not occur, and correspond to holes. These holes themselves form the faces of n -cubes; translates of one such cube by the sublattice $\Lambda_{(1^n)}$ (generated by the vectors $(\varepsilon_1, \dots, \varepsilon_n)$, with $\varepsilon_i = \pm 1$ for $i = 1, \dots, n$) give all the holes. The 2-faces of F_n are then the remaining square faces of the cubic tiling $\{4, 3^{n-2}, 4\}$.

Note that F_n is also the typical n -face of $\{4, 3^{m-2}, 4\}^{k_1}$, for each $m > n$.

We end this part by saying a little more about the abstract group of the apeirotope $\{4, 3^{n-2}, 4\}^{k_1}$. By definition, its vertex-figure is $\{3^{n-2}, 4\}^{k_0}$. Its (abstract) facet $\mathcal{F}^n$, say, is a face of $\{4, 3^{m-2}, 4\}^{k_1}$ for each $m \geq n$, as we have remarked, and so the general facial structure is specified by this. In fact, it is universal, in the following sense.

Theorem 6.2. *For $n \geq 3$, the facet $\mathcal{F}^n$ of $\{4, 3^{n-2}, 4\}^{k_1}$ is the universal polytope*

$$\mathcal{F}^n = \{\mathcal{F}^{n-1}, \{3^{n-2}\} \diamond \{ \}\},$$

with starting point $\mathcal{F}^3 = \{4, 6|4\}$.

Proof. To see this, we use the circuit criterion [7, Theorem 2F4]; roughly speaking, this says that the group of a regular polytope is given by that of its vertex-figure, together with relations arising from its edge-circuits. The vertex-figure is undoubtedly $\{3^{n-2}\} \diamond \{ \}$. This has the same vertices as the cross-polytope $\{3^{n-2}, 4\}$; however, the edges of some pair of opposite facets $\{3^{n-2}\}$ correspond to square holes $\{4\}$ (those of 3-faces $\{4, 6|4\}$), while the remaining edges correspond to faces $\{4\}$. Now the concrete realization F_n of $\mathcal{F}^n$ has all the vertices and edges of the cubic tiling $\{4, 3^{n-2}, 4\}$, while the square faces of the tiling are either faces of F_n or holes. It follows at once that any edge-circuit can be contracted over the faces or holes, and so no extra relations arise from edge-circuits, apart from those already given by those of $\{4, 6|4\}$. The claim of the theorem follows at once. $\square$

The group of $\{4, 3^{n-2}, 4\}^{\kappa_1}$ admits a similar description.

Theorem 6.3. *With $\mathcal{F}^n$ as previously the facet of $\{4, 3^{n-2}, 4\}^{\kappa_1}$, the apeirotope itself is*

$$\{4, 3^{n-2}, 4\}^{\kappa_1} = \{\mathcal{F}^n, \{3^{n-2}, 4\}^{\kappa_0}\},$$

the universal polytope of its type.

Proof. The argument is just like that for Theorem 6.2; once again, the edge-circuits can be contracted over squares and holes of 3-faces $\{4, 6 \mid 4\}$ (in fact, only the squares are needed here), and an appeal to the circuit criterion [7, Theorem 2F4] finishes the proof. $\square$

Remark 6.4. We can rephrase the results of Theorems 6.2 and 6.3 in the following way. The facet $\mathcal{F}^n$ of $\{4, 3^{n-2}, 4\}^{\kappa_1}$ is universal with 3-face $\{4, 6 \mid 4\}$ and vertex-figure $\{3^{n-2}\} \diamond \{\}$, while (as the proof of the latter theorem shows) $\{4, 3^{n-2}, 4\}^{\kappa_1}$ itself is universal with 2-face $\{4\}$ and vertex-figure $\{3^{n-2}, 4\}^{\kappa_0}$.

We now consider the alternative case, when R_0 is a point. Necessarily, then, we are in the situation of Theorem 4.2, since R_0 must be the base vertex of the vertex-figure. The final result of this section is thus the following.

Theorem 6.5. *Each of the six regular n -polytopes P of (5.1) and Theorems 5.2–5.4 is the vertex-figure of a corresponding discrete regular $(n+1)$ -apeirotope $\text{apeir } P$.*

Proof. Indeed, each of the potential vertex-figures P is rational (as is extremely easy to check), and so, by Theorem 4.2, yields a regular apeirotope $\text{apeir } P$. $\square$

It is of interest to describe the structure of these apeirotopes in a little detail. The easiest way to present $\text{apeir}\{3^{n-1}\}$ and its relative $\text{apeir}(\{3^{n-1}\} \diamond \{\})$ is to work in the hyperplane $H := \{(\xi_0, \dots, \xi_n) \in \mathbb{E}^{n+1} \mid \xi_0 + \dots + \xi_n = 0\}$, and, for even greater simplicity, to take the vertices to lie in the intersection $\mathbb{Z}^{n+1} \cap H$.

We may take the $n+1$ vertices of $\{3^{n-1}\}$ to be the points $\frac{1}{2}((n+1)e_j - u)$ for $j = 0, \dots, n$, with $\{e_0, \dots, e_n\}$ the standard basis of $\mathbb{E}^{n+1}$ and $u := (1^{n+1}) = e_0 + \dots + e_n$; the initial vertex of $\text{apeir}\{3^{n-1}\}$ is the centre $v := o$ of $\{3^{n-1}\}$, and that of the vertex-figure is $w := \frac{1}{2}((n+1)e_0 - u)$. The translation lattice Λ (as in Theorem 4.2) is thus generated by all $(n+1)(e_j - e_k)$, so that $\Lambda = (n+1)\mathbb{Z}^{n+1} \cap H$, and the vertex-set of the apeirotope is $\Lambda + \{o, (n+1)e_0 - u\}$. The case $n = 3$ corresponds to the well-known *diamond net*. We end by remarking that we already showed in Theorem 7F13 of [7] (see also [6] for the three-dimensional case) that

$$\text{apeir}\{3^{n-1}\} = \{\text{apeir}\{3^{n-2}\}, \{3^{n-1}\}\}$$

is the universal apeirotope of its kind. The starting point is

$$\text{apeir}\{3\} = \{\infty, 3\}_6 = \{6, 3\}^\pi.$$

The vertex-set of $\text{apeir}(\{3^{n-1}\} \diamond \{\})$ (again in H) is the lattice generated by all $(n+1)e_j - u$ for $j = 0, \dots, n$, which is the image of $(n+1)\mathbb{Z}^{n+1}$ under orthogonal projection on H .

Taking the vertices of the (centrally symmetric) cross-polytope $\{3^{n-2}, 4\}$ to be all $\pm \frac{1}{2}e_j$ for $j = 1, \dots, n$, we see that the vertex-set of $\text{apeir}\{3^{n-2}, 4\}$ (and its translation group) is just $\mathbb{Z}^n$. Indeed, its symmetry group $\langle S_0, \dots, S_n \rangle$ is just the symmetry group $\langle R_0, \dots, R_n \rangle = [4, 3^{n-2}, 4]$ of the cubic lattice; it is obtained by means of the involutory mixing operation

$$\kappa_{02}: (R_0, \dots, R_n) \mapsto (R_0(R_2 \cdots R_n)^{n-1}, R_1, \dots, R_n) =: (S_0, \dots, S_n).$$

In the same way, the group of

$$\text{apeir}(\{3^{n-2}, 4\}^{\kappa_0}) = (\text{apeir}\{3^{n-2}, 4\})^{\kappa_1} = \{4, 3^{n-2}, 4\}^{\kappa_1 \kappa_{02}}$$

is also $[4, 3^{n-2}, 4]$. Since the vertex-figure of the latter is flat, so is the apeirotope; it has 2^{n-1} facets.

