A Small Cover for Convex Unit Arcs

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Abstract. An arc in the plane is *convex* if it is simple (i.e., one–one except that its endpoints may coincide) and lies on the boundary of its convex hull. We describe a compact convex plane set of area about 0.2466 that contains a congruent copy of each convex plane arc of unit length, a reduction of about 1.1% from the smallest such set previously known.

1. Introduction

In 1965 Besicovitch [2] observed that an equilateral triangle of side 1, whose area is $\sqrt{3}/4 \approx 0.4330$, contains a congruent copy of every convex arc of length 1. A few years later Wetzel noticed that an isosceles right triangle of hypotenuse 1, with area 0.2500, also has this property, and he showed that a slightly smaller such region with area about 0.2492 can be formed by clipping the tiny isosceles right triangle with altitude $\frac{1}{2} - \frac{1}{3}\sqrt{2} \approx 0.0286$ from its right-angle vertex (see pp. 218–219 of [5]). In view of Leo Moser’s well-known and still unsolved *worm problem*, which asks for the convex set of least area that contains a congruent copy of *every* arc of length 1 (see [4], [5], and [6]), it is natural to seek the convex set of least area that contains a congruent copy of every convex arc of length 1. In this article we describe such a cover whose area is less than 0.2466.

A convex cover for the family $\mathcal{C}$ of all convex unit arcs must have diameter at least 1 to accommodate the unit line segment, and its minimum width must be at least $b_0 \approx 0.4389$, the minimum width of the unit broadworm. (The *broadworm*, shown in Fig. 1(a), is the

Fig. 1. Two unit arcs: (a) the broadworm, (b) a drape.

unique arc of length 1 whose minimum width is as large as possible (see [9], [1], or [8].) It follows that the area of a convex cover for $\mathcal{C}$ must be at least $(1 \times b_0)/2 \approx 0.2194$. It seems clear that the smallest possible area for a convex cover for $\mathcal{C}$ lies closer to 0.24 than to this weak lower bound.

2. Convex Arcs and Drapes

By an *arc* we mean the continuous image in the plane of the closed unit interval $[0, 1]$ under a (parametric) function of bounded variation. We write $\ell(\gamma)$ for the length of the arc γ , and when $\ell(\gamma) \leq 1$ we call γ a *unit arc*.

Definition. An arc is *convex* if it is simple (i.e., one–one except that its endpoints may coincide) and it lies on the boundary of its convex hull. A convex arc δ with distinct endpoints P and Q is a *drape* if it lies entirely in one of the two closed half-strips bounded by the line segment PQ and the lines perpendicular to PQ at the endpoints P and Q (Fig. 2). Let $\mathcal{D}$ be the family of all drapes of unit length.

Definition. An arc γ *fits* in a compact convex set K (or K *accommodates* or *covers* γ) if there is an isometry μ so that $\mu(\gamma) \subset K$ (i.e., $\mu(\gamma(t)) \in K$ for each t in $[0, 1]$).

We note first that every convex unit arc lies in the convex hull of a unit drape.

Lemma 1. *If γ is a convex unit arc, then there is a unit drape δ whose convex hull contains γ .*

Proof. Suppose the convex arc γ has endpoints P and Q , and let s be the line PQ if $P \neq Q$ and any support line of γ through P if $P = Q$. Suppose the support lines perpendicular to s touch γ at E and F and meet s at U and V , as pictured in Fig. 2. Define an arc δ by replacing the subarc PE of γ by the (shorter) line segment UE and the subarc FQ by the (shorter) line segment FV . Then δ is a drape, $\ell(\delta) \leq \ell(\gamma) \leq 1$, and γ lies in the convex hull of δ . $\square$

Corollary 2. *Every closed unit arc lies in the convex hull of a unit drape.*

Fig. 2. Proof of Lemma 1.

Proof. The boundary curve of the convex hull of a closed unit arc is a convex unit arc (see, for example, [7]). $\square$

It follows that a convex cover for the family $\mathcal{D}$ of all unit drapes is also a cover for the family $\mathcal{C}$ of all convex unit arcs.

