Gram Matrix Analysis of Finite Distance Spaces in Constant Curvature

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Abstract. We develop the utility of Gram matrix machinery as a tool to treat the geometry of simplices in space forms. A formula relating the determinant of a normalized Gram matrix to the geometry of the simplex it represents is presented. We then apply the tools to *leaf spaces*, i.e. the set of degree 1 vertices of a metric tree. One main result is that for a given metric space X there exists a constant $\kappa_0 < 0$, such that X embeds into all hyperbolic spaces of curvature less than κ_0 , if and only if X is a leaf space.

Introduction

This paper is based on a part of my Ph.D. thesis [9], with some improvements and corrections.

It is well known that the notions of subspaces and rank of finite subsets, i.e. the dimension of a minimal subspace containing a given set of points, are exactly the same in spherical, Euclidean and hyperbolic geometry. Less known is that these concepts can be described using essentially the same algebraic methods in all three geometries. These concepts even remain valid if we broaden the category to include also indefinite, i.e. semi-Riemannian, spaces of constant curvature.

For a given finite subset X of a space form, the central object under study in this theory is a matrix $C_\kappa(X)$ whose entries are a trigonometric function of the distances between points in X . This matrix can be considered as a *normalized Gram matrix* for the set of points. Models for the constant curvature geometries are provided by hyperquadrics in semi-Euclidean spaces and each entry of $C_\kappa(X)$ can be regarded as a normalized scalar product of the respective position vectors of the points in the ambient space. Via this matrix many concepts from linear algebra of the ambient space translate into intrinsic geometry in the space form.

The theory discussed in this paper suggests joining the two hyperquadrics of different causal character in an ambient indefinite semi-Euclidean space into a single object,

as is also done in, e.g. [15] and [16]. Thus hyperbolic space, which is isometric to a hyperquadric of negative sign in the semi-Euclidean space of index 1, is joined with the hyperquadric of positive sign. This gives a disconnected space which is more “complete” in an algebraic sense; something which should be clear later.

The viewpoint in this paper is mainly intrinsic and we try to abstract from the concept of “ambient space”. Hence we extend the usual Riemannian distance concept to the indefinite space forms such that the distance interpretation of the $\mathbf{C}_\kappa$ -matrix carries on to these spaces as well. However, the $\mathbf{C}_\kappa$ -matrix can also be given another interpretation, namely as *dual* to the matrix of cosines to *exterior dihedral angles*, assuming the points of X span a simplex in the space form under consideration. This matrix of cosines is also, in the classical setting, sometimes called a Gram matrix, see [10] and [4]. See below for a clarification of these concepts.

In the 1930s Schoenberg, Blumenthal and Menger (among others) studied the question: *when is a given metric space realizable as a subset of a (simply connected) space form?* See, e.g. [14]. This question can be answered in terms of the $\mathbf{C}_\kappa$ -matrix (Theorem 1 below). A fundamental source is [2], where the basic concepts of the theory are developed from an intrinsic viewpoint.

The first section of this paper is devoted to notations and preliminary results. In the second section we develop further the geometric significance of the *Gram matrix machinery*. A geometric formula for the determinant of a $\mathbf{C}_\kappa$ -matrix is derived. This is just a natural extension of a well-known formula for the Cayley–Menger determinant. Also a formula for the dihedral angles of a simplex is presented. These geometric formulas establish the power of the *Gram matrix machinery*, developed in this paper, as a tool to treat the geometry of simplices, or polytopes, in high-dimensional spaces.

We treat the geometry in a unified manner independent of curvature. We choose here to exclude the case of zero curvature since this case requires some special attention and makes things slightly more cumbersome. See, e.g. [9], where this case is also treated.

In the third and final section we characterize the finite metric spaces, that can be isometrically embedded into a real hyperbolic space in the limit where curvature goes to $-\infty$. One main result is that for a given finite metric space X there exists a constant $\kappa_0 < 0$, such that X embeds into all hyperbolic spaces of curvature less than κ_0 , if and only if X is a so-called *leaf space*, i.e. the set of degree 1 vertices of a weighted tree, or a subset of the line. The “only if” part is already established in the framework of δ -hyperbolic spaces, see [6], where a metric space, which satisfies a weaker condition than embedability, is shown to be isometric to a subset of a weighted tree.

The main tools of this section are combinatorial, and we develop several results about the $\mathbf{C}_\kappa$ -matrix of a metric tree, e.g. a result on the behavior of the eigenvalues as curvature goes to $-\infty$. These results are then combined in the final subsection with the tools of the second section to yield results on the “limiting” geometry.

1. Notations and Preliminaries

We use the notation $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, $\mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\} = \mathbb{Z} \cap \mathbb{R}_+$. For $x \in \mathbb{R}$ the convention $\sqrt{x} \in \mathbb{R}_+ \cup \mathbf{i}\mathbb{R}_+$ is used; here $\mathbf{i}$ denotes (a fixed choice) of $\sqrt{-1} \in \mathbb{C}$. The complex conjugate of $z \in \mathbb{C}$ is denoted $\bar{z}$.

Trigonometric Functions. For $\kappa \in \mathbb{R}^*$ and $x \in \mathbb{R}$ we define

$$\mathbf{c}_\kappa(x) := \cos(\sqrt{\kappa}x), \quad (1)$$

$$\mathbf{s}_\kappa(x) := \sin(\sqrt{\kappa}x). \quad (2)$$

Note that for $z \in \mathbb{C}$, $\cos(\mathbf{i}z) = \cosh(z)$, while $\sin(\mathbf{i}z) = \mathbf{i} \sinh(z)$.

We associate to the trigonometric functions a subset of $\mathbb{C}$. Define

$$\mathbb{C}_1 := [0, \pi] \cup \mathbf{i}\mathbb{R}_+ \cup (\pi - \mathbf{i}\mathbb{R}_+) \cup \left(\frac{\pi}{2} + \mathbf{i}\mathbb{R}\right)$$

and for $\kappa \in \mathbb{R}^*$ define $\mathbb{C}_\kappa \subset \mathbb{C}$ as

$$\mathbb{C}_\kappa := \frac{1}{\sqrt{\kappa}} \mathbb{C}_1.$$

Then $\mathbf{c}_\kappa$ maps $\mathbb{C}_\kappa$ bijectively onto $\mathbb{R} \cup \mathbf{i}\mathbb{R}$, so for a given $y \in \mathbb{R} \cup \mathbf{i}\mathbb{R}$ the equation $\mathbf{c}_\kappa(x) = y$ has a unique solution in $\mathbb{C}_\kappa$.

$\mathbb{C}_\kappa$ -Distance Spaces. In the following the term $\mathbb{C}_\kappa$ -distance space is used to denote a pair (X, d) , where X is a set and $d: X \times X \rightarrow \mathbb{C}_\kappa$ is a symmetric function such that $d(p, p) = 0$ for all $p \in X$. As usual we drop the (X, d) notation most of the time and just write X with the distance implicitly included.

A finite $\mathbb{C}_\kappa$ -distance space $X = \{p_1, \dots, p_n\}$ is completely characterized by its $n \times n$ symmetric distance matrix $\mathbf{D}_X = [d_{ij}] \in \text{Sym}_n(\mathbb{C}_\kappa)$, where $d_{ij} := d(p_i, p_j)$. For such a space, we introduce the following fundamental object:

Definition 1. For $\kappa \in \mathbb{R}^*$ and X a finite $\mathbb{C}_\kappa$ -distance space with n points and distance matrix $\mathbf{D}_X = [d_{ij}] \in \text{Sym}_n(\mathbb{C}_\kappa)$ define the $n \times n$ -matrix:

$$\mathbf{C}_\kappa(X) := [\mathbf{c}_\kappa(d_{ij})] \in \text{Sym}_n(\mathbb{R} \cup \mathbf{i}\mathbb{R}). \quad (3)$$

Semi-Euclidean Spaces. Given $m, v \in \mathbb{N}_0$, with $v \leq m$, $\mathbb{R}_v^m$ denotes the semi-Euclidean space of dimension m and index v , i.e. $\mathbb{R}^m$ equipped with the scalar product:

$$\langle x, y \rangle := - \sum_{i=1}^v x_i y_i + \sum_{i=v+1}^m x_i y_i. \quad (4)$$

The associated “norm” of a vector $x \in \mathbb{R}_v^m$ is written

$$r(x) := \sqrt{\langle x, x \rangle} \in \mathbb{R}_+ \cup \mathbf{i}\mathbb{R}_+.$$

The causal character of a vector $x \in \mathbb{R}_v^m$ is *spacelike* if $r(x) \in \mathbb{R}_+^*$ or $x = 0$, *timelike* if $r(x) \in \mathbf{i}\mathbb{R}_+^*$ and *lightlike* if $x \neq 0$ and $r(x) = 0$.

On $\mathbb{R}_v^m$ we have the equivalence relation

$$x \sim y \quad \text{iff} \quad x = ky \quad \text{for some} \quad k \in \mathbb{R}_+^*.$$

The projection onto the set of equivalence classes is written

$$\Pi: \mathbb{R}_v^m \rightarrow \mathbb{R}_v^m / \mathbb{R}_+^*.$$

It is the action of the group $\mathbb{R}_+^*$ that is factored out.

Now we are ready to define the basic entity in which the geometry is to take place. In the spirit of [15] and [16] we introduce a $\mathbb{C}_\kappa$ -distance on the set $\Pi(\mathbb{R}_v^{n+1} \setminus r^{-1}(0))$, where $r^{-1}(0)$ denotes the r -preimage of 0, i.e. the vectors of norm zero.

Definition 2. For $\kappa \in \mathbb{R}^*$ and $p, q \in \Pi(\mathbb{R}_v^{n+1} \setminus r^{-1}(0))$ let $d_\kappa(p, q)$ be the unique element in $\mathbb{C}_\kappa$ such that

$$\mathbf{c}_\kappa(d_\kappa(p, q)) = \frac{\langle \tilde{p}, \tilde{q} \rangle}{r(\tilde{p})r(\tilde{q})}, \quad (5)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of the ambient space and $\tilde{p}, \tilde{q} \in \mathbb{R}_v^{n+1}$ are representatives of p, q .

For $\kappa \in \mathbb{R}^*$ and $n, v \in \mathbb{N}_0$ with $v \leq n$, $\mathbb{M}(n, v, \kappa)$ denotes $\Pi(\mathbb{R}_v^{n+1} \setminus r^{-1}(0))$ equipped with the $\mathbb{C}_\kappa$ -distance d_κ .