If we take the vertex-set of the cube $\{4, 3^{n-2}\}$ to consist of all $\frac{1}{2}(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_j = \pm 1$ for $j = 1, \dots, n$, then the vertex-set V of $\text{apeir}\{4, 3^{n-2}\}$ is the n -dimensional analogue of the body-centred cubic lattice, so that

$$V = \{(\zeta_1, \dots, \zeta_n) \in \mathbb{Z}^n \mid \zeta_1 \equiv \dots \equiv \zeta_n \pmod{2}\}.$$

Again, we have a universality result from Theorem 7F13 of [7] (see also [6] for the case $n = 3$):

$$\text{apeir}\{4, 3^{n-2}\} = \{\text{apeir}\{4, 3^{n-3}\}, \{4, 3^{n-2}\}\},$$

with starting point

$$\text{apeir}\{4\} = \{\infty, 4\}_4 = \{4, 4\}^\pi.$$

The translation group is V , since the cube is centrally symmetric.

When n is even, the alternative cube $\{4, 3^{n-2}\}^{\text{alt}} = \{4, 3^{n-2}\}^{\kappa_0}$ has the same vertices as the cube, and the same group. Thus $\text{apeir}(\{4, 3^{n-2}\}^{\text{alt}})$ has the same vertices and the same group as $\text{apeir}\{4, 3^{n-2}\}$. However, when n is odd, $\{4, 3^{n-2}\}_n = \{4, 3^{n-2}\}^{\kappa_0}$ is not centrally symmetric. In this case the translation lattice Λ is generated by all $2(e_j \pm e_k)$ with $1 \leq j < k \leq n$, and the vertex-set is $\Lambda + \{o, u\}$, with $u := (1^n) = e_1 + \dots + e_n$.

We may notice that the same analysis applied to the case $n = 2$ has nearly completed the enumeration of the regular apeirohedra. The only rational polygons are $\{3\}$, $\{4\}$ and $\{6\} = \{3\} \diamond \{\}$, and so these yield the only discrete apeirohedra $\text{apeir}\{q\}$ (with $q > 2$ rational) – they fit into the scheme we have just described. However, if $\langle R_0, R_1, R_2 \rangle$ is the symmetry group of any other regular apeirohedron P in $\mathbb{E}^2$, then each of R_0 , R_1 and R_2 must be a line. But now the Petrie operation π results in the mirror $S_0 = R_0 R_2$ being a point, and so we are back in the former case. Thus a regular apeirohedron of Schläfli type $\{p, q\}$ (with p, q finite) must have $q = 3, 4$ or 6 , and then $p = 6, 4$ or 3 , respectively. Of course, this is a somewhat unusual way to approach the classification problem.

Remark 6.6. Note that the faces of our new polytopes (and apeirotopes) are *of nearly full rank*, in the sense that rank and dimension are the same (in the context of Theorem 3.1). In fact, this is the case with all regular polytopes of full rank: the faces are either of full rank themselves, or of nearly full rank (we have as yet no general explanation for this).

7. Three Dimensions

Although we have already enumerated the regular polytopes and apeirotopes of full rank in $\mathbb{E}^3$ in [6] (see also Sections 7E and 7F of [7]), it is useful briefly to revisit the problem with a little extra insight gained by our new methods.

For the polytopes, in effect we have already carried out the analysis; however, we did not fit the information together in the manner of Section 14. We did observe Theorem 3.5, and that if the basic mirror R_0 is a line (rather than a plane), then both π and κ_0 lead to a group, all of whose generating mirrors are planes. The diagrams of Section 14 are, however, new; with duality δ , they show how these polytopes fit into just two families. In particular, within each family, π and κ_0 group together those polyhedra whose vertex-figures have the same geometric symmetry (thus we distinguish between $\{5\}$ and $\{\frac{5}{2}\}$). For $\{3, 4\}$, applying π or κ_0 yields the same polyhedron, but with different basic generating reflexions. Note, by the way, that the diagram listing the pentagonal polyhedra can be extended at the bottom by δ to wrap back around to the top.

For the apeirotopes, we need make only two remarks. First, since $\{6, 3\}_4 = \{3, 3\}^{\kappa_0} = \{3, 3\} \diamond \{ \} = \{4, 3\}^\pi$, two families, which in general dimensions are distinct, fit together in $\mathbb{E}^3$. Second, since (as in Section 6) $\{4, 3, 4\}$ and $\text{apeir}\{3, 4\}$ are related by κ_{02} , and so have the same group, it therefore follows that the apeirotopes fit into two families. Note that it was shown in [6] (see also Corollary 7F6 of [7]) that

$$\{4, 3, 4\}^\pi = \{\{4, 6|4\}, \{6, 3\}_3\}$$

is the universal apeirotope of its kind. The analysis of Section 6 shows that our enumeration is complete, because all rational polyhedra are accounted for.

8. Polytopes in Four Dimensions

In this section and the next, we classify the regular polytopes and apeirotopes of full rank in $\mathbb{E}^4$.

For the polytopes, our first observation is that the groups of the regular 3-polytopes of full rank are all generated by plane reflexions (and so products of these with $\pm I$ are not needed). This takes care of the group $\mathcal{G}_0$ of the vertex-figure. Additionally, as we remarked in Section 3, if $\dim R_0 = 1$ (rather than 3), then we can replace R_0 by $-R_0$ (an application of κ_0). Our conclusion is the following.

Proposition 8.1. *The symmetry group of a regular 4-polytope of full rank is either a group $\mathcal{G}$ generated by hyperplane reflexions, or a product $\mathcal{G} \times \mathcal{C}_2$ of such a group with the group $\mathcal{C}_2$ generated by the central inversion.*

Thus our procedure is straightforward. We can start with a reflexion group $\mathcal{G}$ which is the symmetry group of a classical regular polytope, and try to apply to it operations κ_0 , κ_1 or π to obtain another C-group.

We deal with the pentagonal family first. Proposition 2.1 has excluded the possibility of applying π to any of them already. By the same token, this excludes κ_1 as well, as the classification of the 3-polytopes shows (any vertex-figure obtainable by κ_0 is obtainable – from another polyhedron – by π). There remains κ_0 . However, it is easy to see that this always works. Consider any such polytope $P = \{p, q, r\}$, with group $\mathcal{G} = \langle R_0, \dots, R_3 \rangle$. Thus, with K_0 the central inversion as in Section 3, the group of the new polytope $Q := P^{\kappa_0}$ is given by

$$(R_0, \dots, R_3) \mapsto (R_0 K_0, R_1, R_2, R_3) =: (S_0, \dots, S_3).$$

The new facet will be $\{p, q\} \diamond \{\}$, with twice as many vertices; the new edges will join vertices u, w , with $u \in F$ (where F is an original facet of P) and $w \in -F$ (the opposite facet in P), such that $\{u, -w\}$ is an original edge of P (in F). If v is the base vertex of P , which is also the base vertex of Q , then an element of $\langle S_0, S_1, S_2 \rangle$ which fixes v must take F into itself (rather than $-F$), and hence actually belongs to $\langle R_0, R_1, R_2 \rangle$. Thus

$$\langle S_0, S_1, S_2 \rangle \cap \langle S_1, S_2, S_3 \rangle = \langle R_0, R_1, R_2 \rangle \cap \langle R_1, R_2, R_3 \rangle = \langle R_1, R_2 \rangle = \langle S_1, S_2 \rangle,$$

which verifies the intersection property. Thus κ_0 always yields another polytope; observe that the group remains the same.