3. The Isosceles Right Triangle

In the mid-1970s Wetzel observed that an isosceles right triangle with hypotenuse one is a cover $\mathcal{C}$. We begin by recalling that argument.

Lemma 3 (Wetzel (see pp. 218–19 of [5])). *Each unit drape $\delta \in \mathcal{D}$ fits in the isosceles right triangle T_0 with its endpoints on the hypotenuse AB .*

Proof. Encage the given drape δ in an isosceles right triangle $T' = A'B'C'$ with hypotenuse $A'B'$ on the line PQ as pictured in Fig. 3, with points X and Y of contact on $A'C'$ and $B'C'$ as shown. Reflect the subarcs PX and YQ across $A'C'$ and $B'C'$, respectively, to form the arcs $P'X$ and YQ' . Then

$$\begin{aligned} A'B' &\leq P'Q' \leq \ell(P'X) + \ell(XY) + \ell(YQ') \\ &= \ell(PX) + \ell(XY) + \ell(YQ) = \ell(\delta) \leq 1. \end{aligned}$$

Consequently T' fits into T_0 with $A'B'$ on AB , and the result follows. $\square$

Fig. 3. The reflection argument.

Fig. 4. Two unit arcs: (a) a zigzag, (b) a staple.

In 1992 Norwood et al. [6] described a convex set of area about 0.2752 that contains a congruent copy of *every* arc of length 1, the smallest such region known to date. The isosceles right triangle T_0 is smaller, but it is not a cover for all unit arcs. For example, the centrally symmetric zigzag unit arc $\omega = AUVW$ pictured in Fig. 4(a), formed by three line segments of length $\frac{1}{3}$ with $\angle VAU = \arctan(\frac{1}{3})$ and $VW \parallel AU$, clearly does not fit in T_0 . In this regard, it would be of considerable interest to find the shortest arc that does not fit in the interior of T_0 . (The length of the shortest arc that does not fit in the interior of a region has been called the *escape length* of the region because of its connection with a well-known problem of Bellman. Bellman's problem, which remains unsolved for triangles, has been solved for only a few regions (see [3]).)

The *staple* σ with *leg* s is the three segment unit arc $IJKL$ formed by the three sides IJ , JK , and KL of a rectangle, with $IJ = s$ and $JK = 1 - 2s$ (Fig. 4(b)). Staples with $s \leq \frac{1}{3}$ fit tightly in T_0 , suggesting that the lower part of T_0 might be close to optimal. Since tall unit drapes can be turned, it should be possible to pare T_0 a bit in a neighborhood of the vertex C to form a smaller cover for $\mathcal{D}$. Wetzel showed that the quadrilateral formed by clipping T_0 down to the height $\sqrt{2}/3 \approx 0.4714$ (indicated in Fig. 4(b) by a dashed line) is a cover for $\mathcal{D}$ whose area is about 0.2492. Since we do better below, we omit the argument.

4. A Smaller Cover

The middle point R of a unit drape δ divides δ into two convex subarcs α and β . Strengthening Lemma 3, we show that δ fits in T_0 with its endpoints on the hypotenuse and its middle point on the median of T_0 (Fig. 5).

Fig. 5. Midpoint on median.

Fig. 6. The cover T_1 .

Lemma 4. *Each unit drape δ fits in T_0 with its endpoints on AB and its middle point R on the median CM of T_0 , where M is the midpoint of AB .*