Given $p \in \mathbb{M}(n, v, \kappa)$ we use the notation $\tilde{p} \in \mathbb{R}_v^{n+1}$ to denote a representative of p , i.e. $\Pi(\tilde{p}) = p$. Thus $p \mapsto \tilde{p}$ is a *section* of the projection Π .

Remark 1 (Gram Matrices and Normalization). Given a finite set of points $X = \{p_1, \dots, p_m\} \subset \mathbb{M}(n, v, \kappa)$ the matrix $\mathbf{C}_\kappa(X)$ should thus be interpreted as a *normalized Gram matrix* of a set of representatives of the points. If $\mathbf{X}$ denotes an $(n+1) \times m$ matrix whose i th column equals (the coordinates of) a representative $\tilde{p}_i$ for p_i , then

$$\mathbf{C}_\kappa(X) = \Lambda \mathbf{X}^t \mathbf{I}_v \mathbf{X} \Lambda, \quad (6)$$

where $\Lambda = (r(\tilde{p}_1)^{-1}, \dots, r(\tilde{p}_m)^{-1})$ is a diagonal “normalization” matrix and

$$\mathbf{I}_v := \text{diag}(\underbrace{-1, \dots, -1}_{v \text{ times}}, \underbrace{1, 1, \dots, 1}_{n+1-v \text{ times}}). \quad (7)$$

Note that the normalization matrix Λ is uniquely determined by the criterion that the diagonal entries of $\mathbf{C}_\kappa(X)$ equals 1. This normalization map $N: [a_{ij}] \mapsto [a_{ij}/(\sqrt{a_{ii}}\sqrt{a_{jj}})]$ gives an equivalence relation on the set of symmetric complex matrices with real diagonal entries: $\mathbf{A} \sim \mathbf{B}$ iff $N(\mathbf{A}) = N(\mathbf{B})$. Clearly, $\Lambda \mathbf{A} \Lambda \sim \mathbf{A}$ for Λ a diagonal matrix with entries in $\mathbb{R} \cup i\mathbb{R}$.

Topology. We may consider $\mathbb{M} := \mathbb{M}(n, v, \kappa)$ as a *hyperquadric* in $\mathbb{R}_v^{n+1}$ by choosing a specific section $\zeta: \mathbb{M} \rightarrow \mathbb{R}_v^{n+1}$ such that $|r \circ \zeta|$ is constant. We equip $\mathbb{M}$ with the manifold structure induced by any such section. The topology is then

$$\mathbb{M}(n, v, \kappa) \underset{\text{homeo}}{\cong} \mathbb{S}^{n-v} \times \mathbb{R}^v + \mathbb{S}^{v-1} \times \mathbb{R}^{n+1-v},$$

where $\mathbb{S}^n$ is interpreted as the empty set for $n < 0$. A compactification of $\mathbb{M} = \mathbb{M}(n, \nu, \kappa)$ is

$$\overline{\mathbb{M}} := \Pi(\mathbb{R}_\nu^{n+1} \setminus \{0\}) \underset{\text{homeo}}{\cong} \mathbb{S}^n.$$

We think of the boundary $\partial\mathbb{M} := \overline{\mathbb{M}} \setminus \mathbb{M}$ as *points at infinity*. We have that $\partial\mathbb{M} = \Pi(\mathcal{L}) \underset{\text{homeo}}{\cong} \mathbb{S}^{\nu-1} \times \mathbb{S}^{n-\nu}$, where $\mathcal{L} := r^{-1}(0) \setminus \{0\}$ is the *lightcone* in $\mathbb{R}_\nu^{n+1}$.

Maps. A linear map $F: \mathbb{R}_\nu^{n+1} \rightarrow \mathbb{R}_\nu^{n+1}$ such that $r(x) \neq 0 \rightarrow r(F(x)) \neq 0$ "restricts" to a map $\Pi(F): \mathbb{M}(n, \nu, \kappa) \rightarrow \mathbb{M}(n, \nu, \kappa)$ as $\Pi \circ F \circ \zeta$, where ζ is any section. In particular any element of $O(n+1, \nu)$, the linear maps preserving $\langle \cdot, \cdot \rangle$, restricts to $\mathbb{M}(n, \nu, \kappa)$.

Isometries. An *isometry* $\sigma: X \rightarrow Y$ between $\mathbb{C}_\kappa$ -spaces X and Y is a *bijective* map preserving the distance $d_Y(\sigma(x), \sigma(y)) = d_X(x, y)$. The group of isometries on a $\mathbb{C}_\kappa$ -space X is denoted $\text{Isom}(X)$.

From the definition of d_κ , Definition 2, it is clear that any isometry of the ambient semi-Euclidean space restricts to an isometry of $\mathbb{M} := \mathbb{M}(n, \nu, \kappa)$. It is also not difficult to show that in fact any element of $\text{Isom}(\mathbb{M})$ is the restriction of an isometry of the ambient space. Hence the group $\text{Isom}(\mathbb{M})$ coincides with $O(n+1, \nu)$. See, e.g. the discussion in Chapter 9 of [11].

Antipodality. One particular isometry which occurs often is the *antipodal map*, which we for reasons that become clear later chose to denote $R_\emptyset \in \text{Isom}(\mathbb{M})$. It is given as the restriction to $\mathbb{M}$ of the antipodal map of $\mathbb{R}_\nu^{n+1}$, $x \mapsto -x$. We choose to call two points $p, q \in \mathbb{M}$ *antipodal* if $p = R_\emptyset(q)$. Without proof we state the following result:

Proposition 1. *Two points $p, q \in \mathbb{M} := \mathbb{M}(n, \nu, \kappa)$ are antipodal iff*

$$d(p, r) + d(r, q) = d(p, q) = \frac{\pi}{\sqrt{\kappa}} \quad \text{for all } r \in \mathbb{M}.$$

Remark 2 (Choice of Values of the Complex Distance). Except for a slightly different choice of values of the distance, $\mathbb{M}(3, 1, -1)$ is similar to the space $\tilde{\text{HS}}^3$ discussed in [16]. In fact, the choice of values for the distance used here, $\mathbb{C}_{-1}$, is exactly the conjugate of the set of values for the distance used in [16].

The main reason for choosing $\mathbb{C}_\kappa$ here is that we want antipodal points to have distance $\pi/\sqrt{\kappa}$ (note that $1/i = -i$). Another important property of $\mathbb{C}_\kappa$ is that $\mathbb{C}_\kappa = \pi/\sqrt{\kappa} - \mathbb{C}_\kappa$.

In [15] and [16] the viewpoint is towards hyperbolic geometry. Here our curvature parameter is free to vary in $\mathbb{R}^*$, with no preference for hyperbolic or spherical geometry.

With our conventions we have

$$(\mathbb{M}(n, \nu, |\kappa|), d_{|\kappa|}) \underset{\text{isom}}{\cong} (\mathbb{M}(n, \nu, -|\kappa|), \text{id}_{-|\kappa|})$$

and for $\nu \geq 1$,

$$(\mathbb{M}(n, \nu, \kappa), d_\kappa) \underset{\text{isom}}{\cong} (\mathbb{M}(n, n+1-\nu, \kappa), d_\kappa)$$

via the restriction of the map $\psi: \mathbb{R}_v^{n+1} \rightarrow \mathbb{R}_{n+1-v}^{n+1}$, $x \mapsto r^2(x)x$, which reverses causal character and maps a timelike vector to a vector proportional to its antipode.

Connection with the Semi-Riemannian Geometry. As in [11] we use the following notation:

$$\begin{aligned} S_v^n(r_0) &:= \{x \in \mathbb{R}_v^{n+1} \mid r(x) = r_0\}, \\ H_v^n(r_0) &:= \{x \in \mathbb{R}_{v+1}^{n+1} \mid r(x) = \mathbf{i}r_0\}. \end{aligned}$$

$S_v^n(r_0)$ is called the *pseudosphere* of radius r_0 and $H_v^n(r_0)$ the *pseudohyperbolic space* of radius r_0 . Equipped with the metric tensor inherited from the ambient space the two hyperquadrics become semi-Riemannian manifolds of dimension n , index v and (for $n \geq 2$) constant curvature $1/r_0^2$ and $-1/r_0^2$, respectively. Note that $S_v^n(r_0)$ and $H_{n-v}^n(r_0)$ are *anti-isometric*.

In each case equip $\mathbb{M} := \mathbb{M}(n, v, \kappa)$ with the metric tensor

$$g = \kappa \zeta^*(r^2\langle \cdot, \cdot \rangle),$$

where $\zeta: \mathbb{M} \hookrightarrow \mathbb{R}_v^{n+1}$ is a section/embedding such that $|r^2(\zeta(p))| = 1/|\kappa|$ and $\mathbb{R}_v^{n+1}$ is considered as a semi-Riemannian manifold in the usual way. Thus on the component of $\mathbb{M}$ where the sign of $r^2 \circ \zeta$ differs from κ , we choose to reverse the sign of the metric tensor inherited from the overlying space.

It is easily seen that an isometry $\sigma: \mathbb{M} \rightarrow \mathbb{M}$ in the sense discussed above is also an isometry in the semi-Riemannian sense, meaning that the differential σ_* preserves the metric tensor on each component of $\mathbb{M}$. In terms of semi-Riemannian isometries we have

$$\mathbb{M}(n, v, \kappa) \underset{\text{isom}}{\cong} \begin{cases} S_v^n\left(\frac{1}{\sqrt{\kappa}}\right) + S_{n+1-v}^n\left(\frac{1}{\sqrt{\kappa}}\right) & \text{for } \kappa > 0, \\ H_{n-v}^n\left(\frac{1}{\sqrt{|\kappa|}}\right) + H_{v-1}^n\left(\frac{1}{\sqrt{|\kappa|}}\right) & \text{for } \kappa < 0, \end{cases}$$

where $S_{n+1}^n(r)$ and $H_{-1}^n(r)$ are interpreted as the empty set. Then for $n > 1$, $\mathbb{M}(n, v, \kappa)$ has constant semi-Riemannian sectional curvature κ .

Notation 1. Sometimes we more specifically write $\mathbb{S}(n, \kappa)$ for the sphere of curvature $\kappa > 0$ and $\mathbb{H}(n, \kappa)$ for the hyperbolic space of curvature $\kappa < 0$; the latter is isometric to a connected component of $H_0^n(1/\sqrt{|\kappa|}) \subset \mathbb{M}(n, 1, \kappa)$.

On the spheres and on the hyperbolic space part of $\mathbb{M}(n, 1, \kappa)$, d_κ coincides with the usual formula for the Riemannian distance on the hyperquadric models. So it is only on the purely semi-Riemannian parts that we obtain anything nonstandard.

