The Alexandroff Dimension of Quotients of $\mathbb{R}^{2*}$

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Abstract. We generalize results of our previous paper [12], showing that the minimal grid of a fenestration of the plane, whose elements are open discs with frontiers which are Jordan curves, has Alexandroff dimension 2. We give examples to show why this condition on the frontier is required and that the methods employed here are not applicable in higher dimensions.

1. Introduction and Terminology

In [8] Kronheimer, generalizing the construction of the Khalimsky line, proposed the construction of a digital space $X(\mathcal{W})$ as a quotient space of an arbitrary topological space Y , starting with a collection $\mathcal{W}$ (called a *fenestration*) of pairwise disjoint open subsets of Y whose union is dense and then extending this family to a partition of Y which is topologized with the quotient topology. The collection $\mathcal{W}$ is identified with an open discrete dense subspace D of Y . The idea behind this, motivated, for example, by the need for topological modelling of the supports of digital images, is to construct a “digital” space which “represents” D and mirrors properties of Y . Throughout the paper cl_X , int_X and fr_X refer respectively to closure, interior and frontier in the space X but if $X = \mathbb{R}^n$ (in particular, when $X = \mathbb{R}^2$), then we simply write cl , int and fr , respectively. We note also that $\subset$ will always denote a proper subset. All undefined topological terms can be found in [3].

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A space (X, τ) is said to be an *Alexandroff space* if each point of X has a minimal open neighbourhood or, equivalently, if τ is closed under arbitrary intersections. A topology τ on a set X determines a preorder $\leq_\tau$ on X as follows:

$$x \leq_\tau y \iff x \in \text{cl}_X(\{y\}),$$

and it is easy to see that if (X, τ) is a T_0 -space, then $\leq_\tau$ is a partial order, usually called the *specialization order* of τ . It is not hard to see that a function between Alexandroff spaces is continuous if and only if it preserves the specialization order.

Conversely, each partial order $\leq$ on a set X determines a unique T_0 Alexandroff topology $\tau_\leq$ on X whose minimal open sets are of the form

$$\{y : y \geq x\} \ (x \in X).$$

These are classical results of Alexandroff [1]; however, even more is true (see [5]): Let $\mathcal{P}$ denote the category of partially ordered sets and order-preserving functions and let $\mathcal{A}_0$ denote the category of Alexandroff T_0 -spaces and continuous maps.

Theorem 1.1. *The categories $\mathcal{A}_0$ and $\mathcal{P}$ are isomorphic; moreover, the functors F and G defined by*

$$F((X, \tau)) = (X, \leq_\tau) \quad \text{and} \quad G((X, \leq)) = (X, \tau_\leq),$$

and which preserve maps, are (mutually inverse) isomorphisms.

For completeness, we recall the (recursive) definition of the *small inductive dimension* $\text{ind}(X)$ of a topological space X :

- (i) $\text{ind}(\emptyset) = -1$.
- (ii) $\text{ind}(X) \leq n$ if each point $x \in X$ has a local base $\mathcal{B}_x$ of open neighbourhoods such that if $U \in \mathcal{B}_x$, then $\text{ind}(\text{fr}(U)) \leq n - 1$.
- (iii) $\text{ind}(X) = n$ if $\text{ind}(X) \leq n$ but $\text{ind}(X) \not\leq n - 1$.
- (iv) $\text{ind}(X) = \infty$ if for all $n \in \omega$, $\text{ind}(X) \not\leq n$.

For more details (and the definition of the other topological dimension functions Ind and dim) we refer the reader to [3] or [10]—we note only that all of these functions coincide in separable metric spaces and $\text{ind}(\mathbb{R}^n) = n$. The *partial order dimension* (see for example Chapter 1 of [2]) of a poset $(X, \leq)$ is defined as

$$\sup\{|C| - 1 : C \text{ is a chain of distinct elements in } (X, \leq)\},$$

which may be a non-negative integer or ∞ .

The following result was proved in [11]:

Proposition 1.2. *If (X, τ) is an Alexandroff T_0 -space and $\leq_\tau$ is its specialization order, then the small inductive dimension of (X, τ) is equal to the partial order dimension of $(X, \leq_\tau)$.*

In light of Proposition 1.2, in future we will not distinguish between the partial order dimension of $(X, \leq)$ and the small inductive dimension of the corresponding Alexandroff T_0 -space $(X, \tau_\leq)$. Both will be referred to as the *Alexandroff dimension* of X and written $\text{Adim}(X)$.

Definition 1.3. A *fenestration* of $\mathbb{R}^n$ is a collection of pairwise disjoint non-empty proper regular open subsets (that is to say, open subsets U such that $\text{int}(\text{cl}(U)) = U$) of $\mathbb{R}^n$, whose union is dense in $\mathbb{R}^n$. If X is a partition of $\mathbb{R}^n$ that contains a fenestration $\mathcal{W}$ and is such that the projection map from $\mathbb{R}^n$ to X (with the quotient topology) is open, then X is called a $\mathcal{W}$ -grid of $\mathbb{R}^n$.

That $\mathcal{W}$ -grids exist can easily be checked; indeed, $\mathcal{W} \cup \{\{x\} : x \in \mathbb{R}^n \setminus \bigcup \mathcal{W}\}$ is one such. Given a fenestration $\mathcal{W}$ of $\mathbb{R}^n$, in general there exist many $\mathcal{W}$ -grids, however, the following property defined in [8] distinguishes a special $\mathcal{W}$ -grid.