For the simplex $\{3, 3, 3\}$, the same arguments show that neither π nor κ_1 can be applied, but, as in Section 5, applying κ_0 gives the mix $\{3, 3, 3\} \diamond \{\}$.

The analysis for the cross-polytope $\{3, 3, 4\}$ (which we already saw in Section 5) and the 24-cell $\{3, 4, 3\}$ is again the same. In particular, π cannot be applied, neither can κ_1 to $\{3, 3, 4\}$, but κ_0 can in both cases.

We finally come to the most interesting case, that of the cube $\{4, 3, 3\}$. We saw in Section 5 that applying κ_0 yields the isomorphic cube $\{4, 3, 3\}^{\text{alt}}$ with the same vertices. Its facets are, of course, 3-cubes $\{4, 3\}$. However, geometrically, they occur in the form $\{4, 3\}_3 \diamond \{\}$. To obtain one such, in a facet of the usual 4-cube $\{4, 3, 3\}$ inscribe a tetrahedron or, rather, its Petrial $\{4, 3\}_3$. In the opposite facet of the 4-cube is a translate of $\{4, 3\}_3$. The edges of the new facet $\{4, 3\}_3 \diamond \{\}$ then switch between these two copies.

In contrast to the other cases, π does apply to $\{4, 3, 3\}$ (note that Proposition 2.1 is not relevant here). The new facets are tori $\{4, 4|4\}$; we thus obtain a polytope of type $\{\{4, 4|4\}, \{4, 3\}_3\}$. In terms of the group generators, the operation is

$$(R_0, \dots, R_3) \mapsto (R_0, R_1 R_3, R_2, R_3) =: (S_0, \dots, S_3).$$

The “hole” relation $(S_0 S_1 S_2 S_1)^4 = E$ for the torus derives from $(R_0 R_1)^4 = E$. The Petrie operation is involutory, and it is straightforward to check that we recover the relations for the group of the 4-cube from the abstract relations for $\{\{4, 4|4\}, \{4, 3\}_3\}$. In other words, the new 4-polytope is universal.

Naturally, if we apply κ_0 to $\{\{4, 4|4\}, \{4, 3\}_3\}$, or π to $\{4, 3, 3\}^{\text{alt}}$, we obtain an isomorphic copy $\{\{4, 4|4\}, \{4, 3\}_3\}^{\text{alt}}$ of the former.

Finally, we must consider κ_1 . The analysis in Section 5 does not immediately exclude the case $\{4, 3, 3\}^{\kappa_1}$, because we actually obtain a finite group (in fact, it is $[3, 4, 3]$).

However, the resulting “polytope” would have vertex-figure $\{3, 3\}^{\kappa_0} = \{6, 3\}_4$ (whose vertices are those of the cube $\{4, 3\}$), and at least the edges of $\{4, 3, 3\}$, and so would have to have the edges of $\{3, 4, 3\}$. With these edges and vertex-figure, we then see that we have reached $\{3, 4, 3\}^\pi$, which is not permitted by Proposition 2.1.

Note, by the way, that we can apply κ_1 to $\{3, 4, 3\}$, but, in accord with halving the order of the group of the vertex-figure, we pass to one of the three copies of $\{\{4, 4|4\}, \{4, 3\}_3\}$ which is inscribed in it (as three copies of $\{4, 3, 3\}$ are).

9. Apeirotopes in Four Dimensions

We now move on to the regular apeirotopes; naturally, we bear in mind the classification of the finite regular 4-polytopes in $\mathbb{E}^4$ which we have just completed. We can discard the pentagonal polytopes as putative vertex-figures, and so there remain just the ten rational ones. Since the free abelian apeirotope construction “apeir” applies to each of them, we are left to consider the 5-apeirotopes for which the initial reflexion mirror R_0 is a hyperplane.

We can immediately eliminate $\{3, 3, 3\}$ and $\{3, 3, 3\} \diamond \{ \}$ from consideration, since there is no discrete infinite group $[p, 3, 3, 3]$ in $\mathbb{E}^4$ (again, the argument is like that of Theorem 5.2, in that $(R_0 R_1)^2$ would be a rotation through an irrational angle). We are thus reduced to considering $\{3, 3, 4\}$, $\{4, 3, 3\}$, $\{3, 4, 3\}$, and the various polytopes derived from them by κ_0 and π (in the second case only). We shall see that all yield regular apeirotopes.

We have already treated $\{4, 3, 3, 4\}$ in Section 6, but we repeat the facts that the 3-faces of $\{4, 3, 3, 4\}^{\kappa_1}$ are Petrie–Coxeter apeirohedra $\{4, 6|4\}$, and that the whole apeirotope has eight facets.

The same analysis as in Section 6 shows that the 3-faces of $\{3, 3, 4, 3\}^{\kappa_1}$ are $\{6, 6|3\}$, while those of $\{3, 4, 3, 3\}^{\kappa_1}$ are $\{6, 4|4\}$. We describe these two apeirotopes in greater detail.

For the first, we take the symmetry group of $\{3, 3, 4, 3\}$ in the form

$$\begin{aligned} R_0: & x \mapsto (1 + \xi_2, -1 + \xi_1, \xi_3, \xi_4), \\ R_1: & x \mapsto (\xi_1, \xi_3, \xi_2, \xi_4), \\ R_2: & x \mapsto (\xi_1, \xi_2, \xi_4, \xi_3), \\ R_3: & x \mapsto (\xi_1, \xi_2, \xi_3, -\xi_4), \\ R_4: & x \mapsto x(I - \tfrac{1}{2}u^\top u), \end{aligned}$$

where $u := (1, 1, 1, 1)$. (For the last, note that the reflexion in the hyperplane through o with unit normal v is $I - 2v^\top v$ in matrix form.) We choose this form for the generators, because then the vertices of $\{3, 3, 4, 3\}$ form the lattice

$$\Lambda := \{(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{Z}^4 \mid \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \equiv 0 \pmod{2}\} \quad (9.1)$$

generated by the vectors $e_j \pm e_k$ for $1 \leq j < k \leq 4$, and so have integer cartesian coordinates.

The symmetry group $\langle S_0, \dots, S_4 \rangle$ of $\{3, 3, 4, 3\}^{\kappa_1}$ is then given by $S_j := R_j$ for $j \neq 1$, and

$$x S_1 = -x R_1 = (-\xi_1, -\xi_3, -\xi_2, -\xi_4).$$

The base edge joins o to $e_1 - e_2 = (1, -1, 0, 0)$. The base 3-face $F_3 = \{6, 6|3\}$ lies in the hyperplane L_3 through o with normal u . Since $(S_1 S_2)^3 = K$ is the point-reflexion in o , its conjugate by S_0 is the point-reflexion in $e_1 - e_2$, and so their product $((S_1 S_2)^3 S_0)^2$ is the translation (in L_3) by $2(e_1 - e_2)$. The conjugates of this under $\langle S_0, S_1, S_2 \rangle$, namely, all $2(e_j - e_k)$ with $1 \leq j < k \leq 4$, actually generate the whole translation subgroup of F_3 .

Since $u S_3 = (1, 1, 1, -1)$, transforming further by $\langle S_0, \dots, S_3 \rangle$ shows that the base facet F_4 is composed of copies of F_3 in hyperplanes with normals all $(1, \pm 1, \pm 1, \pm 1)$. Further, we have already noted that $K = (S_1 S_2)^3 \in \langle S_0, \dots, S_3 \rangle$, so that $\langle R_0, \dots, R_3 \rangle \leq \langle S_0, \dots, S_3 \rangle$. It follows that the point-reflexion in the centre e_1 of the base facet of $\{3, 3, 4, 3\}$ belongs to $\langle S_0, \dots, S_3 \rangle$. Thus the product of K with this, which is the translation by $2e_1$, is in the group; this and its conjugates generate the translation subgroup $2\mathbb{Z}^4$ of F_4 . Observe that the ridges (images of F_3) lie in parallel hyperplanes spanned by vertices at minimal distance apart.