With these conventions Proposition 38 on p. 149 in [11] translates directly into a distance statement:

Proposition 2. *Let $p, q \in \mathbb{M}(n, v, \kappa)$ be distinct nonantipodal points lying in the same connected component of $\mathbb{M}(n, v, \kappa)$.*

1. If $\kappa > 0$ and $d_\kappa(p, q) \in (0, \pi/\sqrt{\kappa})$ or $\kappa < 0$ and $d_\kappa(p, q) \in (0, \infty)$ then p, q lie on a unique spacelike geodesic.
  2. If $d_\kappa(p, q) = 0$, then p, q lie on a unique lightlike geodesic.
  3. If $\kappa > 0$ and $d_\kappa(p, q) \in \mathbf{i}\mathbb{R}_+$ or $\kappa < 0$ and $d_\kappa(p, q) \in \mathbf{i}(-\pi/\sqrt{|\kappa|}, 0)$, then p, q lie on a unique timelike geodesic.
- Otherwise p and q are not geodesically connected.

Points in the same connected component that are not geodesically connected either have distance $\pi/\sqrt{\kappa}$ or distance in $\mathbb{C}_\kappa \setminus \mathbb{R}_+ \cup \mathbf{i}\mathbb{R}_+$. Two points which lie in components of $\mathbb{M}(n, \nu, \kappa)$ of different causal character have distance in $\pi/2\sqrt{\kappa} + \sqrt{-\kappa}\mathbb{R}$, as is easily checked.

Realizability Criteria. In this context a *realization* of one $\mathbb{C}_\kappa$ -space X in another Y is a map $\sigma: X \rightarrow Y$ such that $d_Y(\sigma(x), \sigma(y)) = d_X(x, y)$. In general, a realization need not be injective. We write $X \xrightarrow{\text{isom}} Y$ if there exists a realization $\sigma: X \rightarrow Y$.

There are various equivalent formulations of the criterion for when a finite metric space X is realizable as a subset of one of the Riemannian space forms: Euclidean space, the sphere and hyperbolic space. They go back to the work of Schoenberg, Menger and Blumenthal in the 1930s, see [2] and [14], but seem to be rediscovered from time to time in different contexts. The next theorem may be seen as a generalization of the ideas in [14].

Theorem 1. *Let $\kappa \in \mathbb{R}^*$ and $X = \{p_1, \dots, p_n\}$ be a finite $\mathbb{C}_\kappa$ -distance space, then $X \xrightarrow{\text{isom}} \mathbb{M}(m, \nu, \kappa)$ if and only if there exists a diagonal matrix $\Lambda \in \text{Diag}_n(\{\mathbf{i}, 1\})$ such that $\Lambda \mathbf{C}_\kappa(X) \Lambda \in \text{Sym}_n(\mathbb{R})$ and the signature of this matrix satisfies*

$$n_- \leq \nu \quad \text{and} \quad n_+ \leq m + 1 - \nu,$$

where n_- and n_+ denotes the number of negative and positive eigenvalues of $\Lambda \mathbf{C}_\kappa(X) \Lambda$, respectively. Furthermore, the minimal m such that $X \xrightarrow{\text{isom}} \mathbb{M}(m, \nu, \kappa)$ is equal to $\text{rank}(\mathbf{C}_\kappa(X)) - 1$.

Proof. The only if part follows from the definition of the distance d_κ , see Remark 1, and Sylvester's Law of Inertia.

For the other direction suppose a diagonal matrix Λ exists such that the criteria are fulfilled. It is basic linear algebra, see Chapter 1 of [9], that, via a diagonalization of $\mathbf{A} := \Lambda \mathbf{C}_\kappa(X) \Lambda$, we may obtain a set of n vectors $Y \subset \mathbb{R}_\nu^{m+1}$ such that $\mathbf{A}$ equals the Gram matrix of the set Y . Thus $\mathbf{C}_\kappa(X) = \Lambda^{-1} \mathbf{A} \Lambda^{-1} = \mathbf{C}_\kappa(\Pi(Y))$. Hence $\Pi(Y)$ is a realization of X in $\mathbb{M}(n, \nu, \kappa)$.

By adding the two signature inequalities we see that $\text{rank}(\mathbf{C}_\kappa(X)) - 1 = n_+ + n_- - 1 \leq m$. $\square$

We can add more specific versions of the theorem above, stating when a space is realizable in one of the classical Riemannian ingredients. The following results are well known, see [2]:

Theorem 2. Let $X = \{p_1, \dots, p_n\}$ be a finite $\mathbb{C}_\kappa$ -distance space with distance matrix in $\text{Sym}_n(\mathbb{C}_\kappa \cap \mathbb{R}_+)$, then for $\kappa < 0$,

$$X \xrightarrow{\text{isom}} \mathbb{H}(m, \kappa) \quad \text{if and only if} \quad n_-(\mathbf{C}_\kappa(X)) \leq m - 1 \quad \text{and} \quad n_+(\mathbf{C}_\kappa(X)) = 1$$

and for $\kappa > 0$,

$$X \xrightarrow{\text{isom}} \mathbb{S}(m, \kappa) \quad \text{if and only if} \quad n_-(\mathbf{C}_\kappa(X)) = 0 \quad \text{and} \quad n_+(\mathbf{C}_\kappa(X)) \leq m + 1.$$

Remark 3. An important observation following from Theorem 1 is that for a finite subset $X \subset \mathbb{M}(n, \nu, \kappa)$, $\mathbf{C}_\kappa(X)$ is *singular* if $|X| > n + 1$.

Matrix Notation. Sometimes the $|\cdot|$ notation is used for determinants in order to save space. Recall that the *cofactor* matrix $\text{cof}(\mathbf{A}) = [c_{ij}]$ of a square matrix $\mathbf{A}$ is defined as $c_{ij} := (-1)^{i+j} |\mathbf{A}_j^i|$, where $|\mathbf{A}_j^i|$ is the minor obtained by deleting column i and row j . It satisfies

$$\mathbf{A} \text{cof}(\mathbf{A}) = \text{cof}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I}.$$

The following formula¹ for a symmetric matrix $\mathbf{A}$, will prove useful:

$$|\mathbf{A}| |\mathbf{A}_{ij}| = |\mathbf{A}_i| |\mathbf{A}_j| - |\mathbf{A}_j^i|^2. \quad (8)$$

Here $\mathbf{A}_i$ is the principal minor of $\mathbf{A}$ obtained by deleting row and column i , while $\mathbf{A}_{ij}$ is obtained by deleting rows and columns i, j , etc. Note that $\mathbf{A}_i = c_{ii}$ and that taking principal submatrices of $\mathbf{C}_\kappa(X)$ corresponds to “deleting” points of X .

2. Geometry of Simplices in $\mathbb{M}(n, \nu, \kappa)$

In this section we treat the geometry of “simplices” in $\mathbb{M}(n, \nu, \kappa)$, which we define below as a natural extension of the usual concept in spherical and hyperbolic geometry. The main result of this section is Theorem 3, which can be seen as a generalization of a classical formula for the Cayley–Menger determinant of a simplex in $\mathbb{R}^n$, see Remark 5 below.

Subspaces. First we need to discuss subspaces of $\mathbb{M} := \mathbb{M}(n, \nu, \kappa)$. We can define these in many ways, one of which is:

Definition 3. A subspace $S \subseteq \mathbb{M}$ is the intersection $S = \Pi(V) \cap \mathbb{M}$, where $V \subseteq \mathbb{R}_\nu^{n+1}$ is a linear subspace of the ambient space. A subspace $S = \Pi(V) \cap \mathbb{M}$ is called *nondegenerate* if V is a nondegenerate linear subspace.

¹ This formula is easily proven for a diagonalizable matrix, and hence holds in general by density of these. The formula also appears in [2] and in Exercise 9.14.23. in [1].

Here *nondegenerate* means for a linear subspace of $\mathbb{R}_v^{n+1}$ that the scalar product restricted to the subspace is nondegenerate, see [11]. We include here the trivial subspace $\{0\} \subset \mathbb{R}_v^{n+1}$ among the nondegenerate subspaces. That a linear subspace $\{0\} \neq V \subseteq \mathbb{R}_v^{n+1}$ is nondegenerate is equivalent to the criterion that *the Gram matrix of any basis for V is nonsingular*. This fact is important in the program of translating the linear geometry of the ambient space into intrinsic distance geometry of $\mathbb{M}$ via the $\mathbf{C}_\kappa$ -matrix, see Remark 1.

Remark 4. Note that the empty set $\emptyset$ and $\mathbb{M}$ in itself are subspaces of $\mathbb{M}$; these two subspaces are called *trivial*.

It is easily seen that the following holds:

Each connected component of a nontrivial and nondegenerate subspace $S \subset \mathbb{M}$ is a semi-Riemannian submanifold of the connected component of $\mathbb{M}$ it belongs to. S is in itself isometric to $\mathbb{M}(m, \mu, \kappa)$, for some $m < n$ and $\mu \leq \nu$.

A one-dimensional subspace is then either homeomorphic to $\mathbb{S}^1$ and as a set equal to the image of a single timelike or spacelike geodesic *or* disconnected with each connected component the image of a geodesic homeomorphic to $\mathbb{R}$. A zero-dimensional subspace always consists of two antipodal points.

Definition 4 (Generalized Simplex). A finite subset $X \subset \mathbb{M}(n, \nu, \kappa)$ is said to constitute the vertices of a nondegenerate simplex iff $\det(\mathbf{C}_\kappa(X)) \neq 0$.

From now on the word *simplex*, unless otherwise indicated, refers to a generalized nondegenerate simplex. A subset $Y \subseteq X$ is said to constitute the vertices of, or span, a *face* of codimension $|X \setminus Y|$. We often loosen the terminology and just call $Y \subseteq X$ a face. A face $Y \subseteq X$ with $\det(\mathbf{C}_\kappa(Y)) \neq 0$ is a simplex in its own right by the definition above.

For any subset $X \subseteq \mathbb{M} = \mathbb{M}(n, \nu, \kappa)$ there is a unique minimal subspace $S(X) \subseteq \mathbb{M}$ such that $X \subseteq S(X)$. In the extrinsic picture we have $S(X) = \pi(\text{span}(\tilde{X}))$. The condition that a finite subset $X \subset \mathbb{M}$ spans a simplex is equivalent to $S(X)$ being nondegenerate and that X is a *basis* for $S(X)$: *for any proper subset $\emptyset \neq Y \subset X$ we have $S(Y) \subset S(X)$* . We may and do always assume that for a simplex $X \subset \mathbb{M}(n, \nu, \kappa)$ we have $|X| = n + 1$, i.e. $\mathbb{M} = S(X)$.