Definition 1.4. A $\mathcal{W}$ -grid X of $\mathbb{R}^n$ is called *minimal* if any continuous open map of X onto some $\mathcal{W}$ -grid, which is injective on $\mathcal{W}$, is a homeomorphism.

It was shown in Theorem 4.4 of [8] that for any fenestration $\mathcal{W}$, and for any $\mathcal{W}$ -grid X , there exists a continuous open map $f_{\min}$ of X onto a minimal $\mathcal{W}$ -grid $X_{\min}$, which is injective on $\mathcal{W}$ and satisfies $f_{\min} \circ \pi = \pi_{\min}$, where $\pi, \pi_{\min}$ are the quotient maps from $\mathbb{R}^n$ onto X and $X_{\min}$, respectively. This minimal $\mathcal{W}$ -grid is then clearly unique up to homeomorphism and is a T_0 Alexandroff space if $\mathcal{W}$ is locally finite.

In a previous paper [12] we studied locally finite fenestrations of $\mathbb{R}^n$ whose elements are bounded convex sets; in that paper we showed that if $\mathcal{W}$ is such a fenestration of $\mathbb{R}^n$, then the minimal $\mathcal{W}$ -grid has Alexandroff dimension n . Although the question was left open as to whether this result can be generalized to non-convex sets, we gave examples to show that this result is false for arbitrary locally finite fenestrations of $\mathbb{R}^2$ (see Examples 2.7 and 2.8 in [12]). In both examples, the elements of the fenestration are non-convex; they are bounded in the first example, and unbounded in the second. Our aim here is to extend the above-mentioned theorem to fenestrations of $\mathbb{R}^2$ whose elements are not convex but whose closures are homeomorphic to closed discs. Although many of our results are applicable in $\mathbb{R}^n$, there are simple examples to show that the construction described in Theorems 2.13 and 2.14 is not applicable to higher dimensions.

Definition 1.5. We say that a locally finite fenestration $\mathcal{W}$ of $\mathbb{R}^n$ is *spherical* if each element is (homeomorphic to) an open n -disc and whose frontier is homeomorphic to S^{n-1} .

It follows that if $\mathcal{W}$ is a spherical fenestration and $x \in \text{fr}(W)$ for some $W \in \mathcal{W}$, then $\text{fr}(W) \setminus \{x\}$ is homeomorphic to $\mathbb{R}^{n-1}$. Moreover, $\text{cl}(W)$ is bounded and hence compact, and W is regular open. We note in passing that neither Example 2.7 nor 2.8 of [12] has this property.

Furthermore, in case $n = 2$, it is a consequence of Schoenflies' theorem (see, for example, p. 175 of [7]) that the property of being spherical is equivalent to the existence, for each $W \in \mathcal{W}$, of a homeomorphism f_W of $\mathbb{R}^2$ onto $\mathbb{R}^2$ such that $f_W(\text{cl}(W)) = D$,

where D is a closed (two-dimensional) disc, and $f_W(W) = \text{int}(D)$. However, if $n \geq 3$, then the existence of wild embeddings of S^{n-1} in $\mathbb{R}^n$ renders this false.

2. Locally Finite Spherical Fenestrations

In this section $\mathcal{W}$ always denotes a locally finite, spherical fenestration of $\mathbb{R}^n$. We begin with a definition and some technical results which will be needed for our main theorems.

Lemma 2.1 [4, Theorem 6.8.11]. *If $M \subseteq \mathbb{R}^n$, then the topological dimension (ind , Ind or dim) of M is n if and only if $\text{int}(M) \neq \emptyset$.*

As a simple corollary, we then have:

Corollary 2.2. *If S is a set homeomorphic to an $(n-1)$ -sphere in $\mathbb{R}^n$ and $M \subseteq S$, then $\text{ind}(M) = n-1 \Leftrightarrow \text{int}_S(M) \neq \emptyset$.*

Proof. Clearly, $S \setminus \{s\} \cong \mathbb{R}^{n-1}$ for any $s \in S$. Suppose first that $\text{int}_S(M) \neq \emptyset$. If $M = S$, then trivially $\text{ind}(M) = n-1$. If $M \neq S$, then there is $s \in S \setminus M$, and then $M \subseteq S \setminus \{s\} \cong \mathbb{R}^{n-1}$. Since any subset of M open in S is also open in $S \setminus \{s\}$, it follows that $\text{int}_{S \setminus \{s\}}(M) \neq \emptyset$. If g is the homeomorphism of $S \setminus \{s\}$ onto $\mathbb{R}^{n-1}$, then $\text{int}_{\mathbb{R}^{n-1}} g(M) \neq \emptyset$ and hence $\text{ind}(g(M)) = n-1$, implying $\text{ind}(M) = n-1$, since ind is a topological invariant.

Now suppose that $\text{ind}(M) = n-1$. If $M = S$, then trivially $\text{int}_S(M) = M \neq \emptyset$, so suppose $M \neq S$, and let $s \in S \setminus M$, so $M \subseteq S \setminus \{s\}$. If g is the homeomorphism of $\mathbb{R}^{n-1}$ onto $S \setminus \{s\}$, then it follows as in the previous paragraph that $\text{int}_S(M) = \text{int}_{S \setminus \{s\}}(M) = \text{int}_{\mathbb{R}^{n-1}} g(M) \neq \emptyset$. $\square$

The following is now obvious:

Corollary 2.3. *For any $W \in \mathcal{W}$ and $M \subseteq \text{fr}(W)$ it follows that $\text{ind}(M) = n-1 \Leftrightarrow \text{int}_{\text{fr}(W)}(M) \neq \emptyset$.*

Definition 2.4.