There is a nice picture of the facet F_4 . Its vertex-figure is the mix $\{3, 4\} \diamond \{\}$ of an octahedron and a segment. The octahedra are opposite facets of the vertex-figure $\{3, 4, 3\}$ of $\{3, 3, 4, 3\}$, and their edges correspond to holes $\{3\}$ of the 3-faces $\{6, 6|3\}$. These holes fit together to form 4-cross-polytopes $\{3, 3, 4\}$; these cross-polytopes, in turn, are those of the truncated tiling $\{^4_{3,3,4}\}$ whose vertices are the mid-points of the edges of $\{4, 3, 3, 4\}$. Moreover, the 2-faces of F_4 are hexagonal cross-sections of the cuboctahedra $\{^4_3\}$ of this truncated tiling, and typical 3-faces lie in sections of this tiling spanned by the tetrahedral faces of the cross-polytopal holes.

Finally, the facets of the whole apeirotope occur in three different orientations (corresponding to the three essentially different sets of orthogonal axes spanned by opposite vertices of $\{3, 4, 3\}$, each facet being associated with two of these sets), with $8 = [\Lambda : 2\mathbb{Z}^4]$ translates in each set, giving $3 \cdot 8 = 24$ facets in all. The index 8 also counts the number of distinct ridges in each hyperplane which they span.

We next look at $\{3, 4, 3, 3\}^{K_1}$. To preserve the underlying symmetry that we had before, we take the generators of $[3, 4, 3, 3]$ in a similar form to those of $[3, 3, 4, 3]$, but in the reverse order. We thus take

$$\begin{aligned} R_0: \quad x &\mapsto u + x(I - \tfrac{1}{2}u^T u), \\ R_1: \quad x &\mapsto (-\xi_1, \xi_2, \xi_3, \xi_4), \\ R_2: \quad x &\mapsto (\xi_2, \xi_1, \xi_3, \xi_4), \\ R_3: \quad x &\mapsto (\xi_1, \xi_3, \xi_2, \xi_4), \\ R_4: \quad x &\mapsto (\xi_1, \xi_2, \xi_4, \xi_3), \end{aligned}$$

where $u := (1, 1, 1, 1)$ as before. However, we have doubled the previous scale, since we wish the underlying vertex-set of $\{3, 4, 3, 3\}$, namely,

$$\{(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{Z}^4 \mid \zeta_1 \equiv \zeta_2 \equiv \zeta_3 \equiv \zeta_4 \pmod{2}\} \setminus 2(\Lambda + e_4),$$

with Λ as in (9.1), to consist of integer points. For later reference to the geometry, we note that the base facet $\{3, 4, 3\}$ has vertices

$$2(e_4 \pm e_j) \quad (j = 1, 2, 3, 4), \quad \pm e_1 \pm e_2 \pm e_3 + (2 \pm 1)e_4,$$

a standard form shifted by $2e_4$. Further, the translation subgroup of the group is 2Λ , generated by all $2(e_j \pm e_k)$ for $1 \leq j < k \leq 4$. With these coordinates, the group $\mathcal{G} = \langle S_0, \dots, S_4 \rangle$ of $\{3, 4, 3, 3\}^{\kappa_1}$ has $S_j := R_j$ for $j \neq 1$, and

$$S_1: x \mapsto (\xi_1, -\xi_2, -\xi_3, -\xi_4).$$

Like the original ridge $\{3, 4\}$ of $\{3, 4, 3, 3\}$, the new base ridge $F_3 = \{6, 4|4\}$ spans the hyperplane L_3 through o with normal $e_3 - e_4$. We can first check that $(S_0 S_1)^3$ is the point-reflexion in $2e_1$; under $\langle S_1, S_2 \rangle$ we similarly obtain point-reflexions in $-2e_1$ and $2e_2$, and hence translations in $\langle S_0, S_1, S_2 \rangle$ by $4(e_1 \pm e_2)$. The conjugate of that by $4(e_1 + e_2)$ under S_0 is the translation by $4(e_3 + e_4)$; these three translations generate the subgroup of translation symmetries of F_3 . However, intrinsically, F_3 has twice as many translations as this, namely, those generated by $2((e_1 + e_2) \pm (e_1 - e_2) \pm (e_3 + e_4)) = 4e_j \pm 2(e_3 + e_4)$ for $j = 1, 2$. In fact, in the subgroup $\langle S_0, S_1, S_2 \rangle$, these correspond to glide reflexions, consisting of the product of the translation with the reflexion in L_3 (we do not reproduce the details).

An idea of how a 3-face (such as F_3) fits may be useful. The edges of F_3 occur among those of $\{3, 4, 3, 3\}$, and lie in its section by the affine hull of an octahedron $\{3, 4\}$. This section is, in fact, a copy of the truncated tiling $\{4_{3,4}\}$; the hexagonal faces $\{6\}$ of $\{6, 4|4\}$ are central sections of the cuboctahedra $\{4_3\}$, while the holes $\{4\}$ are central sections of the octahedra $\{3, 4\}$. However, these faces and holes form a somewhat sparse subset. The easiest starting point is an octahedron. This is met by eight cuboctahedra at its triangular faces, and then by six further octahedra at its vertices alone. If we delete this initial octahedron and the edges which meet it, we see that it is surrounded by eight diametral hexagons of cuboctahedra, and six diametral squares of cuboctahedra; these form a truncated octahedron. In a similar fashion, we delete the octahedra which we find by going across the hexagonal faces; carrying on like this, we obtain a tiling of a 3-space by truncated octahedra; its hexagons are those of $\{6, 4|4\}$. Note that each diametral square which survives meets eight diametral hexagons of cuboctahedra; these are the hexagons of $\{6, 4|4\}$ which meet the hole. It should be clear that exactly a quarter of the octahedra are used to form the truncated octahedra; thus the translation subgroup of F_3 has index 4 in that of the original section of the tiling $\{3, 4, 3, 3\}$.

Moving now to the base facet F_4 , with group $\langle S_0, \dots, S_3 \rangle$, we readily see that it contains ridges in hyperplanes with normals all $e_j \pm e_k$ for $1 \leq j < k \leq 4$. The hexagons of F_3 lie in parallel planes, and these are met by images of F_3 alternately at angles $\pm\pi/3$ (corresponding to the glide reflexions mentioned earlier). The translation subgroup of $\langle S_0, \dots, S_3 \rangle$ is now generated by all $4(e_j \pm e_k)$ for $1 \leq j < k \leq 4$, and so is 4Λ . Figure 1 gives a partial view of a typical facet, projected along the affine hull of a hexagonal face, which illustrates this glide reflexion. Only the facets whose normals lie in a plane are shown; the 3-faces project into lines.

Since the translation subgroup of a facet has index $[2\Lambda : 4\Lambda] = 16$ in the whole translation group, we now see fairly easily that all facets are translates of F_4 , and that there are 16 of them.

We have already remarked that we can also apply π to $\{3, 4, 3, 3\}$ and $\{3, 4, 3, 3\}^{\kappa_1}$. The ridges (3-faces) are preserved, and so they remain octahedra $\{3, 4\}$ or Petrie–Coxeter apeirohedra $\{6, 4|4\}$, respectively. In both cases the vertex-figure is $\{\{4, 4|4\}, \{4, 3\}_3\}$

Fig. 1. The facet of $\{3, 4, 3, 3\}^{\kappa_1}$.