Reflections and Altitudes. The reflection in (or symmetry of) a nondegenerate subspace $S \subseteq \mathbb{M}$ is the unique isometry $R_S \in \text{Isom}(\mathbb{M})$ such that

$$S = \{p \in \mathbb{M} \mid R_S(p) = p\} \quad \text{and} \quad R_S \circ R_S = id_{\mathbb{M}}.$$

If V is the unique nondegenerate linear subspace of $\mathbb{R}_v^{n+1}$ such that $S = \pi(V) \cap \mathbb{M}$, then R_S is the restriction to $\mathbb{M}$ of the reflection in V in the ambient space. This is given

explicitly as the linear map in $O(n+1, \nu)$ such that

$$v \mapsto \pi(v) - \pi^\perp(v). \quad (9)$$

Here $\pi: \mathbb{R}_\nu^{n+1} \rightarrow V$ is the orthogonal projection onto V and $\pi^\perp: \mathbb{R}_\nu^{n+1} \rightarrow V^\perp$ is the projection onto the orthogonal complement (see p. 50 of [11]). The reflection in $\emptyset \subset \mathbb{M}$ is the antipodal isometry $R_\emptyset$, while the reflection in $\mathbb{M}$ is simply the identity.

Definition 5. We define the altitude from $p \in \mathbb{M} = \mathbb{M}(n, \nu, \kappa)$ onto a nondegenerate subspace $S \subseteq \mathbb{M}$ as

$$d_\kappa(p, S) := \frac{1}{2}d_\kappa(p, R_S(p)) \in \frac{1}{2}\mathbb{C}_\kappa. \quad (10)$$

Example 1. For $\mathbb{M} = \mathbb{S}(n, 1)$ and $S = \{a, b\}$, where a and b are antipodal, R_S is the rotation in $SO(n+1)$ which fixes a, b and corresponds to the antipodal map on the equatorial $\mathbb{S}(n-1, 1)$. Hence the altitude $d_\kappa(p, S)$ is equal to the “latitude” of p , the distance to the closest “pole”, which is then $\pi/2$ on the equator and in $[0, \pi/2)$ on the two open hemispheres. Note that for p not on the equator, there is a well-defined map taking p to the closest point in S , this is the *orthogonal projection*.

Lemma 1. *The altitude $d_\kappa(p, S)$ onto a nondegenerate subspace $S \subseteq \mathbb{M}(n, \nu, \kappa)$ is in $\mathbb{C}_\kappa \cap \frac{1}{2}\mathbb{C}_\kappa$.*

Proof. Since R_S is an isometry, p and $R_S(p)$ have the same causal character. Thus $d_\kappa(p, R_S(p)) \in \mathbb{C}_\kappa \setminus (\pi/2\sqrt{\kappa} + \sqrt{-\kappa}\mathbb{R}^*)$ and $\frac{1}{2}d_\kappa(p, R_S(p)) \in \mathbb{C}_\kappa$. $\square$

The orthogonal projection of a point p onto a subspace $S \subseteq \mathbb{M}(n, \nu, \kappa)$ may not always exist, e.g. when p and $R_S(p)$ are antipodal there is no unique way to define the projection of p . Assuming this is not the case, the one-dimensional subspace spanned by p and $R_S(p)$ will either intersect S in two points² or will be disjoint from S . In the latter case we can think of the orthogonal projection as being a *point at infinity*.

Lemma 2. *Let $X \subset \mathbb{M}(n, \nu, \kappa)$ span a simplex. Then for every $Y \subset X$ and $p \in X \setminus Y$ such that the faces spanned by Y and $Y \cup \{p\}$ are nondegenerate, the altitude from p onto the subspace spanned by Y is nonvanishing: $d_\kappa(p, S(Y)) \neq 0$.*

Proof. Let V be the nondegenerate linear subspace of $\mathbb{R}_\nu^{n+1}$ such that $\Pi(V) \cap \mathbb{M} = S(Y)$, and let $R_V \in O(n+1, \nu)$ be the reflection in V , see (9). That $d_\kappa(p, S(Y)) = 0$ means that $\langle \tilde{p}, R_V(\tilde{p}) \rangle / r(\tilde{p})r(R_V(\tilde{p})) = 1$, which implies $\langle \tilde{p}, R_V(\tilde{p}) \rangle = r(\tilde{p})r(R_V(\tilde{p})) = r(\tilde{p})^2$. From this we deduce that $\langle \tilde{p} - R_V(\tilde{p}), \tilde{p} - R_V(\tilde{p}) \rangle = 0$, meaning that $\tilde{p} - R_V(\tilde{p})$ is lightlike. However, it is easily seen that this vector is in the orthogonal complement of V in the subspace $W = \text{span}(\tilde{p} \cup V)$, which is nondegenerate. This is impossible since the orthogonal complement in W of the codimension one subspace V is nondegenerate. $\square$

² We can take as a definition that the projection is the point having distance $d_\kappa(p, S)$ to p .

We have the following interesting trigonometric formula, which is one of the main results of this paper:

Theorem 3. *Let $X \subset \mathbb{M}(n, \nu, \kappa)$ span a simplex and assume that there is an ordering of the vertices $X = \{p_1, \dots, p_{n+1}\}$ such that the faces $X^i := \{p_1, \dots, p_i\}$, $i = 0 \dots n+1$, are nondegenerate. Then*

$$\det(\mathbf{C}_\kappa(X)) = \prod_{i=1}^{n+1} \mathbf{s}_\kappa(h_{<i})^2, \quad (11)$$

where $h_{<i}$ is the altitude from p_i onto $S(X^{i-1})$.

Proof. Let p_{n+2} be the reflection of p_{n+1} in the hypersurface $S(X_{n+1})$ spanned by $X_{n+1} := X \setminus \{p_{n+1}\} = X^n$. Then $d(p_{n+1}, p_{n+2}) = 2h_{n+1}$ and $d(p_i, p_{n+2}) = d(p_i, p_{n+1})$ for $i \neq n+1$, where h_{n+1} denotes the altitude from p_{n+1} onto $S(X_{n+1})$. Let $\hat{X}$ be $X \cup \{p_{n+2}\}$. Since $|\hat{X}| = n+2$, and $\hat{X} \subset \mathbb{M}(n, \nu, \kappa)$, the $\mathbf{C}_\kappa(\hat{X})$ -matrix must be singular. Now the two last columns and the two last rows of $\mathbf{C}_\kappa(\hat{X})$ are identical, except for inside the principal 2×2 submatrix which involves p_{n+1} and p_{n+2} :

$$\begin{pmatrix} 1 & \mathbf{c}_\kappa(2h_{n+1}) \\ \mathbf{c}_\kappa(2h_{n+1}) & 1 \end{pmatrix}. \quad (12)$$

We see that $|\mathbf{C}_\kappa(\hat{X})_{n+1}| = |\mathbf{C}_\kappa(\hat{X})_{n+2}| = |\mathbf{C}_\kappa(X)|$ and $|\mathbf{C}_\kappa(\hat{X})_{n+1}^{n+2}| = |\mathbf{C}_\kappa(X)| + (\mathbf{c}_\kappa(2h_{n+1}) - 1)|\mathbf{C}_\kappa(X_{n+1})|$ (simply expand the minor). Hence applying formula (8), we get (since $|\mathbf{C}_\kappa(\hat{X})| = 0$)

$$|\mathbf{C}_\kappa(X)|^2 = (|\mathbf{C}_\kappa(X)| + (\mathbf{c}_\kappa(2h_{n+1}) - 1)|\mathbf{C}_\kappa(X_{n+1})|)^2.$$

We have $(\mathbf{c}_\kappa(2h_{n+1}) - 1)|\mathbf{C}_\kappa(X_{n+1})| \neq 0$ since X_{n+1} spans a nondegenerate simplex and $\mathbf{c}_\kappa(2h_{n+1}) \neq 1$ by the previous lemma. Using this we obtain, from the equation above,

$$\frac{1}{2}(1 - \mathbf{c}_\kappa(2h_{n+1}))|\mathbf{C}_\kappa(X_{n+1})| = |\mathbf{C}_\kappa(X)|.$$

We have the usual formula

$$1 - \mathbf{c}_\kappa(2h_{n+1}) = 2\mathbf{s}_\kappa(h_{n+1})^2, \quad (13)$$

valid for complex numbers as well (by analytic continuation). Inserting this we obtain

$$\mathbf{s}_\kappa(h_{n+1})^2 |\mathbf{C}_\kappa(X_{n+1})| = |\mathbf{C}_\kappa(X)|. \quad (14)$$

The result follows by induction, since $|\mathbf{C}_\kappa(\{p_1\})| = 1 = \mathbf{s}_\kappa(h_1)$, where $h_1 = \pi/2\sqrt{\kappa}$ is the altitude of p_1 onto $X_0 = \emptyset$. $\square$

Remark 5. It is interesting to note that since the left-hand side of the formula is independent of the ordering of the points (as long as the nondegeneracy criterion is fulfilled), this is also true for the right-hand side.

Also note that everything is “antipodally invariant”: mapping a point p_i to its antipode $R_\theta(p_i)$ will alter the altitude from p_i as $h_i \mapsto \pi/\sqrt{\kappa} - h_i$, but this does not change $\mathbf{s}_\kappa(h_i)$, hence does not change the determinant; this is also seen directly since mapping p_i to its antipode corresponds to changing a sign in row and column i of $\mathbf{C}_\kappa(X)$.

We have the following formula for the volume V of a simplex spanned by $n+1$ points $X \subset \mathbb{R}^n$, see p. 98 of [2]:

$$V^2 = \frac{1}{(n!)^2} \prod_{i=2}^{n+1} h_{<i}^2 = \frac{(-1)^{n+1}}{2^n (n!)^2} \det(\mathbf{CM}(X)), \quad (15)$$

where $\mathbf{CM}(X)$ is the Cayley–Menger matrix of X . It is possible to prove this by the methods of Theorem 3, including the case $\kappa = 0$ in the general framework as is done in [9]. However, it can also be derived from (11) by a power series expansion in κ . The lowest-order term in an expansion of $\det(\mathbf{C}_\kappa(X))$ is proportional to $\det(\mathbf{CM}(X))$.