- (i) For $x \in \mathbb{R}^n$: $N_x = \{W \in \mathcal{W} : x \in \text{cl}(W)\}$.
- (ii) For $W \in \mathcal{W}$: $N_W = \{W' \in \mathcal{W} : W' \neq W, \text{cl}(W) \cap \text{cl}(W') \neq \emptyset\}$.

Since $\mathcal{W}$ is assumed to be locally finite, then it is clear that for each $x \in \mathbb{R}^n$, N_x is a finite set. Furthermore, since the elements of $\mathcal{W}$ are mutually disjoint and open, it is clear that if $W, W' \in \mathcal{W}$, then $\text{cl}(W) \cap W' = \emptyset$ and thus $\text{cl}(W) \cap \text{cl}(W') = \text{fr}(W) \cap \text{fr}(W')$.

We need the following two simple lemmas.

Lemma 2.5.

- (i) For each $x \in \mathbb{R}^n$, $x \in \text{int}(\bigcup \{\text{cl}(W) : W \in N_x\})$.

- (ii) If $x \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ is open such that $U \subseteq \bigcup \{\text{cl}(W) : W \in N_x\}$, then $U \cap \text{cl}(W) = \emptyset$ for any $W \in \mathcal{W} \setminus N_x$.

Proof. (i) Let $x \in \mathbb{R}^n$; by the definition of N_x , $x \in M = \bigcup \{\text{cl}(W) : W \in N_x\}$ and M is closed since N_x is finite. Suppose $x \in \text{fr}(M)$. Then any open neighbourhood of x intersects $\mathbb{R}^n \setminus M$. Since $\mathcal{W}$ is locally finite (hence so is $\mathcal{W} \setminus N_x$), $\mathbb{R}^n = \text{cl}(\bigcup \mathcal{W}) = \text{cl}(\bigcup N_x) \cup \text{cl}(\bigcup (\mathcal{W} \setminus N_x))$, which implies that $\mathbb{R}^n \setminus M \subseteq \text{cl}(\bigcup (\mathcal{W} \setminus N_x))$, and hence $x \in \text{cl}(\bigcup (\mathcal{W} \setminus N_x)) = \bigcup \{\text{cl}(W) : W \in \mathcal{W} \setminus N_x\}$, which contradicts the definition of N_x . In consequence $x \in \text{int}(M)$.

(ii) Let $U \subseteq \bigcup \{\text{cl}(W) : W \in N_x\}$ be open. Suppose that there exists $V \in \mathcal{W} \setminus N_x$ such that $U \cap \text{cl}(V) \neq \emptyset$, implying immediately that $U \cap V \neq \emptyset$; thus we can choose $z \in U \cap V$. However, $U \cap V \subseteq V \cap \bigcup \{\text{cl}(W) : W \in N_x\}$ and so there is some $W' \in N_x$ such that $z \in \text{cl}(W') \cap V$, a contradiction. $\square$

Lemma 2.6. *For each $W \in \mathcal{W}$, N_W is non-empty and finite.*

Proof. Since $\mathcal{W}$ is locally finite, $\mathbb{R}^n = \bigcup \{\text{cl}(W) : W \in \mathcal{W}\}$. Since all sets of the form $\text{cl}(W)$ are bounded, $\mathcal{W}$ has an infinite number of elements. However, if $N_W = \emptyset$, then since $\bigcup \mathcal{W}$ is dense in $\mathbb{R}^n$, it follows that $\text{cl}(W)$ is open in $\mathbb{R}^n$, hence $\text{cl}(W) = \mathbb{R}^n$, a contradiction. The fact that N_W is finite is a simple consequence of the fact that for each $W \in \mathcal{W}$, $\text{fr}(W)$ is compact and if this set were to meet an infinite number of sets of the form $\text{cl}(W')$ for $W' \in \mathcal{W}$, then there would exist some point of $\text{fr}(W)$ all of whose neighbourhoods meet an infinite number of sets $W' \in \mathcal{W}$, contradicting the local finiteness of $\mathcal{W}$. $\square$

Up to now, all our results are valid in $\mathbb{R}^n$ for each positive integer n . However, from this point on, we consider only the space $\mathbb{R}^2$. Thus, in what follows, $\mathcal{W}$ will always represent a spherical fenestration of $\mathbb{R}^2$.

Theorem 2.7. *Let $\mathcal{W} = \{W_n : n \in \omega\}$ be a spherical fenestration of $\mathbb{R}^2$. If $W, W' \in \mathcal{W}$ are such that $M = \text{cl}(W) \cap \text{cl}(W') \neq \emptyset$, then M has a finite number of components $\{C_i : 1 \leq i \leq k\}$ and $\text{int}_{\text{fr}(W)}(M) = \bigcup \{\text{int}_{\text{fr}(W)}(C_i) : 1 \leq i \leq k\}$.*