Proof. Let R be the middle point of a given drape δ with endpoints P and Q . Take RM perpendicular to the line PQ , and, using $\pm 45^\circ$ support lines, encage δ in an isosceles right triangle with hypotenuse $A'B'$ along PQ and vertex C' on RM (Fig. 5). At least one of the two subarcs α and β formed by splitting δ at its midpoint R meets $A'C' \cup B'C'$, and we may arrange the notation so that α meets $A'C'$ at a point X . Let P' be the foot of the perpendicular from X to $A'B'$ and R' the foot of the perpendicular from X to MC' . Then

$$\begin{aligned} \frac{1}{2}A'B' &= A'M = A'P' + P'M \\ &= P'X + XR' \leq PX + XR \leq \ell(\alpha) \leq \frac{1}{2}, \end{aligned}$$

so that $A'B' \leq 1$. The result follows. $\square$

To describe the new cover T_1 (Fig. 6) requires a little analytic geometry. Let ABC be an isosceles right triangle with hypotenuse $AB = 1$, and let D and E be the trisection points of AB , so that $AD = DE = EB = \frac{1}{3}$. Suppose the perpendiculars to AB at D and E meet AC and BC , respectively, at F and G . Introduce rectangular coordinates with the origin at the midpoint M of AB and B on the positive x -axis, so that A , B , C , D and E have coordinates $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(-\frac{1}{6}, 0)$ and $(\frac{1}{6}, 0)$, respectively. Then the locus Γ^- of points (x, y) in the third quadrant whose distances $-x$ to the y -axis and $[(x + \frac{1}{6})^2 + y^2]^{1/2}$ to the point D have sum $\frac{1}{2}$ is the parabolic arc with equation

$$\Gamma^-: 9y^2 - 6x - 2 = 0, \quad (1)$$

whose focus is $D(-\frac{1}{6}, 0)$ and whose directrix is the line $x = -\frac{1}{2}$. For $-\frac{1}{6} \leq x \leq 0$ this arc connects the point $F(-\frac{1}{6}, \frac{1}{3})$ to the point $H(0, \sqrt{2}/3)$ on the y -axis, and its slope decreases from $+1$ at F to $1/\sqrt{2}$ at H . The symmetric parabolic arc Γ^+ for $x \geq 0$, with equation

$$\Gamma^+: 9y^2 + 6x - 2 = 0,$$

similarly joins H to G . The resulting region $T_1 = ABGHF$ (Fig. 6) is convex, and its area is $(4\sqrt{2} + 1)/27 \approx 0.2466$.

Proof. Let X be a point of α that lies on Γ^- , and let R' be the foot of the perpendicular from X to the median CM (Fig. 7). Then $\ell(\alpha) \geq PX + XR > DX + XR' = \frac{1}{2}$. $\square$

Lemma 6. *A rotation through 135° carries the quadrilateral AEGC into T_1 .*

Proof. The rotated quadrilateral fits as pictured (with E and F carried to H and M , respectively), because $MH = EF = \sqrt{2}/3$. $\square$

Now the proof that T_1 is a cover is straightforward.

Theorem 7. *The region T_1 is a cover for $\mathcal{D}$ (and consequently for $\mathcal{C}$) with area $(4\sqrt{2} + 1)/27 \lesssim 0.2466$.*

Proof. Place a unit drupe δ in T_0 with its endpoints P and Q on AM and MB , respectively, and its midpoint R on the median CM of T_0 (Lemma 4). If $P \in AD$ with $P \neq D$ and $Q \in EB$ with $Q \neq E$, then δ cannot meet either parabolic arc Γ^- or Γ^+ (by

Fig. 9. Two possible covers: (a) T_2 , (b) T_3 .

Lemma 5); and hence δ already lies in T_1 . If P and Q are both in the (closed) segment AE , then according to Lemma 6 a suitable rotation carries δ into T_1 ; and similarly if P and Q are both in the (closed) segment DB . In every case, δ fits in T_1 . $\square$

5. Improvements?

Since every arc of length 1 fits in a strip of width $b_0 \approx 0.4389$ (the minimum width of the unit broadworm), one might hope that the portion T_2 of the cover T_1 lying in a strip of width b_0 (Fig. 9(a)) would be a cover for $\mathcal{D}$ and hence for $\mathcal{C}$. We conjecture that it is, but we have not been able to prove or disprove this conjecture. Its area is about 0.24509.

The even smaller set T_3 pictured in Fig. 9(b) might also be a cover for $\mathcal{C}$. Its area is about 0.23982.

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