(in the latter case, in its alternative form); since the vertex-figure is flat, so are the whole apeirotopes, and each therefore has three facets. We give a few more details about each.

The (base) facet of the former is a locally toroidal regular apeirotope with vertex-figure the toroid $\{4, 4|4\}$, and so is of type $\{\{3, 4\}, \{4, 4|4\}\}$; in fact, it is the universal regular polytope of its type (see Chapter 10 of [7]). There is an interesting further feature of the apeirotope. Ignoring the finer structure, it is of type $\{3, 4, 4, 3\}$. Now clearly it is not self-dual (it has infinitely many vertices, but only three facets). However, the middle reflexion S_2 of its symmetry group has a two-dimensional mirror, and it turns out that, if we replace this mirror by a suitable orthogonal mirror (of the same dimension), then we are, in effect, constructing the reflexion mirrors for another copy, but in dual form (that is, with the order of the reflexions reversed).

In $\{3, 4, 3, 3\}^{\kappa_1\pi}$ the new distinguished generator S_2 is now

$$S_2: x \mapsto (\xi_2, \xi_1, \xi_4, \xi_3).$$

What were formerly, in the base ridge $F_3 = \{6, 4|4\}$, only glide reflexions inducing its intrinsic translation group, now become genuine translations. In contrast to what happens in $\{3, 4, 3, 3\}^{\kappa_1}$, while here the full translation group 2Λ preserves the base facet F_4 , the hyperplanes spanned by the ridges in it now have normals only $e_1 \pm e_2$, $e_1 \pm e_3$, $e_2 \pm e_4$ and $e_3 \pm e_4$, giving two out of three of the sets of orthogonal axes spanned by vertices of $\{3, 4, 3\}$. Figure 2 gives a partial view of this, illustrating those facets whose normals lie in a plane; the projection is to the same scale as that of Fig. 1, and again is a projection along the affine hull of a hexagonal face.

10. Two Non-Examples

There are two possible examples of finite regular polytopes of full rank which we have not so far considered, because we have neglected to look at the following. In our enumeration hitherto, we have tacitly assumed that an application of κ_j to a co- $(j - 1)$ -face to turn a

Fig. 2. The facet of $\{3, 4, 3, 3\}^{\kappa_1 \pi}$.

mirror S_j of dimension $(j + 1)$ into a hyperplane will result in no smaller a group of this co-face. However, of course, this need not be the case, although the sole examples are

$$\{4, 3^{n-2}\}_n \xrightarrow{\kappa_0} \{4, 3^{n-2}\} = \{4, 3^{n-2}\}_n \diamond \{ \},$$

when n is odd.

Now we have already observed that this $\{4, 3^{n-2}\}_n$ has the same vertices as the half-cube $h\gamma_n$ (that is, alternate vertices of the n -cube) and the same symmetry group. However, when $n = 5$ or 7 , $h\gamma_n$ is actually the vertex-figure of a Gosset–Elte polytope $2_{(n-3)1}$, which has a very high degree of symmetry (see Section 11.8 of [2]). It is natural to ask whether replacing the vertex-figure $h\gamma_n$ of $2_{(n-3)1}$ by $\{4, 3^{n-2}\}_n$ (for $n = 5, 7$) can lead to faithfully realized regular polytopes. Notice that, in each case, the mirror of the reflexion S_1 in the corresponding symmetry group $\langle S_0, \dots, S_{n-1} \rangle$ will be a two-dimensional plane (the others will be hyperplane reflexions).

In fact, the answer is negative in both cases. We briefly sketch the reasons.

First, consider 2_{21} ; we take its edge-length to be $\sqrt{2}$ (we soon see why). The geometry tells us that the original edge-length of the vertex-figure $h\gamma_5$, thought of as inscribed in the unit cube, is $\sqrt{2}$, giving triangular 2-faces. The $\{4, 3, 3, 3\}_5$ with the same vertices has edge-length $\sqrt{4} = 2$, and so we expect our putative 6-polytope to have square 2-faces. Indeed, the 3-faces are now going to be tori $\{4, 4|3\}$, whose 2-holes (of length 3) correspond to 2-faces of $h\gamma_5$. Moreover, the facets will then have to be quotients of 5-polytopes of type $\{4, 4, 3, 3\}$, with these toroidal 3-faces. Suppose that the group of the facet is $\langle S_0, \dots, S_4 \rangle$. Consider the mixing operation

$$(S_0, \dots, S_4) \mapsto (S_0, S_1 S_2 S_1, S_1 S_0 S_1, S_2, S_3, S_4) =: (T_0, \dots, T_5).$$

We recover the original group if we twist by S_1 (thus we have a semidirect product), and so the subgroup is of index 1 or 2. (In fact, the group is the same, because S_1 is induced by an inner automorphism.) Then $\langle T_0, \dots, T_5 \rangle$ is a group generated by hyperplane reflexions which satisfies the relations for the group $[3^{2,2,1}]$ of 2_{21} itself, and so must actually be that group. Obviously this contradicts the intersection property for C-groups, since the trivial relation $S_5 \notin \langle S_0, \dots, S_4 \rangle$ is vitiated. The situation is illustrated by the following

diagram, with a label j on a node (or on the twist) given by the suffix of the corresponding generator S_j :

Second, consider 2_{41} , again with edge-length $\sqrt{2}$. The edge-length of $h\gamma_7$ is $\sqrt{2}$, while that of the inscribed $\{4, 3, 3, 3, 3, 3\}_7$ is $\sqrt{6}$; thus we expect the 2-faces to be hexagons. We perform the analogue of the operation used in the previous case on the facet, namely,

$$(S_0, \dots, S_6) \mapsto (S_0, S_1 S_2 S_1, S_1 S_0 S_1, S_2, \dots, S_6) =: (T_0, \dots, T_7).$$

Once again, we obtain a hyperplane reflexion group $\langle T_0, \dots, T_7 \rangle$; the supposed twist is actually by the inner automorphism S_1 . Its form is a little concealed, but a standard change of generators enables us to recognize it as the whole group $[3^{4,2,1}]$ of 2_{41} . (The mark “2” on the pentagonal circuit in the diagram indicates the period of a certain product of the associated reflexions; see Chapter 9 of [7] for further details, and compare particularly (9A18).) In other words, we have $S_7 \in \langle S_0, \dots, S_6 \rangle$, contrary to the intersection property. The situation is illustrated in the following diagram, with the same conventions as before:

11. Chiral Polytopes

Roughly speaking, chiral polytopes have half as many possible automorphisms as have regular polytopes. More technically, the n -polytope $\mathcal{P}$ is *chiral* if it has two orbits of flags under its group $\Gamma(\mathcal{P})$, with adjacent flags in different orbits. A chiral n -polytope $\mathcal{P}$ is then identified with a group of the form $\Gamma = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$, on which there are relations

$$\begin{cases} \sigma_j^{p_j} = \varepsilon, & j = 1, \dots, n-1, \\ (\sigma_j \sigma_{j+1} \cdots \sigma_k)^2 = \varepsilon, & 1 \leq j < k \leq n-1. \end{cases} \quad (11.1)$$

We again refer to $\{p_1, \dots, p_{n-1}\}$ as the *Schläfli type* of $\mathcal{P}$.

The relationship between the group and the corresponding (abstract) polytope is a little less obvious than is the case for regular polytopes. The distinguished generator σ_j permutes the $(j-1)$ - and j -faces cyclically in the appropriate section of the base flag $\Phi = \{F_0, F_1, \dots, F_{n-1}\}$; if F'_j replaces F_j in the adjacent flag Φ^j , then $F'_{j-1}\sigma_j = F_{j-1}$ and $F_j\sigma_j = F'_j$. The vertices of $\mathcal{P}$ are (identified with) the right cosets of the subgroup $\Gamma_0 := \langle \sigma_2, \dots, \sigma_{n-1} \rangle$, with $F_0 = \Gamma_0$ itself the base vertex. The involutory element $\tau := \sigma_1\sigma_2$ interchanges the two vertices of the base edge, taking Φ into $\Phi^{02} = \Phi^{20}$; it is often useful to replace σ_1 as a generator by τ (compare [8]).