Proof. It is a corollary of the Jordan Curve Theorem that for two closed discs $D_1, D_2 \subseteq \mathbb{R}^2$, $D_1 \cap D_2$ has a finite number of components if and only if $\mathbb{R}^2 \setminus (D_1 \cup D_2)$ has a finite number of components (see p. 116 of [6]). Thus to prove our first assertion, we need only show that $\mathbb{R}^2 \setminus (\text{cl}(W) \cup \text{cl}(W'))$ has a finite number of components. Suppose to the contrary that the open set $\mathbb{R}^2 \setminus (\text{cl}(W) \cup \text{cl}(W'))$ has an infinite number of components $\{V_n : n \in \omega\}$; note that each set V_n is open, since $\mathbb{R}^2$ is locally connected. Since $\text{cl}(W) \cup \text{cl}(W')$ is bounded, it follows that exactly one component of $\mathbb{R}^2 \setminus (\text{cl}(W) \cup \text{cl}(W'))$, say V_0 , is unbounded. Thus for $n \geq 1$, V_n is a bounded open set and since the elements of $\mathcal{W}$ are mutually disjoint and their union is dense, it follows that for each $n \geq 1$, we can find a subset $F_n \subset \omega$ such that for each $k \in F_n$, $W_k \subseteq V_n$ and $\bigcup \{W_k : k \in F_n\}$ is dense in V_n . Furthermore, since each such V_n is bounded and $\mathcal{W}$ is locally finite, it follows that for each $n \geq 1$, F_n is finite. Now since $\mathbb{R}^2$ is connected and $\text{cl}(V_n) \cap V_m = \emptyset$ whenever

$m \neq n$, it follows that $\text{cl}(V_n) \cap (\text{cl}(W) \cup \text{cl}(W')) \neq \emptyset$ and so for each $n \geq 1$, there is some $k(n) \in F_n$ such that $\text{cl}(W_{k(n)})$ meets $\text{cl}(W) \cup \text{cl}(W')$. Thus there is an infinite set of elements of $\mathcal{W}$ whose closures meet $\text{cl}(W) \cup \text{cl}(W')$ and so there is an infinite set of elements of $\mathcal{W}$ whose closures meet either $\text{cl}(W)$ or $\text{cl}(W')$, contradicting Lemma 2.6.

For the second assertion, note first that since $M = \bigcup \{C_i : 1 \leq i \leq k\}$, it follows that $\text{int}_{\text{fr}(W)}(M) \supset \bigcup \{\text{int}_{\text{fr}(W)}(C_i) : 1 \leq i \leq k\}$. For the inverse inclusion, we note that $\text{fr}(W)$ is homeomorphic to S^1 and since each connected subset of S^1 is locally connected, it follows that M is locally connected and each C_i is clopen in M . $\square$

Lemma 2.8. *If $\mathcal{W}$ is a spherical fenestration of $\mathbb{R}^2$ and $W_1, W_2 \in \mathcal{W}$ are distinct, then $\text{cl}(W_1) \cap \text{cl}(W_2)$ is a proper subset of $\text{fr}(W_1)$.*

Proof. $\text{fr}(W_2) \supset \text{fr}(W_1)$ is not possible since no proper subset of S^1 is homeomorphic to S^1 . Additionally, $\text{fr}(W_2) = \text{fr}(W_1)$ is not possible since W_2 would then be a subset of the unbounded component C of the complement of the Jordan curve $J = \text{fr}(W_1)$ and has J as its frontier. Thus the open set W_2 can have no frontier points in the connected subset C and so it must be equal to C contradicting the boundedness of the elements of $\mathcal{W}$. The result follows. $\square$

The following theorem characterizing those sets N_x with exactly two elements is the key to our main result. For the proof, we recall the property that if C_1, C_2 are the two components of the complement of a Jordan curve J and $a, b \in J$, then for $k = 1, 2$, there is an arc A_k with endpoints a and b such that $A_k \setminus \{a, b\} \subset C_k$ (see pp. 161–164 of [9]).

Theorem 2.9. *If $z \in \mathbb{R}^2$ and $W_1, W_2 \in \mathcal{W}$, then $z \in \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$ if and only if $N_z = \{W_1, W_2\}$.*

Proof. For the necessity, suppose that $z \in \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$; it follows from Corollary 2.3 that $\text{ind}(\text{cl}(W_1) \cap \text{cl}(W_2)) = 1$. Since, by Theorem 2.7, $\text{cl}(W_1) \cap \text{cl}(W_2)$ has only a finite number of components and the zero-dimensional components are singletons, there is some one-dimensional component D of $\text{cl}(W_1) \cap \text{cl}(W_2)$ such that $z \in \text{int}(D)$. Also, by Lemma 2.8, D is a proper subset of $\text{fr}(W_1)$ and hence is an arc. Now we choose an open disc U centred at z such that the component (an open arc) E of $U \cap \text{fr}(W_1)$ which contains z is a proper subset of D not containing the endpoints of D . Choose $a, b \in E$ so that z separates a and b in E and let $A_0 \subset E$ be the arc from a to b ; finally choose $b' \in D \setminus A_0$. As mentioned in the paragraph prior to this theorem, since both $\text{fr}(W_1)$ and $\text{fr}(W_2)$ are Jordan curves, for $k = 1, 2$, there is an arc A_k with endpoints a and b such that $A_k \setminus \{a, b\} \subset W_k$. Then $A_1 \cup A_2, A_1 \cup A_0$ and $A_2 \cup A_0$ are Jordan curves and it is clear that z lies in the open bounded component C of $\mathbb{R}^2 \setminus (A_1 \cup A_2)$, while b' lies in the unbounded component. We claim that C is contained in $\text{cl}(W_1) \cup \text{cl}(W_2)$.