As a simple application of the theorem above, we have the following result which we choose to call *Heron’s formula* because of its relation to a classical formula:

Proposition 3 (Heron’s Formula). *Let $X = \{p_1, p_2, p_3\}$ be a $\mathbb{C}_\kappa$ -distance space, with distances $d(p_1, p_2) = a$, $d(p_2, p_3) = b$, $d(p_3, p_1) = c$ and put $s = \frac{1}{2}(a + b + c)$. Then*

$$\det(\mathbf{C}_\kappa(X)) = 4 \mathbf{s}_\kappa(s) \mathbf{s}_\kappa(s - a) \mathbf{s}_\kappa(s - b) \mathbf{s}_\kappa(s - c). \quad (16)$$

Hence if $X = \{p_1, p_2, p_3\} \subset \mathbb{M}(n, v, \kappa)$ and $\mathbf{s}_\kappa(a) \neq 0$, then

$$\mathbf{s}_\kappa(a)^2 \mathbf{s}_\kappa(h)^2 = 4 \mathbf{s}_\kappa(s) \mathbf{s}_\kappa(s - a) \mathbf{s}_\kappa(s - b) \mathbf{s}_\kappa(s - c), \quad (17)$$

where h is the altitude from p_3 .

Proof. For (16) simply expand the determinant and apply the usual identities for $\mathbf{c}_\kappa$ and $\mathbf{s}_\kappa$. Actually we do not need X to be a $\mathbb{C}_\kappa$ -space here, the result holds for symmetric unidiagonal complex matrices.

The second statement is then just an application of Theorem 3, noting that $\mathbf{s}_\kappa(a) \neq 0$ means that the face spanned by p_1, p_2 is nondegenerate. $\square$

2.1. Half-Spaces, Angles and Duality

Angles can be defined in various ways in semi-Riemannian geometry, see, e.g. [17]. Here we shall as for distances define angles with complex values.

The orthogonal complement duality, $V \mapsto V^\perp \subseteq \mathbb{R}_v^{n+1}$, of linear subspaces of the ambient space translates into a duality of subspaces of $\mathbb{M} = \mathbb{M}(n, v, \kappa)$.

Definition 6. If $S \subseteq \mathbb{M}$ is a subspace, the dual subspace S^* is the set

$$S^* := \left\{ p \in \mathbb{M} \mid d_\kappa(p, q) = \frac{\pi}{2\sqrt{\kappa}}, \forall q \in S \right\}. \quad (18)$$

For a nondegenerate codimension 1 subspace, $S \subset \mathbb{M}$, the dual subspace consists of two antipodal points $S^* = \{p, q\}$. S divides $\mathbb{M}$ into two “components” or *half-spaces* which are interchanged by the reflection R_S . We use these two dual points to distinguish the two half-spaces having S as boundary.

Definition 7. Let $S \subset \mathbb{M}$ be a nondegenerate codimension 1 subspace and let p be one of the two points such that $d_\kappa(p, S) = \pi/2\sqrt{\kappa}$, then the half-space $H \subset \mathbb{M}$ containing p is defined as the set

$$H := \{q \in \mathbb{M} \mid d_\kappa(p, q) \in \mathbb{C}_\kappa \cap \frac{1}{2}\mathbb{C}_\kappa\}. \quad (19)$$

Assume that $X = \{p_1, \dots, p_{n+1}\} \subset \mathbb{M}(n, \nu, \kappa)$ spans a simplex with nondegenerate codimension 1 faces. Then for each face $X_i := X \setminus \{p_i\}$ define the *interior* dual point $p_i^* \in \mathbb{M}$ to be the point dual to $S(X_i)$ such that p_i is in the half-space determined by p_i^* . Then we define the *exterior dihedral angle*, along which two codimension 1 faces X_i and X_j intersect, as:

Definition 8. The exterior dihedral angle along a face $X_{ij} = X_i \cap X_j$ of a simplex $X \subset \mathbb{M}(n, \nu, \kappa)$ is the element $\theta_{ij} \in \mathbb{C}_1$ such that

$$\theta_{ij} := \sqrt{\kappa} d_\kappa(p_i^*, p_j^*), \quad (20)$$

where p_i^*, p_j^* are the interior dual points to $S(X_i)$ and $S(X_j)$, respectively. The *interior* dihedral angle is defined as $\pi - \theta_{ij}$.

Note that with this definition $\theta_{ii} = 0$.

Proposition 4. Let $X = \{p_1, \dots, p_{n+1}\}$ be points of $\mathbb{M}(n, \nu, \kappa)$ that span a simplex. Let $[c_{ij}]$ be the elements of the cofactor matrix $\text{cof}(\mathbf{C}_\kappa(X))$. The exterior dihedral angles θ_{ij} satisfy

$$\cos(\theta_{ij}) = \frac{c_{ij}}{\sqrt{c_{ii}}\sqrt{c_{jj}}}. \quad (21)$$

Proof. This proof uses the extrinsic picture. Let $\tilde{X} = \{\tilde{p}_1, \dots, \tilde{p}_{n+1}\} \subset \mathbb{R}_\nu^{n+1}$ denote representatives of the points in X and let $\mathbf{X}$ denote the $(n+1) \times (n+1)$ matrix having $\tilde{X}$ as columns. Then $\mathbf{C}_\kappa(X) = \Lambda \mathbf{X}' \mathbf{I}_\nu \mathbf{X} \Lambda$, where Λ is the diagonal matrix with entries $\lambda_{ii} = r(\tilde{p}_i)^{-1}$, see Remark 1. Let $\mathbf{X}^*$ be the matrix having the dual basis of $\tilde{X}$ as columns, normalized such that $\mathbf{X}' \mathbf{I}_\nu \mathbf{X}^* = \Lambda^{-2}$. Then the columns of $\mathbf{X}^*$, denote them by $\tilde{p}_i^*$, are representatives of the interior dual points of X ; $\tilde{p}_i^*$ has the same causal character as $\tilde{p}_i$, being orthogonal to the same hyperplane, and the scalar product $\langle \tilde{p}_i, \tilde{p}_i^* \rangle$ has the same sign as $r^2(\tilde{p}_i)$. This translates into that $d_\kappa(p_i, \Pi(\tilde{p}_i^*)) \in \mathbb{C}_\kappa \cap \frac{1}{2}\mathbb{C}_\kappa$.

We deduce that $\mathbf{X}^* = \mathbf{I}_\nu \mathbf{X}^{-1,t} \Lambda^{-2}$ and hence that the Gram matrix of $\mathbf{X}^*$ is $\Lambda^{-2} \mathbf{X}^{-1} \mathbf{I}_\nu \mathbf{X}^{-1,t} \Lambda^{-2} = \Lambda^{-1} \mathbf{C}_\kappa(X)^{-1} \Lambda^{-1}$. Thus the normalized Gram matrix of the interior dual points equals the normalization of $\mathbf{C}_\kappa(X)^{-1}$ which again equals the normalization of $\text{cof}(\mathbf{C}_\kappa(X))$, see Remark 1. Hence $c_{ij}/(\sqrt{c_{ii}}\sqrt{c_{jj}}) = \mathbf{c}_\kappa(d_\kappa(p_i^*, p_j^*)) = \cos(\theta_{ij})$. $\square$

Fig. 1. A triangle.

In this constant curvature setting it is useful to define angles intrinsically via distances and the *cosine relation*. A triangle in $\mathbb{M}(n, \nu, \kappa)$ consists of three points p_1 , p_2 and p_3 , which do not have to span a simplex (Fig. 1). Assuming that the two one-dimensional faces containing p_1 are nondegenerate, we define the *angle* at p_1 to be the element $\theta_{23} \in \mathbb{C}_1$ such that

$$\cos(\theta_{23}) = \frac{\mathbf{c}_\kappa(d_{23}) - \mathbf{c}_\kappa(d_{12})\mathbf{c}_\kappa(d_{13})}{\mathbf{s}_\kappa(d_{12})\mathbf{s}_\kappa(d_{13})}. \quad (22)$$

To connect Proposition 4 with the more intrinsic definition of angles given above, we have:

Proposition 5. *Let $X = \{p_1, \dots, p_{n+1}\}$ be points of $\mathbb{M}(n, \nu, \kappa)$ that span a simplex with nondegenerate codimension 1 faces and for $i \neq j$ suppose that the orthogonal projection of p_i onto $S(X_{ij})$ exists; denote this by $\hat{p}_{ij}$ and let $\hat{p}_i$ denote the reflection of p_i in $S(X_i)$. Then the angle α at the vertex $\hat{p}_{ij}$ of the triangle $p_i, \hat{p}_i, \hat{p}_{ij}$ equals either twice the exterior dihedral angle θ_{ij} or twice the interior dihedral angle $\pi - \theta_{ij}$.*

Proof. From the cosine relation we get that

$$\cos(\alpha) = \frac{\mathbf{c}_\kappa(2h_i) - \mathbf{c}_\kappa(h_{ij})^2}{\mathbf{s}_\kappa(h_{ij})^2},$$

where $2h_i$ is the distance from p_i to $\hat{p}_i$ and h_{ij} is the distance from p_i to $\hat{p}_{ij}$. Applying the usual trigonometric identities, e.g. (13), we can derive

$$\cos\left(\frac{\alpha}{2}\right)^2 = \frac{\mathbf{s}_\kappa(h_{ij})^2 - \mathbf{s}_\kappa(h_i)^2}{\mathbf{s}_\kappa(h_{ij})^2}.$$

From (14) we see $\mathbf{s}_\kappa(h_i)^2 = |\mathbf{C}_\kappa(X)|/|\mathbf{C}_\kappa(X_i)|$ and similarly $\mathbf{s}_\kappa(h_{ij})^2 = |\mathbf{C}_\kappa(X_j)|/|\mathbf{C}_\kappa(X_{ij})|$. Inserting and multiplying through we obtain

$$\begin{aligned} \cos\left(\frac{\alpha}{2}\right)^2 &= \frac{|\mathbf{C}_\kappa(X_i)||\mathbf{C}_\kappa(X_j)| - |\mathbf{C}_\kappa(X)||\mathbf{C}_\kappa(X_{ij})|}{|\mathbf{C}_\kappa(X_i)||\mathbf{C}_\kappa(X_j)|} \\ &= \frac{|\mathbf{C}_\kappa(X)_j^i|^2}{|\mathbf{C}_\kappa(X_i)||\mathbf{C}_\kappa(X_j)|} = \cos(\theta_{ij})^2, \end{aligned}$$

where the second equality follows from (8) and the last follows from Proposition 4. $\square$

Definition 9 (Duality). For $X = \{p_1, \dots, p_{n+1}\} \subset \mathbb{M}(n, \nu, \kappa)$ spanning a simplex with nondegenerate codimension 1 faces, the *dual simplex* of X is the simplex having $X^* = \{p_1^*, \dots, p_{n+1}^*\}$ as vertices.