There is a similar realization theory; if $\mathcal{G} = \langle S_1, \dots, S_{n-1} \rangle$ is a representation of Γ (note the change of notation, in that S_j now has a quite different meaning from before), then the Wythoff space is the fixed set of the subgroup $\mathcal{G}_0 := \langle S_2, \dots, S_{n-1} \rangle$. Indeed, we call a realization P of an abstract polytope $\mathcal{P}$ *chiral* if P has two orbits of flags under its symmetry group $\mathcal{G}(P)$, with adjacent flags lying in different orbits. It is clear that the original polytope $\mathcal{P}$ must be regular or chiral.

It is also helpful to remark that if $\mathcal{P}$ is a regular n -polytope with group $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$, then its rotation subgroup $\Gamma^+(\mathcal{P})$ has generators

$$\sigma_j := \rho_{j-1}\rho_j, \quad j = 1, \dots, n-1.$$

Thus a chiral realization of a polytope may be thought of as having only rotational symmetries. Moreover, if the abstract polytope $\mathcal{P}$ is at least chiral, in that its group Γ contains the automorphisms $\sigma_1, \dots, \sigma_{n-1}$ in the definition of chirality, then $\mathcal{P}$ is actually regular if we can adjoin any one of the involutions ρ_j for $j = 0, \dots, n-1$. (We then have $\rho_i = \sigma_{i+1}\rho_{i+1}$ for $i = 0, \dots, j-1$, or $\rho_i = \rho_{i-1}\sigma_i$ for $i = j+1, \dots, n-1$.)

It is clear that an abstract regular polytope may have chiral realizations, though not necessarily faithful ones; it is an interesting open question whether it could actually have faithful chiral realizations. It is an elementary observation that a realized polygon with full rotational symmetry group is actually regular. Similar arguments to those of Section 3 then yield

Proposition 11.1. *Let P be a chiral realization of a polytope, whose ambient space is a spherical, euclidean or hyperbolic space E . Then $\dim P \geq \text{rank } P - 1$.*

If equality occurs in Proposition 11.1, then (as before) we say that P is *of full rank*. The main result of this section shows that including chiral polytopes does not add any new examples to our classification.

Theorem 11.2. *There are no chiral realizations of polytopes of full rank.*

Proof. Suppose that P is a putative faithful chiral realization (as usual, discrete where the latter is relevant). Then its symmetry group $\mathcal{G}(P)$ contains a subgroup $\langle S_1, \dots, S_{n-1} \rangle$, with S_j representing the automorphism σ_j for $j = 1, \dots, n-1$ in the definition of chirality. We use induction on its rank to show that P is actually regular, so that $\langle S_1, \dots, S_{n-1} \rangle = \mathcal{G}^+(P)$, the rotation subgroup of $\mathcal{G}(P)$. The result is easily seen to hold when $\text{rank } P \leq 2$.

The edge-graph of P , together with the vertex-figure P/v at its initial vertex v , clearly determine the structure of P (compare – in a slightly different context—Theorem 2F4

of [7]). Let $\{v, w\}$ be the initial edge of P . The inductive hypothesis implies that P/v is regular, with group $\mathcal{G}(P/v) = \langle R_1, \dots, R_{n-1} \rangle$, say, where $S_j = R_{j-1}R_j$ for each $j = 2, \dots, n-1$. Thinking of (the vertex-set of) P/v as consisting of w and its images under $\mathcal{G}(P/v)$, and observing that P/v together with v spans E (in the sense we employed earlier), we then readily see that each symmetry of $P/\{v, w\}$ extends globally to an isometry of E , since it gives a local isometry.

Furthermore, $T := S_1S_2$ is an involutory symmetry which interchanges v and w , and hence interchanges P/v and P/w . Now the edge-figure $P/\{v, w\}$ is also regular, with group $\langle R_2, \dots, R_{n-1} \rangle$, and has fixed set exactly the line spanned by v and w . If we look at the effect on the base flag (compare the abstract description above), then (without assuming that R_2 is a symmetry of P itself) we see that $TR_2T = R_2$, and hence that $R_2TR_2 = T$.

We must now show that the isometries in $\mathcal{G}(P/v)$ are actually symmetries of P . By our earlier remarks, it is enough to show that R_2 is a symmetry of the edge-graph of P . We can argue on the length of the (shortest) edge-path

$$v = v_0, v_1, \dots, v_k = x$$

joining v to an arbitrary vertex x of P , showing that $\mathcal{G}(P/v)$ permutes the set V_k of vertices at edge-distance at most k from v , which is clearly true for $k = 1$. So, suppose that $k > 1$, and that the claim holds with $k - 1$ instead of k . It is enough to show that $xR_2 \in V_k$ also. By assumption,

$$v = v_0R_2, v_1R_2, \dots, v_kR_2 = xR_2$$

is such that $v_{k-1}R_2 \in V_{k-1}$; indeed, except possibly for $j = k$, each $\{v_{j-1}R_2, v_jR_2\}$ is an edge of P . Now $v_1R_2 \in V_1$; hence we may choose some element G in the rotation subgroup $\mathcal{G}^+(P/v) = \langle S_2, \dots, S_{n-1} \rangle$ of $\mathcal{G}(P/v)$ such that $v_1R_2G = w$. We next apply T ; using the fact that $R_2\mathcal{G}^+(P/v)R_2 = \mathcal{G}^+(P/v)$, we can write

$$R_2GT = R_2GTR_2 \cdot R_2 = G'TR_2,$$

with $G' := R_2GR_2 \in \mathcal{G}^+(P/v) \leq \mathcal{G}(P)$. Since $v_1G' = w$ and $wT = v$, we see that $xG'T \in V_{k-1}$, and hence, by the inductive hypothesis, that $xR_2GT = xG'TR_2 \in V_{k-1}$ also. Reversing the steps above now establishes that $xR_2 \in V_k$, and that $\{v_{k-1}R_2, xR_2\}$ is an edge of P , as we had to show.

We conclude from this that $R_2 \in \mathcal{G}(P)$. By the remark we made earlier, this completes the proof; note that the last generator of $\mathcal{G}(P)$ is $R_0 := S_1R_1 = TR_2$, which interchanges v and w . Thus P is regular, as was claimed. $\square$

One consequence of Theorem 11.2 is that a chiral polytope which is not regular must have rank and dimension each at least 3. Schulte [8] has recently described three families of chiral apeirohedra in $\mathbb{E}^3$, and thus the restriction is tight.

12. Non-Discrete Apeirotopes

The general problem of classifying the non-discrete regular apeirotopes of full rank is hopelessly hard. A lot of what we have said previously does apply to this case as well,

even though the vertex-figure now no longer need be finite. As a simple example, which begins to hint at the difficulties which will arise, let Q be a regular apeirogon inscribed in the unit circle whose edge subtends an irrational angle, and consider $P := \text{apeir } Q$ (defined in exactly the expected way).

It is clear, therefore, that in order to pose a problem that has any possibility of solution, we will have to impose some extra conditions on our apeirotopes. A natural such condition, particularly from the viewpoint of applications to quasiperiodic tilings, is to insist that the vertex-figure be finite, so that the translation subgroup of the symmetry group of the apeirotope will be finitely generated. In these circumstances, since, in fact, from dimension five on, all finite regular polytopes of full rank are rational, the only non-discrete examples will arise in dimensions two, three and four.