To prove our claim, we assume to the contrary, that there is some $W \in \mathcal{W}$ such that $W \cap C \neq \emptyset$ and since $A_1 \setminus \{a, b\} \subseteq W_1, A_2 \setminus \{a, b\} \subseteq W_2$ and W is connected, it follows that $W \subseteq C$ and, since C is bounded, there can only be a finite number of such W . Clearly, $W \cap A_0 = \emptyset$, for otherwise $W \cap W_1 \neq \emptyset$ and hence either W is subset of the

bounded component C_1 of $\mathbb{R}^2 \setminus (A_0 \cup A_1)$, or is a subset of the bounded component C_2 of $\mathbb{R}^2 \setminus (A_0 \cup A_2)$. The cases are similar and we consider only the first. However, since $A_1 \setminus \{a, b\} \subseteq W_1$, it follows that the open set $W_1 \cap C_1$ is a non-empty subset of the connected open set C_1 and hence if $W_1 \cap C_1 \subset C_1$, then $\text{fr}(W_1 \cap C_1) \cap C_1 \neq \emptyset$ and hence $\text{fr}(W_1) \cap C_1 \neq \emptyset$. Suppose $c \in \text{fr}(W_1) \cap C_1$; the arc (not containing z) of $\text{fr}(W_1)$ from $c \in C_1$ to $b' \notin C_1$ cannot intersect A_0 and hence must intersect $A_1 \setminus \{a, b\}$, giving a contradiction, since $A_1 \setminus \{a, b\} \subseteq W_1$. Thus $W_1 \cap C_1 = C_1$ and so $C \subseteq \text{cl}(W_1) \cup \text{cl}(W_2)$ completing the proof of our claim.

Now, since $C \subseteq \text{cl}(W_1) \cup \text{cl}(W_2)$ is an open set containing z , it follows from Lemma 2.5(ii) that $C \cap \text{cl}(W) = \emptyset$ if $W \notin \{W_1, W_2\}$. Thus $z \in \text{cl}(W)$ if and only if $W \in \{W_1, W_2\}$ and the necessity is proved.

For the converse, we begin by showing that

$$\text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2)) \supseteq \text{int}(\text{cl}(W_1) \cup \text{cl}(W_2)) \cap \text{fr}(W_1).$$

To this end, suppose that $z \in \text{int}(\text{cl}(W_1) \cup \text{cl}(W_2)) \cap \text{fr}(W_1)$; we show that $\text{int}(\text{cl}(W_1) \cup \text{cl}(W_2)) \cap \text{fr}(W_1) \subseteq \text{cl}(W_1) \cap \text{cl}(W_2)$, implying that $z \in \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$. For suppose that $t \in \text{int}(\text{cl}(W_1) \cup \text{cl}(W_2)) \cap \text{fr}(W_1)$, then by Lemma 2.5(ii), with $U = \text{int}(\text{cl}(W_1) \cup \text{cl}(W_2))$, $N_t \subseteq \{W_1, W_2\}$ and $t \notin \text{cl}(W)$ for all $W \in \mathcal{W}$ distinct from W_1 and W_2 . However, since $t \in \text{fr}(W_1)$, there is some $W' \in \mathcal{W}$ distinct from W_1 such that $t \in \text{cl}(W')$ and so we must have $W' = W_2$ and $t \in \text{cl}(W_1) \cap \text{cl}(W_2)$.

Next we show that

$$\text{fr}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2)) \subseteq \text{fr}(\text{cl}(W_1) \cup \text{cl}(W_2)).$$

To this end, let $t \in \text{fr}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2)) = (\text{cl}(W_1) \cap \text{cl}(W_2)) \setminus \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$; however, from the previous paragraph, if $t \in \text{int}(\text{cl}(W_1) \cup \text{cl}(W_2))$, then $t \in \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$ and so $t \in \text{fr}(\text{cl}(W_1) \cup \text{cl}(W_2))$.

Finally suppose that $N_z = \{W_1, W_2\}$; then $z \in \text{cl}(W_1) \cap \text{cl}(W_2) = \text{fr}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2)) \cup \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$. However, if $z \in \text{fr}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$, then by the previous paragraph, $z \in \text{fr}(\text{cl}(W_1) \cup \text{cl}(W_2)) \subseteq \text{cl}(\mathbb{R}^2) \setminus (\text{cl}(W_1) \cup \text{cl}(W_2))$. Thus $z \in \bigcup \{\text{cl}(W) : W \in \mathcal{W} \text{ and } W \neq W_1, W_2\}$, contradicting the hypothesis that $N_z = \{W_1, W_2\}$. $\square$

Corollary 2.10. *If W_1, W_2, W_3 are distinct elements of $\mathcal{W}$, then $\text{ind}(\text{cl}(W_1) \cap \text{cl}(W_2) \cap \text{cl}(W_3)) = 0$.*

Proof. If $\text{ind}(\text{cl}(W_1) \cap \text{cl}(W_2) \cap \text{cl}(W_3)) \geq 1$, then by Corollary 2.2, $\text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2) \cap \text{cl}(W_3)) \neq \emptyset$. Thus if $z \in \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2) \cap \text{cl}(W_3))$ it follows that $z \in \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$ and so from Theorem 2.9 we have that $N_z = \{W_1, W_2\}$, contradicting the fact that $z \in \text{cl}(W_3)$. $\square$

Lemma 2.11. *Suppose that $\mathcal{W}$ is a spherical fenestration of $\mathbb{R}^2$. If $W_1, W_2 \in \mathcal{W}$, then $W_1 \cup W_2 \cup \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$ is open in $\mathbb{R}^2$.*

Proof. Since $W_1 \cup W_2$ is open it suffices to show that $W_1 \cup W_2 \cup \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$ is a neighbourhood of each point $z \in \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$. To this end, let z be

such a point and note that it follows from Theorem 2.9 that $N_z = \{W_1, W_2\}$. Then by Lemma 2.5(i) and part of the proof of Theorem 2.9, we have that $z \in \text{int}(\text{cl}(W_1) \cup \text{cl}(W_2)) \cap \text{fr}(W_1) \subseteq \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$. It is now immediate that the open set $\text{int}(\text{cl}(W_1) \cup \text{cl}(W_2))$ contains z and is contained in $W_1 \cup W_2 \cup \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$. $\square$