Proposition 6. *If $X = \{p_1, \dots, p_{n+1}\} \subset \mathbb{M}(n, v, \kappa)$ spans a simplex with nondegenerate codimension 1 faces, then $X^* \subset \mathbb{M}(n, v, \kappa)$ also spans a simplex with nondegenerate codimension 1 faces and $X^{**} \underset{\text{isom}}{\cong} X$.*

Proof. From Proposition 4 we have

$$\text{cof}(\mathbf{C}_\kappa(X^*)) = \text{cof}(\Lambda \mathbf{C}_\kappa(X)^{-1} \Lambda) = \det(\Lambda)^2 \det(\mathbf{C}_\kappa(X))^{-1} \Lambda^{-1} \mathbf{C}_\kappa(X) \Lambda^{-1}, \quad (23)$$

which has nonzero diagonal entries, hence X^* has nondegenerate codimension 1 faces. Here Λ is the normalization matrix of $\mathbf{C}_\kappa(X)^{-1}$.

From (23) it is also clear that the normalized cofactor matrix of $\mathbf{C}_\kappa(X^*)$, which equals $\mathbf{C}_\kappa(X^{**})$ by Proposition 4, is $\mathbf{C}_\kappa(X)$. Thus X and X^{**} are isometric. $\square$

This is another formulation of a duality studied in a different setting by Rivin and Hodgson in [12], based on Rivin's Ph.D. thesis. Later Schlenker worked with a formulation of the duality in [15] and [16]. The version that resembles our presentation most is the one in the latter reference.

The dual of a spherical simplex is a spherical simplex, while the dual of a hyperbolic simplex is a simplex in $H_{n-1}^n(1/\sqrt{|\kappa|}) \subset \mathbb{M}(n, 1, \kappa)$, which is antiisometric to the *de Sitter sphere* $S_1^n(1/\sqrt{|\kappa|})$. Modulo a factor of $\sqrt{\kappa}$ the duality interchanges dihedral angles and edge lengths.

An interesting aspect about the matrix formulation in Proposition 4 is that *it makes sense for a completely abstract $\mathbb{C}_\kappa$ -space, without reference to any ambient space*. Perhaps it could be useful in other contexts.

2.2. Various Examples

The connection between the algebra of the $\mathbf{C}_\kappa(X)$ -matrix and the geometry of the simplex spanned by X provides us with a geometric interpretation of various formulas from linear algebra.

Example 2. For example, expanding $\det(\mathbf{C}_\kappa(X))$ after row i , we get

$$\det(\mathbf{C}_\kappa(X)) = c_{ii} + \sum_{j \neq i} c_{ij} \mathbf{c}_\kappa(d_{ij}),$$

where $[c_{ij}] = \text{cof}(\mathbf{C}_\kappa(X))$. Using Proposition 4 this becomes

$$\det(\mathbf{C}_\kappa(X)) = c_{ii} + \sum_{j \neq i} \cos(\theta_{ij}) \sqrt{c_{ii}} \sqrt{c_{jj}} \mathbf{c}_\kappa(d_{ij}).$$

Dividing by $c_{ii} = \det(\mathbf{C}_\kappa(X_i))$ and using (14), we arrive at

$$\mathbf{s}_\kappa(h_i)^2 = 1 + \sum_{j \neq i} \cos(\theta_{ij}) \frac{\mathbf{s}_\kappa(h_i)}{\mathbf{s}_\kappa(h_j)} \mathbf{c}_\kappa(d_{ij}), \quad (24)$$

where h_i is the altitude from p_i onto $S(X_i)$.

Example 3. We have from Proposition 4 $\mathbf{C}_\kappa(X^*) = \Lambda \text{cof}(\mathbf{C}_\kappa(X)) \Lambda$, where Λ is the normalization matrix of $\text{cof}(\mathbf{C}_\kappa(X))$. From this we deduce

$$\det(\mathbf{C}_\kappa(X^*)) = \det(\mathbf{C}_\kappa(X))^n \prod_{i=1}^{n+1} \det(\mathbf{C}_\kappa(X_i))^{-1}.$$

However, then using Theorem 3 and (14), we see

$$\det(\mathbf{C}_\kappa(X^*)) = \prod_{i=1}^{n+1} \mathbf{s}_\kappa(h_{<i}^*)^2 = \det(\mathbf{C}_\kappa(X)^{-1}) \prod_{j=1}^{n+1} \mathbf{s}_\kappa(h_j)^2,$$

where h_j is the altitude from p_j onto $S(X_j)$. It is possible to produce many relations like this.

The determinant of $\mathbf{C}_\kappa(X)$ is just one of the invariants appearing in the characteristic polynomial:

$$\chi(\lambda) = \det(\mathbf{C}_\kappa(X) - \lambda \mathbf{I}) = \sum_{i=0}^{n+1} a_i \lambda^i. \quad (25)$$

Here

$$a_i = (-1)^i \Sigma_{|X|-i}(\lambda_1, \dots, \lambda_n) = (-1)^i \sum_{|I|=i} |\mathbf{C}_\kappa(X)_I|,$$

where $\Sigma_{|X|-i}$ is the $(|X| - i)$ th elementary symmetric polynomial, which is evaluated in the eigenvalues $\lambda_1, \dots, \lambda_n$; the last sum is over all principal minors obtained by deleting i rows and columns. Written differently we have

$$a_i = (-1)^i \sum_{|X \setminus Y|=i} \det(\mathbf{C}_\kappa(Y)), \quad (26)$$

a sum over all faces $Y \subseteq X$ of codimension i . Then one has, for example,

$$-\frac{a_1}{a_0} = \text{trace}(\mathbf{C}_\kappa(X)^{-1}) = \sum_{i=1}^{n+1} \frac{1}{\lambda_i} = \sum_{i=1}^{n+1} \frac{1}{\mathbf{s}_\kappa(h_i)^2}, \quad (27)$$

where the λ_i 's are the eigenvalues of $\mathbf{C}_\kappa(X)$. The last equality follows using (14) on $-a_1/a_0$.

Example 4 (Regular Simplices). A regular simplex $X \subset \mathbb{M}(n, v, \kappa)$ is a simplex where all edge lengths d_{ij} are equal to the same constant $l \in \mathbb{C}_\kappa$. The $\mathbf{C}_\kappa(X)$ matrix of such a simplex has the form

$$\begin{pmatrix} 1 & \mathbf{c}_\kappa(l) & \mathbf{c}_\kappa(l) & \dots \\ \mathbf{c}_\kappa(l) & 1 & \mathbf{c}_\kappa(l) & \dots \\ \mathbf{c}_\kappa(l) & \mathbf{c}_\kappa(l) & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (28)$$

Expanding this $(n + 1) \times (n + 1)$ determinant, we obtain

$$\det(\mathbf{C}_\kappa(X)) = (1 - \mathbf{c}_\kappa(l))^n (n\mathbf{c}_\kappa(l) + 1). \quad (29)$$

It can also be checked that this factorization of the determinant corresponds to the eigenvalues of $\mathbf{C}_\kappa(X)$. We see that the matrix is singular when $\mathbf{c}_\kappa(l) = -1/n$. This corresponds to the fact that we can embed a metric space with all distances equal to $l \in \mathbb{R}_+^*$ as a regular simplex in $\mathbb{S}(n, \kappa)$ iff $l < \mathbf{c}_\kappa^{-1}(-1/n)$, and if $l = \mathbf{c}_\kappa^{-1}(-1/n)$ we can embed the space with a *dimension drop* of 1 in $\mathbb{S}(n - 1, \kappa)$. When $-1 \leq \mathbf{c}_\kappa(l) < -1/n$, $\mathbf{C}_\kappa(X)$ has exactly one negative eigenvalue, and X is realizable in $\mathbb{M}(n, 1, \kappa)$.

Using (14), we obtain a formula for the altitude from a vertex of X onto the opposite face:

$$\mathbf{s}_\kappa(h)^2 = (1 - \mathbf{c}_\kappa(l)) \frac{n\mathbf{c}_\kappa(l) + 1}{(n - 1)\mathbf{c}_\kappa(l) + 1}. \quad (30)$$

Note that $\mathbf{s}_\kappa(h)^2$ is negative when h is real and $\kappa < 0$ by the definition of $\mathbf{s}_\kappa$. See a plot of the altitude as a function of curvature in Fig. 2.

From (24) we can find a relation between the dihedral angles θ , altitudes h and edge lengths l of a regular simplex $X \subset \mathbb{M}(n, v, \kappa)$:

$$\mathbf{s}_\kappa(h)^2 = 1 + n \cos(\theta) \mathbf{c}_\kappa(l). \quad (31)$$

Fig. 2. A plot of the absolute value of the altitude in a regular three-dimensional simplex with edge lengths 1. Note that the simplex collapses for $\kappa = \arccos(-\frac{1}{3})^2$, after this the altitude becomes imaginary.

The dual simplex X^* , which by symmetry is also regular, has edge lengths $l^* = \theta/\sqrt{\kappa}$, hence we obtain the symmetric (duality invariant) expression:

$$\mathbf{s}_\kappa(h)^2 = 1 + n\mathbf{c}_\kappa(l^*)\mathbf{c}_\kappa(l) = \mathbf{s}_\kappa(h^*)^2. \quad (32)$$

We see that the regular simplex in $\mathbb{S}(n, \kappa)$ with $l = \pi/2\sqrt{\kappa}$ is *self-dual*.

3. Leaf Spaces Are Hyperbolic

In this section we examine which metric spaces are embeddable in $\mathbb{H}(m, \kappa)$ in the limit where $\kappa \rightarrow -\infty$. In order to economize we introduce the following terminology:

Definition 10. A metric space X is said to satisfy condition $\mathcal{H}$ if there exists an integer m and a $\kappa_0 < 0$ such that $X \xrightarrow{\text{isom}} \mathbb{H}(m, \kappa)$ for all $\kappa < \kappa_0$.

For reasons which should become clear later (see, e.g. Corollary 3 below) it is interesting to consider yet another matrix besides $\mathbf{C}_\kappa(X)$. For a finite metric space X with distance matrix $\mathbf{D} = [d_{ij}]$, and for $t \in \mathbb{R}$, we use the notation

$$\mathcal{E}_t(X) := [\exp(td_{ij})].$$

For $t = 1$, we just write $\mathcal{E}(X)$. For $\kappa < 0$ we have

$$\mathbf{C}_\kappa(X) = [\cosh(\sqrt{|\kappa|}d_{ij})] = \frac{1}{2}(\mathcal{E}_t(X) + \mathcal{E}_{-t}(X)),$$

where $t = \sqrt{|\kappa|}$.