In $\mathbb{E}^2$ we will have two kinds of apeirotope. Let $q > 2$ be rational, as usual thought of as a fraction in its lowest terms. Then there are two apeirotopes with vertex-figure $\{q\}$, namely,

$$\{p, q\} \xleftrightarrow{\pi} \text{apeir}\{q\},$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

In $\mathbb{E}^3$ none of the $6 + 6 = 12$ pentagonal polyhedra P can serve as the vertex-figure of a regular apeirotope with finite faces. We thus only have the twelve apeirotopes $\text{apeir } P$.

In $\mathbb{E}^4$ we have $12 + 12 = 24$ pentagonal regular 4-polytopes P of full rank, with the two sets related by κ_0 . All will serve as vertex-figures for apeirotopes $\text{apeir } P$. In addition (compare 14.14 of [2]), ten of the first twelve (namely, all except $\{5, 3, 3\}$ and $\{\frac{5}{2}, 3, 3\}$) are vertex-figures of apeirotopes with finite (planar) faces; the corresponding ten of the second twelve, with vertex-figures related to those of the first by κ_0 , will also serve. Note that we then have ten sets of four, related by κ_1 and κ_{02} , and two pairs, related by κ_1 .

13. Hyperbolic Spaces

Our treatment in Section 3 means that the core Theorem 3.5 covers the hyperbolic case as well, at least when the vertices are finite or on the absolute. This suggests the notion of pursuing the corresponding problem of classifying the discrete regular apeirotopes of full rank in these spaces. However, a little thought shows that, in the plane at least, the problem is again hopelessly difficult. Indeed, merely noting that any subgroup of a discrete group is also discrete indicates that, as long as there are regular apeirotopes whose symmetry groups are suitable hyperbolic Coxeter groups, we have plenty of scope for applying analogues of the “apeir” construction.

Let us be a little more explicit about this. Consider, for example, a regular tiling $\{p, q\}$ of the hyperbolic plane, so that $p, q \geq 3$ are integers satisfying

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.$$

Let the distinguished generating reflexions of its group have mirrors the lines R_0, R_1, R_2 , with initial vertex v . Whether q is odd or even, it is clear that, proceeding along R_2 from v ,

we shall encounter infinitely many other reflexion mirrors which meet it perpendicularly. Each of these intersections provides a suitable point-reflexion S_0 which, with $S_j = R_j$ for $j = 1, 2$, yields a discrete group $\langle S_0, S_1, S_2 \rangle$.

There are examples in four- and five-dimensional hyperbolic space which permit analogous constructions.

14. The Enumeration

For general reference purposes, we gather together the results of the classifications of this paper and the earlier paper [6] (see also Sections 7E and 7F of [7]). The tables show linkages among polytopes of various kinds.

It should be observed that many of the corresponding tables in [6] and [7] were drawn up with a different classification process in mind. This accounts for our duplicating these entries, rather than merely citing them.

We list the finite polytopes first, and then the apeirotopes. The notation has been explained in previous sections. In many places we have not given an exact description of the polytopes or apeirotopes; rather, where indicated by ^(s), we have often just given the Schläfli type. In three and four dimensions the crystallographic and pentagonal polytopes are listed separately.

Finite regular polytopes

Dimension 0:

point

Dimension 1:

segment { }

Dimension 2:

polygon { p } for each rational $p > 2$

Dimension 3 – crystallographic:

$$\begin{array}{ccccccc}
 \{3, 3\} & \xleftrightarrow{\pi} & \{4, 3\}_3 & & & & \\
 \uparrow \kappa_0 & & \uparrow \kappa_0 & & & & \\
 \{6, 3\}_4 & \xleftrightarrow{\pi} & \{4, 3\} & \xleftrightarrow{\delta} & \{3, 4\} & \xleftrightarrow{\pi} & \{6, 4\}_3
 \end{array}$$

Dimension 3 – pentagonal:

$$\begin{array}{ccc}
 \{3, 5\} & \xleftrightarrow{\pi} & \{10, 5\}_3 \\
 \updownarrow \kappa_0 & & \updownarrow \kappa_0 \\
 \{6, 5\}_{\frac{5}{2}} & \xleftrightarrow{\pi} & \{\frac{5}{2}, 5\} \\
 & \updownarrow \delta & \\
 & \{5, \frac{5}{2}\} & \xleftrightarrow{\pi} \{6, \frac{5}{2}\}_5 \\
 & \updownarrow \kappa_0 & \updownarrow \kappa_0 \\
 & \{\frac{10}{3}, \frac{5}{2}\}_3 & \xleftrightarrow{\pi} \{3, \frac{5}{2}\} \\
 & & \updownarrow \delta \\
 & & \{\frac{5}{2}, 3\} \xleftrightarrow{\pi} \{\frac{10}{3}, 3\}_{\frac{5}{2}} \\
 & & \updownarrow \kappa_0 \quad \updownarrow \kappa_0 \\
 & & \{10, 3\}_5 \xleftrightarrow{\pi} \{5, 3\}
 \end{array}$$

Dimension 4 – crystallographic:

$$\begin{array}{ccccc}
 \{3, 3, 3\} & \xleftrightarrow{\kappa_0} & \{3, 3, 3\} \diamond \{ \} & & \\
 & & & & \\
 \{3, 3, 4\} & \xleftrightarrow{\delta} & \{4, 3, 3\} & \xleftrightarrow{\pi} & \{\{4, 4|4\}, \{4, 3\}_3\} \\
 \updownarrow \kappa_0 & & \updownarrow \kappa_0 & & \updownarrow \kappa_0 \\
 \{3, 3, 4\}^{\kappa_0} & & \{4, 3, 3\}^{\text{alt}} & \xleftrightarrow{\pi} & \{\{4, 4|4\}, \{4, 3\}_3\}^{\text{alt}} \\
 & & & & \\
 \{3, 4, 3\} & \xleftrightarrow{\kappa_0} & \{3, 4, 3\}^{\kappa_0} & &
 \end{array}$$

Dimension 4 – pentagonal:

$$\begin{array}{ccccccc}
 \{3, 3, 5\}^{\kappa_0} & \longleftrightarrow^{\kappa_0} & \{3, 3, 5\} & \longleftrightarrow^{\delta} & \{5, 3, 3\} & \longleftrightarrow^{\kappa_0} & \{5, 3, 3\}^{\kappa_0} \\
 \{3, 5, \frac{5}{2}\}^{\kappa_0} & \longleftrightarrow^{\kappa_0} & \{3, 5, \frac{5}{2}\} & \longleftrightarrow^{\delta} & \{\frac{5}{2}, 5, 3\} & \longleftrightarrow^{\kappa_0} & \{\frac{5}{2}, 5, 3\}^{\kappa_0} \\
 \{5, \frac{5}{2}, 5\}^{\kappa_0} & \longleftrightarrow^{\kappa_0} & \{5, \frac{5}{2}, 5\} & & & & \\
 \{5, 3, \frac{5}{2}\}^{\kappa_0} & \longleftrightarrow^{\kappa_0} & \{5, 3, \frac{5}{2}\} & \longleftrightarrow^{\delta} & \{\frac{5}{2}, 3, 5\} & \longleftrightarrow^{\kappa_0} & \{\frac{5}{2}, 3, 5\}^{\kappa_0} \\
 & & & & \{\frac{5}{2}, 5, \frac{5}{2}\} & \longleftrightarrow^{\kappa_0} & \{\frac{5}{2}, 5, \frac{5}{2}\}^{\kappa_0} \\
 \{5, \frac{5}{2}, 3\}^{\kappa_0} & \longleftrightarrow^{\kappa_0} & \{5, \frac{5}{2}, 3\} & \longleftrightarrow^{\delta} & \{3, \frac{5}{2}, 5\} & \longleftrightarrow^{\kappa_0} & \{3, \frac{5}{2}, 5\}^{\kappa_0} \\
 \{\frac{5}{2}, 3, 3\}^{\kappa_0} & \longleftrightarrow^{\kappa_0} & \{\frac{5}{2}, 3, 3\} & \longleftrightarrow^{\delta} & \{3, 3, \frac{5}{2}\} & \longleftrightarrow^{\kappa_0} & \{3, 3, \frac{5}{2}\}^{\kappa_0}
 \end{array}$$