Given a locally finite spherical fenestration of $\mathbb{R}^2$, we are now in a position to define the minimal $\mathcal{W}$ -grid on $\mathbb{R}^2$. However, we first construct a special grid on $\mathbb{R}^2$. Define

$$\mathcal{W}_1 = \{\text{int}_{\text{fr}(W)}(\text{cl}(V) \cap \text{cl}(W)) : V, W \in \mathcal{W} \wedge V \neq W\}$$

and

$$\mathcal{W}_0 = \left\{ \{x\} : x \notin \left(\bigcup \mathcal{W} \right) \cup \left(\bigcup \mathcal{W}_1 \right) \right\}.$$

Since the elements of $\mathcal{W}_1$ have empty interiors (in $\mathbb{R}^2$), it is immediate that $X = \mathcal{W} \cup \mathcal{W}_1 \cup \mathcal{W}_0$ is a partition of $\mathbb{R}^2$. We endow X with the quotient topology τ induced by the natural projection map π from $\mathbb{R}^2$ onto X and for each $x \in \mathbb{R}^2$ we let $[x]$ denote the element of the partition which contains x . Note that since π is one-to-one when restricted to $\bigcup \mathcal{W}_0$, each point of $\mathcal{W}_0$ is closed in X and, by Theorem 2.7, $\mathcal{W}_0$ is discrete.

Lemma 2.12. $\mathcal{W} = \{[x] : |N_x| = 1\}$, $\mathcal{W}_1 = \{[x] : |N_x| = 2\}$ and $\mathcal{W}_0 = \{[x] : |N_x| \geq 3\}$ and all three sets are non-empty.

Proof. The first equality and the fact that $\mathcal{W} \neq \emptyset$ are clear. The second is an immediate consequence of Theorem 2.9, while the fact that $\mathcal{W}_1 \neq \emptyset$ follows from the local finiteness of $\mathcal{W}$ and Theorem 2.7, which implies that for some $W' \in \mathcal{W}$, $\text{cl}(W) \cap \text{cl}(W')$ must have a non-empty interior in $\text{fr}(W)$.

To prove the third equality, we claim that if W_1, W_2 are distinct elements of $\mathcal{W}$ such that $\text{cl}(W_1) \cap \text{cl}(W_2) \neq \emptyset$, then $\text{fr}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2)) \neq \emptyset$. However, this follows immediately from Lemma 2.8, since $\text{fr}(W_1)$ is connected, being homeomorphic to S^1 , and every proper subset of a connected space has a non-empty frontier. Thus $\text{fr}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2)) \neq \emptyset$ and, by Theorem 2.9, if $x \in \text{fr}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$, then $|N_x| \geq 3$. Conversely, if $|N_x| \geq 3$, say $N_x = \{W_j : 1 \leq j \leq n\}$, for some $n \geq 3$, then clearly $x \notin \bigcup \mathcal{W}$. Further, it follows from Theorem 2.9 that for distinct $j, k \leq n$, $x \notin \text{int}_{\text{fr}(W_j)}(\text{cl}(W_j) \cap \text{cl}(W_k))$. However, this implies that $[x] \notin \mathcal{W} \cup \mathcal{W}_1$ as required, and it follows that $\mathcal{W}_0 \neq \emptyset$. $\square$

Theorem 2.13. (X, τ) is an Alexandroff T_0 -space which is a $\mathcal{W}$ -grid and has Alexandroff dimension 2.

Proof. We leave to the reader the straightforward proof that (X, τ) is an Alexandroff T_0 -space whose minimal open sets are of the following kinds:

- (a) If $[x] \in \mathcal{W}$, then the minimal open neighbourhood of $[x]$ which we denote by $U([x])$ is $\{[x]\}$.

- (b) If $[x] \in \mathcal{W}_1$ and $N_x = \{U, V\}$, then for any $y \in V$ and $z \in W$, $U([x]) = \{[x], [y] = \pi(V), [z] = \pi(W)\}$ since $\pi^{-1}([x]) = U \cup V \cup \text{int}_{\text{fr}(U)}(\text{cl}(U) \cap \text{cl}(V))$.
- (c) If $\{x\} \in \mathcal{W}_0$, then the minimal open neighbourhood of $\{x\}$ is $U(\{x\}) = \{x\} \cup \{[y] \in \mathcal{W} \cup \mathcal{W}_1 : x \in \text{cl}(\pi^{-1}([y]))\}$.

To show that (X, τ) is a $\mathcal{W}$ -grid it suffices to prove that π is an open map. To this end, suppose that $M \subseteq \mathbb{R}^2$ is open and $[x] \in \pi(M) \subseteq X$; then there is some $z \in M$ such that $\pi(z) = [x]$. It suffices to show that $U([x]) \subseteq \pi(M)$. Let $y \in U([x])$; there are three cases to consider:

- (i) If $[x] \in \mathcal{W}$, then $[x] = \pi(W)$ for some $W \in \mathcal{W}$ and so $z \in W$. However, since $U([x]) = [x]$, it follows that $y = [x]$ and so $y \in \pi(M)$.
- (ii) If $[x] \in \mathcal{W}_1$, $[x] = \pi(\text{int}_{\text{fr}(V)}(\text{cl}(V) \cap \text{cl}(W)))$ for some distinct $V, W \in \mathcal{W}$, $U([x]) = \{[x], \pi(V), \pi(W)\}$ and $z \in \pi^{-1}[x] = \text{int}_{\text{fr}(V)}(\text{cl}(V) \cap \text{cl}(W))$. Since M is open and $\pi(z) \in M$, it follows that M intersects both V and W . Choosing $a \in M \cap V$ and $b \in M \cap W$, $U([x]) = \pi(\{x, a, b\}) \subseteq \pi(M)$ and we are done.
- (iii) If $[x] \in \mathcal{W}_0$, then $[x] = \{x\}$ and so $z = x$; furthermore, $U([x]) = \{x\} \cup \{[t] : x \in \text{cl}(\pi^{-1}([t]))\}$. Since M is open it follows that $M \cap A \neq \emptyset$ whenever $x \in \text{cl}(A)$ and hence $y \in \pi(M)$ whenever $y \in U([x])$.