Remark 6. If a metric space is of *negative type*, then $\mathcal{E}_{-t}(X)$ is positive semidefinite for $t \geq 0$, see [3]. It is known that subsets of $\mathbb{H}(m, \kappa)$ are of negative type, see [8], hence $\mathcal{E}_{-t}(X)$ will be positive semidefinite for such spaces, e.g. if X satisfies condition $\mathcal{H}$.

Proposition 7. If $X \xrightarrow{\text{isom}} \mathbb{H}(m, \kappa)$, then $\mathcal{E}_t(X)$ has exactly one positive eigenvalue for $t = \sqrt{|\kappa|}$.

Proof. We have $\mathbf{C}_\kappa(X) = \frac{1}{2}(\mathcal{E}_t(X) + \mathcal{E}_{-t}(X))$, where $t = \sqrt{|\kappa|}$. Since $X \xrightarrow{\text{isom}} \mathbb{H}(m, \kappa)$, $\mathbf{C}_\kappa(X)$ has exactly one positive eigenvalue. However, the matrix $\mathcal{E}_{-t}(X)$ is positive semidefinite, so $\mathcal{E}_t(X)$ can have at most (and hence exactly) one positive eigenvalue. $\square$

Recall that if $G = (V, E, w)$ is a weighted, connected graph, with V the set of vertices, E the set of edges and $w: E \rightarrow \mathbb{R}_+$ the weight function, then we get a natural induced metric on V by setting $d(v_i, v_j) := \inf\{\sum_{e \in \gamma} w(e)\}$, where the infimum is taken over all paths γ joining v_i and v_j . A *tree* is a connected graph which contains no cycles.

A metric space which is isometric to a subset of a weighted tree is called *tree realizable*. It is well known, and easy to prove, that if X is tree realizable, then there is a unique *minimal* weighted tree $T = (V, E, w)$ that realizes X . We always assume that the trees

in discussion are minimal. The points corresponding to vertices v with $\deg(v) > 1$ are called *branch points*, while a point corresponding to a vertex of degree 1 is called a *leaf*. More specific, since the tree T is unique, it makes sense to talk about the degree $\deg(v)$, for $v \in X$.

Definition 11. A leaf space is a finite metric space that can be realized as the set of degree 1 vertices of a weighted tree $T = (V, E, w)$.

The result below follows from Corollaries 1 and 2, and is the main synopsis of the results in this section.

Theorem 4 (Main Theorem). *A finite metric space satisfies condition $\mathcal{H}$ if and only if it is a leaf space or a subset of the line.*

We shall need the following, see, e.g. [6] (and the remark below).

Theorem 5 (The Four-Point Condition/Zero-Hyperbolicity). *A finite metric space X , with $|X| \geq 4$, is isometric to a subspace of a weighted tree if and only if all four-point subsets $\{p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}\} \subseteq X$ satisfy the following condition: among the three sums*

$$s_1 = d_{i_1 i_2} + d_{i_3 i_4}, \quad s_2 = d_{i_1 i_3} + d_{i_2 i_4}, \quad s_3 = d_{i_1 i_4} + d_{i_2 i_3}, \quad (33)$$

two are equal and not smaller than the third one.

Remark 7. The four-point condition is in the language of Gromov's δ -hyperbolic spaces equivalent to zero-hyperbolicity, see [6]. A metric space which satisfies condition $\mathcal{H}$ satisfies the weaker condition of being an *asymptotic subcone* of hyperbolic space of curvature -1 . It is easy to show that an asymptotic subcone of a δ -hyperbolic space must be zero-hyperbolic, and hence a metric space satisfying condition $\mathcal{H}$ must be either a leaf space or a subset of the line (there can be no “branching geodesics”). However we treat the problem in a self-contained way, using the Gram matrix machinery, without reference to δ -hyperbolic spaces.

Proposition 8. *A metric space on four points that satisfies condition $\mathcal{H}$ is a leaf space or a subset of the line.*

Proof. Simply expanding the determinant of $\mathcal{E}_t(X)$, where $X = \{p_1, p_2, p_3, p_4\}$, and collecting possible candidates for a leading-order exponent, reveals

$$\begin{aligned} \det(\mathcal{E}_t(X)) = & -2 \exp(t(d_{12} + d_{23} + d_{34} + d_{41})) \\ & - 2 \exp(t(d_{12} + d_{24} + d_{43} + d_{41})) - 2 \exp(t(d_{13} + d_{32} + d_{24} + d_{41})) \\ & + \exp(2t(d_{12} + d_{34})) + \exp(2t(d_{13} + d_{24})) + \exp(2t(d_{14} + d_{23})) \\ & + \text{lower-order terms.} \end{aligned} \quad (34)$$

Here the first three terms correspond to the six 4-cycles in Σ_4 ; these have negative sign. The last three terms correspond to elements composed of two 2-cycles, which gives

a positive sign. It is easily seen that if one of the three sums s_1, s_2, s_3 as defined in Theorem 5 is strictly larger than the two others, then the leading-order exponent in (34) occurs in one of the last three terms. Hence the sign of the determinant would be positive for t large, and X could not be embeddable in $\mathbb{H}(m, \kappa)$, where $\kappa = -t^2$, by Proposition 7.

Hence s_1, s_2, s_3 do not have a strict maximum, so that X is tree realizable by Theorem 5. Since $\mathbb{H}(m, \kappa)$ does not have “branching geodesics”, it is clear that X must either be a leaf space or a subset of the line (all branch points have degree 2). $\square$

It is geometrically clear that any metric space with $|X| < 4$ is realizable in any $\mathbb{H}(n, \kappa)$ for $n \geq 2$. Also it is easily seen that any such X is realizable as a leaf space. If $|X| \geq 4$ and X satisfies condition $\mathcal{H}$, then the same is true for all four-point subsets, and hence X satisfies the four-point condition, so as a corollary to the proposition we get (with the same argument that $\mathbb{H}(m, \kappa)$ does not have “branching geodesics”)

Corollary 1. *Let X be a finite metric space. If X satisfies condition $\mathcal{H}$, then X is a leaf space or a subset of the line.*

We turn our attention to the opposite direction, are all leaf spaces embeddable in $\mathbb{H}(m, \kappa)$ when the curvature is negative enough? In fact we show a little more:

Theorem 6. *Let $X = \{v_1, \dots, v_n\}$ be a tree realizable metric space with b branch points $\{v_1, \dots, v_b\}$. Then the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of $\mathbf{C}_\kappa(X)$ converge in $\mathbb{R} \cup \{\pm\infty\}$ for $\kappa \rightarrow -\infty$ and can be ordered such that*

$$\lim_{\kappa \rightarrow -\infty} \lambda_n = +\infty \quad \text{and} \quad \lim_{\kappa \rightarrow -\infty} \lambda_i = -\infty \quad \text{for } i = b+1..n-1, \quad (35)$$

$$\lim_{\kappa \rightarrow -\infty} \lambda_i = \frac{\deg(v_i) - 2}{2(\deg(v_i) - 1)} \quad \text{for } i = 1 \dots b. \quad (36)$$

We see that if $T = (V, E, w)$ is a weighted tree, it is clearly possible to construct a procedure using Theorem 6 to recover the combinatorial structure of T from the limits of eigenvalues of the $\mathbf{C}_\kappa$ -matrix of T and certain subspaces of T .

Proposition 9. *Let $T = (V, E, w)$ be a weighted tree. From the limits, as $\kappa \rightarrow -\infty$, of eigenvalues of $\mathbf{C}_\kappa(T)$ and its principal submatrices the combinatorial structure of T can be recovered.*

Using Proposition 10 below, the weight function can also be recovered.

Since a leaf space has no branch points, the $\mathbf{C}_\kappa$ -matrix will for large negative κ have exactly one positive eigenvalue, so by Theorem 1 we get:

Corollary 2. *All leaf spaces satisfy condition $\mathcal{H}$.*

This concludes the proof of Theorem 4 modulo the proof of Theorem 6, which is given below.

Proof of Theorem 6. We work first with the $\mathcal{E}$ -matrix and deal later with the $\mathbf{C}_k$ -matrix. Clearly, there is a unique expansion

$$\det(\mathcal{E}_t(X)) = \sum_{k \in I} c_k \exp(\omega_k t) \quad (37)$$

such that $\omega_k = \omega_l$ iff $k = l$, where I is a finite index set. The c_k 's are integers determined by the combinatorics and each ω_k is a linear combination of the edge lengths in the tree representing X . $\omega_{\max}$ is used to denote the leading-order exponent in the expansion $\omega_{\max} := \max_{k \in I} \omega_k$, and $c_{\max}$ is the corresponding coefficient.

Proposition 10. *Let X be a tree realizable metric space and let $T = (V, E, w)$ be the weighted tree that represents X . Then*

$$\omega_{\max}(X) = 2L(X) \quad \text{and} \quad c_{\max}(X) = (-1)^{|X|+1} \prod_{v \in V \setminus X} (\deg(v) - 1), \quad (38)$$

where $L(X) := \sum_{e \in E} w(e)$ is the total weight. If X is a full tree, i.e. $X = V$, then

$$\det(\mathcal{E}(X)) = \prod_{e \in E} (1 - \exp(2w(e))). \quad (39)$$

Proof. We describe two operations from which any tree realizable metric space can be constructed. Let $T = (V, E, w)$ be a weighted tree and assume that all subsets $X \subseteq V$ satisfy Proposition 10.

Operation A. If $v \in X \subseteq V$ is a leaf, consider the new tree $\hat{T}$, where $k \geq 1$ leaves $\{l_1, \dots, l_k\}$ have been attached to v by edges of weight w_i , $1 \leq i \leq k$. Let $\hat{X}$ consist of $X \cup \{l_1, \dots, l_k\}$. Clearly, $d(l_i, x) = d(v, x) + w_i$ for any $x \in \hat{X} \setminus \{l_i\}$. Order the points of $\hat{X}$ such that l_i is the i th point and v is point number $k+1$ and consider the matrix $\mathcal{E}(\hat{X})$. Now multiply row $k+1$ by $\exp(w_i)$ and subtract the result from row i for $i = 1 \dots k$. This will produce a matrix of the form

$$\begin{pmatrix} \Lambda & O \\ A & \mathcal{E}(X) \end{pmatrix},$$

where Λ is a $k \times k$ diagonal matrix with diagonal entries $\lambda_{i,i} = 1 - \exp(2w_i)$ and O is a $k \times |X|$ matrix of zeros. Hence

$$\det(\mathcal{E}(\hat{X})) = \prod_{i=1}^k (1 - \exp(2w_i)) \det \mathcal{E}(X). \quad (40)$$

We see that $\omega_{\max}(\hat{X}) = 2 \sum_{i=1}^k w_i + \omega_{\max}(X)$ and $c_{\max}(\hat{X}) = (-1)^k c_{\max}(X)$. If X satisfied (38) above, this will then also be true for $\hat{X}$. Hence this operation of *adding branches to a leaf* preserves (38). It is clear that a full tree can be built using only operation A, so (39) follows by induction from (40).