Dimension $n \geq 5$:

$$\begin{array}{ccc}
 \{3^{n-1}\} & \longleftrightarrow^{\kappa_0} & \{3^{n-1}\} \diamond \{ \} \\
 \\
 \{3^{n-2}, 4\}^{\kappa_0} & \longleftrightarrow^{\kappa_0} & \{3^{n-2}, 4\} \\
 & \updownarrow^{\delta} & \\
 \{4, 3^{n-2}\} & \longleftrightarrow^{\kappa_0} & \begin{cases} \{4, 3^{n-2}\}^{\text{alt}}, & \text{if } n \text{ is even,} \\ \{4, 3^{n-2}\}_n, & \text{if } n \text{ is odd.} \end{cases}
 \end{array}$$

Discrete regular apeirotopes

Dimension 1:

$$\text{apeirogon } \{\infty\} = \text{apeir}\{ \}$$

Dimension 2:

$$\{4, 4\} \longleftrightarrow^{\pi} \text{apeir}\{4\}$$

$$\text{apeir}\{6\} \longleftrightarrow^{\pi} \{3, 6\} \longleftrightarrow^{\delta} \{6, 3\} \longleftrightarrow^{\pi} \text{apeir}\{3\}$$

Dimension 3:

$$\begin{array}{ccc}
 \{4, 3, 4\} & \xleftrightarrow{\pi} & \{\{4, 6 \mid 4\}, \{6, 4\}_3\} \\
 \updownarrow \kappa_{02} & & \updownarrow \kappa_{02} \\
 \text{apeir}\{3, 4\} & \xleftrightarrow{\pi} & \text{apeir}\{6, 4\}_3 \\
 \text{apeir}\{3, 3\} & \xleftrightarrow{\pi} & \text{apeir}\{4, 3\}_3 \\
 \updownarrow \kappa_1 & & \updownarrow \kappa_1 \\
 \text{apeir}\{6, 3\}_4 & \xleftrightarrow{\pi} & \text{apeir}\{4, 3\}
 \end{array}$$

Dimension 4:

$$\begin{array}{ccc}
 \{4, 3, 3, 4\} & \xleftrightarrow{\kappa_1} & \{4, 6, 3, 4\}^{(s)} \\
 \updownarrow \kappa_{02} & & \updownarrow \kappa_{02} \\
 \text{apeir}\{3, 3, 4\} & \xleftrightarrow{\kappa_1} & \text{apeir}(\{3, 3, 4\}^{\kappa_0}) \\
 \text{apeir}\{3, 3, 3\} & \xleftrightarrow{\kappa_1} & \text{apeir}(\{3, 3, 3\} \diamond \{ \}) \\
 \text{apeir}\{4, 3, 3\} & \xleftrightarrow{\pi} & \text{apeir}(\{\{4, 4 \mid 4\}, \{4, 3\}_3\}) \\
 \updownarrow \kappa_1 & & \updownarrow \kappa_1 \\
 \text{apeir}\{4, 3, 3\}^{\text{alt}} & \xleftrightarrow{\pi} & \text{apeir}(\{\{4, 4 \mid 4\}, \{4, 3\}_3\}^{\text{alt}})
 \end{array}$$

$$\begin{array}{ccccccc}
 \text{apeir}\{3, 4, 3\} & \xleftrightarrow{\kappa_{02}} & \{3, 3, 4, 3\} & \xleftrightarrow{\delta} & \{3, 4, 3, 3\} & \xleftrightarrow{\pi} & \{3, 4, 4, 3\}^{(s)} \\
 \updownarrow \kappa_1 & & \updownarrow \kappa_1 & & \updownarrow \kappa_1 & & \updownarrow \kappa_1 \\
 \text{apeir}(\{3, 4, 3\}^{\kappa_0}) & \xleftrightarrow{\kappa_{02}} & \{6, 6, 4, 3\}^{(s)} & & \{6, 4, 3, 3\}^{(s)} & \xleftrightarrow{\pi} & \{6, 4, 4, 3\}^{(s)}
 \end{array}$$

Dimension $n \geq 5$:

$$\begin{array}{ccc}
 \{4, 3^{n-2}, 4\} & \xleftrightarrow{\kappa_1} & \{4, 6, 3^{n-3}, 4\}^{(s)} \\
 \updownarrow \kappa_{02} & & \updownarrow \kappa_{02} \\
 \text{apeir}\{3^{n-2}, 4\} & \xleftrightarrow{\kappa_1} & \text{apeir}(\{3^{n-2}, 4\}^{\kappa_0}) \\
 \text{apeir}\{3^{n-1}\} & \xleftrightarrow{\kappa_1} & \text{apeir}(\{3^{n-1}\} \diamond \{ \}) \\
 \text{apeir}\{4, 3^{n-2}\} & \xleftrightarrow{\kappa_1} & \begin{cases} \text{apeir}\{4, 3^{n-2}\}^{\text{alt}}, & \text{if } n \text{ is even,} \\ \text{apeir}\{4, 3^{n-2}\}_n, & \text{if } n \text{ is odd.} \end{cases}
 \end{array}$$

The following table lists the numbers of finite regular polytopes and discrete regular apeirotopes of full rank, according to dimension:

Dimension	Polytopes	Apeirotopes
0	1	—
1	1	1
2	∞	6
3	18	8
4	34	18
≥ 5	6	8

Acknowledgement

We thank Egon Schulte for making many helpful comments on earlier versions of this paper. We are also grateful to one of the referees for several suggestions for clarification.

References

1. H.S.M. Coxeter, Regular skew polyhedra in 3 and 4 dimensions and their topological analogues. *Proc. London Math. Soc.* (2) **43** (1937), 33–62. (Reprinted with amendments in *Twelve Geometric Essays*, Southern Illinois University Press, Carbondale, IL, 1968, pp. 76–105.)
2. H.S.M. Coxeter, *Regular Polytopes* (3rd edition). Dover, New York, 1973.
3. A.W.M. Dress, A combinatorial theory of Grünbaum's new regular polyhedra, I: Grünbaum's new regular polyhedra and their automorphism group. *Aequationes Math.* **23** (1981), 252–265.
4. A.W.M. Dress, A combinatorial theory of Grünbaum's new regular polyhedra, II: complete enumeration. *Aequationes Math.* **29** (1985), 222–243.
5. B. Grünbaum, Regular polyhedra – old and new. *Aequationes Math.* **16** (1977), 1–20.

6. P. McMullen and E. Schulte, Regular polytopes in ordinary space. *Discrete Comput. Geom.* **17** (1997), 449–478.
7. P. McMullen and E. Schulte, *Abstract Regular Polytopes*. Encyclopedia of Mathematics and its Applications, vol. 92. Cambridge University Press, Cambridge, 2002.
8. E. Schulte, Chiral polyhedra in ordinary space, I. *Discrete Comput. Geom.*, this issue, pp. 55–99.

Received May 14, 2003, and in revised form September 3, 2003. Online publication March 22, 2004.