Finally we are required to show that $\text{Adim}(X, \tau) = 2$. That $\text{Adim}(X, \tau) \geq 2$ is clear since by Lemma 2.12 there are $U, V, W \in \mathcal{W}$ such that $\text{cl}(U) \cap \text{cl}(V) \cap \text{cl}(W) \neq \emptyset$. Choosing $x \in \text{cl}(U) \cap \text{cl}(V) \cap \text{cl}(W)$ it is clear that $[x], \text{int}_{\text{fr}(U)}(\text{cl}(U) \cap \text{cl}(V)), V$ form a specialization chain of length 3.

To show that $\text{Adim}(X, \tau) \leq 2$ we note that elements of $\mathcal{W}$ are specialization incomparable, being disjoint open sets in $\mathbb{R}^2$ and the elements of $\mathcal{W}_0$ are specialization incomparable since, they are closed singletons. Finally, the elements of $\mathcal{W}_1$ are specialization incomparable since by Lemma 2.11, if $z = \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2)) \in \mathcal{W}_1$, then $U = W_1 \cup W_2 \cup \text{int}_{\text{fr}(W_1)}(\text{cl}(W_1) \cap \text{cl}(W_2))$ is open and $U \cap \mathcal{W}_1 = \{z\}$. Thus each specialization chain can contain at most one element from each of $\mathcal{W}, \mathcal{W}_1$ and $\mathcal{W}_2$. $\square$

To construct the minimal $\mathcal{W}$ -grid, we now define an equivalence relation $\sim$ on (X, τ) as follows: For $[x], [y] \in X$, let

$$[x] \sim [y] \quad \text{if and only if} \quad N_x = N_y.$$

$X^\sim$ will denote the quotient space $X/\sim$, $\tau^\sim$ the quotient topology, $\pi^\sim$ the quotient map and $[z]^\sim$ the equivalence class of $[z]$. We note that the quotient space $X^\sim$ is obtained from (X, τ) by a map which is one-to-one on $\mathcal{W} \cup \mathcal{W}_1$.

Theorem 2.14. $(X^\sim, \tau^\sim)$ is the minimal $\mathcal{W}$ -grid and $\text{Adim}(X^\sim, \tau^\sim) = 2$.

Proof. For each $W \in \mathcal{W}$, if we denote $\text{cl}(W) \cap (\bigcup \mathcal{W}_0)$ by W_0 , then W_0 is nowhere dense and has a finite number of components in $\mathbb{R}^2$; hence it is a finite set (of closed points) in $\mathbb{R}^2$ and hence so is $\pi(W_0)$ in (X, τ) , since π is one-to-one on $\bigcup \mathcal{W}_0$. Furthermore, from the characterization of the minimal neighbourhoods in (X, τ) given in Theorem 2.13, it is clear that $\pi(W_0)$ is discrete in this space.

We claim first that $\pi^\sim$ is an open map. Suppose that U is open in X and let $[z]^\sim \in \pi^\sim(U)$; if $(\pi^\sim)^{-1}([z]^\sim) \cap \mathcal{W}_0 = \emptyset$, then $(\pi^\sim)^{-1}([z]^\sim) = [z]$ and since $\mathcal{W}_0$ is closed and discrete, there is some open neighbourhood V of $[z]$ contained in U which misses $\mathcal{W}_0$; hence $\pi^\sim(V)$ is an open neighbourhood of $[z]^\sim$ contained in $\pi^\sim(U)$. If on the other hand $(\pi^\sim)^{-1}([z]^\sim) \cap \mathcal{W}_0 \neq \emptyset$, then we choose x in this latter set and let V be the minimal neighbourhood of x in (X, τ) . Again since $\mathcal{W}_0$ is closed and discrete in X , it is easy to see that $\pi^\sim$ is one-to-one on V and hence $\pi^\sim(V)$ is open and $z \in \pi^\sim(V) \subseteq \pi^\sim(U)$, proving our claim. Thus $\pi^\sim \circ \pi : \mathbb{R}^2 \rightarrow X^\sim$ is an open map, completing the proof that $(X^\sim, \tau^\sim)$ is a $\mathcal{W}$ -grid.

The proof that $\text{Adim}(X^\sim, \tau^\sim) = 2$ is a consequence of the fact that $\pi^\sim$ is one-to-one on $\mathcal{W} \cup \mathcal{W}_1$; we omit the details.

Finally, to show that $(X^\sim, \tau^\sim)$ is the minimal $\mathcal{W}$ -grid, it suffices to note that $(X^\sim, \tau^\sim)$ is the decomposition $\Delta \times$ introduced by Kronheimer in Section 6 of [8]. Since the projection from $\mathbb{R}^2 \rightarrow (X^\sim, \tau^\sim)$ is open, it is an immediate consequence of Theorem 6.2 of [8] that $(X^\sim, \tau^\sim)$ is the minimal $\mathcal{W}$ -grid. $\square$

3. Three Examples

In an attempt to justify our rather narrow focus on the Euclidean space $\mathbb{R}^2$, we give first an example to show that a number of the above results are false in $\mathbb{R}^3$ and higher dimensions. Specifically, the definition of the partition given prior to Theorem 2.14 cannot be applied in $\mathbb{R}^3$; there are fenestrations of $\mathbb{R}^3$, each of whose elements has a frontier homeomorphic to S^2 which give rise neither to a grid nor to a quotient with Alexandroff dimension 3.

Example 3.1. We define $\mathcal{W}$ as follows: Whenever $k, m, n \in \mathbb{Z} \setminus \{-1, 0\}$, then $(k, k + 1) \times (m, m + 1) \times (n, n + 1) \in \mathcal{W}$. The other four elements of $\mathcal{W}$ are:

$$\begin{aligned} W_1 &= (-1, 0) \times (-1, 1) \times (-1, 1), \\ W_2 &= (0, 1) \times (0, 1) \times (-1, 0), \\ W_3 &= \{(x, y, z) : x, y, z \in (0, 1) \text{ and } x < z\}, \\ W_4 &= (-1, 1) \times (-1, 1) \times (-1, 1) \setminus \text{cl}(W_1 \cup W_2 \cup W_3). \end{aligned}$$

This is a spherical fenestration of $\mathbb{R}^3$ and only W_4 is non-convex. After sketching a diagram it is easy to check that:

- (a) $N_{(0,y,0)} = \{W_1, W_2, W_3, W_4\}$ for each $y \in [0, 1)$.
- (b) $N_{(0,0,z)} = \{W_1, W_3, W_4\}$ for each $z \in (0, 1)$.
- (c) $N_{(0,1,0)} \supset \{W_1, W_2, W_3, W_4\}$.
- (d) $N_{(0,0.5,0.5)} = \{W_1, W_3\}$.

Using the construction of the $\mathcal{W}$ -grid described above produces the specialization chain $[(0, 1, 0)] < [(0, 0, 0)] < [(0, 0, 0.5)] < [(0, 0.5, 0.5)] < W_1$, implying that the Alexandroff dimension of the quotient space is at least 4. We leave the reader to check that the quotient mapping is not open and hence the construction does not give a $\mathcal{W}$ -grid.

This example shows that when dealing with Euclidean spaces of dimension greater than 2, the partition used to define the minimal grid must be different from that given

prior to Theorem 2.14. One possibility that suggests itself is to use the minimal elements of the set of all Boolean combinations of the closures of all $\sim$ -equivalence classes as the required partition. This remains to be studied, but, as noted at the end of Section 1, the existence of wild embeddings of $(n - 1)$ -spheres in $\mathbb{R}^n$ whenever $n > 2$ may be a complicating factor.

The reader might also wonder whether the condition that the elements of $\mathcal{W}$ have frontiers homeomorphic to S^1 is necessary. The following example, which is a slight variation on Example 2.8 of [12], illustrates the importance of this condition.

Example 3.2. Let $W_1 = (-1, 0) \times (0, 2)$, $W_2 = (-1, 0) \times (-2, 0)$, $W_3 = \{(x, y) \in \mathbb{R}^2 : 1 > x > 0, 2 > y > \sin(1/x)\}$ and $W_4 = \{(x, y) \in \mathbb{R}^2 : 1 > x > 0, -2 < y < \sin(1/x)\}$. The rest of $\mathbb{R}^2$ may have the standard fenestration (see Example 2.6 of [12]). If we again define the partition as in Section 2, $N_{(0,0)} = \{W_1, W_2, W_3, W_4\}$, $N_{(0,0.5)} = \{W_1, W_3, W_4\}$, $N_{(4/\pi, \sqrt{2}/2)} = \{W_3, W_4\}$ and hence $N_{(0,0)} < N_{(0,0.5)} < N_{(4/\pi, \sqrt{2}/2)} < W_4$, and so the quotient space has Alexandroff dimension 3. Note that only W_3 and W_4 fail to have frontiers homeomorphic to S^1 .

Finally, in contrast with convex fenestrations, we note that the minimal $\mathcal{W}$ -grid of a given spherical fenestration $\mathcal{W}$ constructed prior to Theorem 2.14 need not have a base of regular open sets (that is, it need not be *semiregular*), as the following example shows.

Example 3.3. Let $\mathcal{W}$ be defined as follows:

$$\begin{aligned} W_1 &= (0, 1) \times (0, 1), \\ W_2 &\text{ is the open half disc } \{p \in \mathbb{R}^2 : d(p, (0.25, 1)) < 0.25\} \setminus \text{cl}(W_1), \text{ where } d \text{ denotes} \\ &\quad \text{the Euclidean metric,} \\ W_3 &= \{p \in \mathbb{R}^2 : d(p, (0.5, 1)) < 0.5\} \setminus \text{cl}(W_1 \cup W_2), \\ W_4 &= (-1, 2) \times (0, 2) \setminus \text{cl}(W_1 \cup W_2 \cup W_3), \end{aligned}$$

and in the rest of the plane we suppose the standard fenestration. So we have a spherical fenestration of $\mathbb{R}^2$, only W_3 and W_4 are non-convex, and the digital space constructed as in Theorem 2.14 has Alexandroff dimension equal to 2. However, it is evident that for $x = (0, 1)$, $[x] = \{x\}$, and that, for example, the point $(0.5, 1)$ is contained in $\text{int}(\text{cl}(U([x]))) \setminus U([x])$, so, the space $X^\sim$ is not semiregular.

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