Operation B. Now consider again a leaf $v \in X \subseteq V$ and attach $k \geq 2$ new leaves, but let $\hat{X}$ consist of $(X \setminus \{v\}) \cup \{l_1, \dots, l_k\}$, so that the branch point v is not included in $\hat{X}$. Order the points so that l_i is the i th point of $\hat{X}$. Consider the two first leaves l_1, l_2 , and let x be any other point of $\hat{X}$, then $d(l_1, x) = d(l_2, x) + w_1 - w_2$. Hence multiplying row 2 of $\mathcal{E}(\hat{X})$ by $\exp(w_1 - w_2)$ and subtracting the result from row 1 will produce zeros in the first row beyond the second column. Do the same thing for the first and second column. The following happens to the principal submatrix involving l_1 and l_2 :

$$\begin{pmatrix} 1 & e^{w_1+w_2} \\ e^{w_1+w_2} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 - e^{2w_1} & e^{w_1+w_2} - e^{w_1-w_2} \\ e^{w_1+w_2} & 1 \end{pmatrix} \\ \sim \begin{pmatrix} 1 - 2e^{2w_1} + e^{2(w_1-w_2)} & e^{w_1+w_2} - e^{w_1-w_2} \\ e^{w_1+w_2} - e^{w_1-w_2} & 1 \end{pmatrix}.$$

Expanding the whole determinant of

$$\begin{pmatrix} 1 - 2e^{2w_1} + e^{2(w_1-w_2)} & e^{w_1+w_2} - e^{w_1-w_2} & 0 & \dots \\ e^{w_1+w_2} - e^{w_1-w_2} & 1 & e^{d_{23}} & \dots \\ 0 & e^{d_{23}} & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we get

$$\det \mathcal{E}(\hat{X}) = (1 - 2e^{2w_1} + e^{2(w_1-w_2)}) \det \mathcal{E}(\hat{X}_1) - (e^{w_1+w_2} - e^{w_1-w_2})^2 \det \mathcal{E}(\hat{X}_{12}),$$

where $\hat{X}_1$ denotes the subspace of $\hat{X}$ with the point l_1 deleted, and $\hat{X}_{12}$ has l_1, l_2 deleted. Collecting maximal terms we get

$$\det \mathcal{E}(\hat{X}) = -2c_{\max}(\hat{X}_1)e^{2w_1+\omega_{\max}(\hat{X}_1)} - c_{\max}(\hat{X}_{12})e^{2(w_1+w_2)+\omega_{\max}(\hat{X}_{12})} \\ + \text{lower-order terms.} \quad (41)$$

For $k = 2$, $\hat{X}_1$ is combinatorically the same as X except that the length of the edge that contains l_2 , which corresponds combinatorically to $v \in X$, has been increased by w_2 . $\hat{X}_{12}$ is the subspace of X in which the leaf v has been removed, hence by induction assumption $\omega_{\max}(\hat{X}_{12}) = 2L(X) - 2w(e_v)$, where v_e is the edge of T that contains the leaf v , and also $\omega_{\max}(\hat{X}_1) = 2L(X) + 2w_2$. Hence we see that $\omega_{\max}(\hat{X}) = 2L(X) + 2w_2 + 2w_1 = 2L(\hat{X})$ and $c_{\max}(\hat{X}) = -2c_{\max}(\hat{X}_1) = -2c_{\max}(X)$. The last equality because $\hat{X}_1$ and X are combinatorially equivalent. Now for $k > 2$, by induction in k , the two exponents in (41) are both equal to $2L(\hat{X})$, and $c_{\max}(\hat{X})$ becomes $-2c_{\max}(\hat{X}_1) - c_{\max}(\hat{X}_{12})$ which (by induction) is easily seen to be $k(-1)^{k-1}c_{\max}(X)$.

Since both operations A and B preserve (38), and any tree realizable metric space X can be built using these two operations, the result follows by induction. $\square$

Remark 8. This maximal “surviving” frequency in the expansion of $\det(\mathcal{E}_t(X))$, $\omega_{\max}(X)$, does seem to be an interesting number connected to a metric space. $(1/n)\omega_{\max}(X)$ is some kind of combinatorial mean distance, a measure of the size of X .

From simple combinatorial considerations, using the definition of a determinant, it can be seen directly, that each exponent in

$$\det[\exp(td_{ij})] = \sum_{k \in I} c_k \exp(\omega_k t),$$

that involves the distance to a leaf l_i , must contain the weight w_i of the edge containing l_i with a factor of 2. That is $\omega = 2w_i + \text{terms without } w_i$, for any exponent involving w_i . This was not used in the proof above, but is utilized in:

Corollary 3. *The $\mathcal{E}(X)$ -matrix of a tree realizable metric space is regular and has one positive eigenvalue and $|X| - 1$ negative eigenvalues.*

Proof. From Proposition 10 it follows that if V is the vertex set of a tree $T = (V, E, w)$, then $\mathcal{E}(V)$ is regular and the determinant has sign $(-1)^{|V|+1}$, since $|E| = |V| - 1$.

There is a sequence of subtrees with vertex sets V_i such that $|V_i| = i$, $i = 1 \cdots |V|$ (e.g. by “peeling off leaves”). Hence $\mathcal{E}(V)$ has an alternating sequence of increasing principal minors. The result for $\mathcal{E}(V)$ then follows from linear algebra, see [5].

However, then for any subspace $X \subseteq V$, since the $\mathcal{E}(X)$ -matrix is a principal minor it can have at most one positive eigenvalue. Hence it must have exactly one positive eigenvalue, and the determinant will have sign $(-1)^{|X|+1}$ if nonvanishing. So if $\det(\mathcal{E}(X)) = 0$, then $(d/dw_i) \det(\mathcal{E}(X)) = 0$, where w_i is the weight of some edge in the tree. If w_i is the weight of an edge that contains a leaf $l_i \in X$, then

$$\frac{d}{dw_i} \det(\mathcal{E}(X)) = \frac{d}{dw_i} \sum_{k \in I} c_k \exp(\omega_k) = -2w_i \det(\mathcal{E}(X_i)),$$

where X_i is the subspace with l_i deleted. This is because all the terms that does not include w_i give the expansion of $\det \mathcal{E}(X_i)$, hence the terms which include w_i add up to $\det(\mathcal{E}(X)) - \det(\mathcal{E}(X_i)) = -\det(\mathcal{E}(X_i))$. Now the result follows by induction, assuming $\mathcal{E}(X_i)$ is regular. $\square$

Lemma 3. *Let $p: \mathbb{R}_+ \rightarrow P_n(\mathbb{R})$, $t \mapsto p_t(\lambda) = \sum_{i=0}^n a_i(t)\lambda^i$, be a curve of polynomials of degree n . Assume that all the fractions $a_i(t)/a_0(t)$ converge for $t \rightarrow \infty$, and let j be the maximal index such that*

$$\lim_{t \rightarrow \infty} \frac{a_i(t)}{a_0(t)} = 0 \quad \text{for } i > j.$$

Then p_t will have $n - j$ roots that converge to ∞ in absolute value, and j bounded roots which will converge to the roots of $q(\lambda) = \sum_{i=0}^j b_i \lambda^i$, where $b_i = \lim_{t \rightarrow \infty} (a_i(t)/a_0(t))$, $i = 0 \cdots j$.

Proof. This is just a special case of a more general principle for homotopies, or sequences, of holomorphic functions, which can be stated as: during a homotopy of holomorphic functions, the zeros move continuously in $\mathbb{C}$, and no zeros disappear except to ∞ .

Alternatively, consider the polynomial $\tilde{p}_t(\lambda) := (\lambda^n/a_0(t))/p_t(1/\lambda)$, and use that (assuming $a_0(t) \neq 0$) there is a 1–1 correspondence between the roots of p_t and $\tilde{p}_t$: $\lambda_i \mapsto 1/\lambda_i$. $\square$

Lemma 4. *Let $X = \{v_1, \dots, v_n\}$ be a tree realizable metric space with b branch points $\{v_1, \dots, v_b\}$. Then the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of $\mathcal{E}_t(X)$ converge in $\mathbb{R} \cup \{\pm\infty\}$ for $t \rightarrow \infty$ and can be ordered such that*

$$\lim_{t \rightarrow \infty} \lambda_n = +\infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \lambda_i = -\infty \quad \text{for } i = b+1 \dots n-1, \quad (42)$$

$$\lim_{t \rightarrow \infty} \lambda_i = \frac{-1}{\deg(v_i) - 1} \quad \text{for } i = 1 \dots b. \quad (43)$$

Proof. We have $\det(\mathcal{E}_t(X)) = c_{\max}(X) \exp(\omega_{\max}(X)t) + o(\exp(\omega_{\max}(X)t))$, and $\omega_{\max}(X) = 2L(X)$, by Proposition 10. Let $\{v_1, \dots, v_b\}$ be the branch points of X .

It is clear that for $j \leq b$ and $Y = X \setminus \{v_1, \dots, v_j\}$, we have $L(Y) = L(X)$. So $\omega_{\max}(Y) = \omega_{\max}(X)$ and $c_{\max}(Y) = c_{\max}(X)(-1)^j \prod_{k=1}^j (\deg(v_k) - 1)$. On the other hand, removing a leaf $l \in X$ will decrease $L(X \setminus \{l\})$ by the weight of the edge that contains l . Hence any principal minor of $\mathcal{E}_t(X)$ corresponding to having removed a leaf will be $o(\det \mathcal{E}_t(X))$, whereas for any minor corresponding to having removed branch points $\{v_1, \dots, v_j\}$, we have

$$\lim_{t \rightarrow \infty} \frac{\det \mathcal{E}_t(Y)}{\det \mathcal{E}_t(X)} = \frac{c_{\max}(Y)}{c_{\max}(X)} = (-1)^j \prod_{k=1}^j (\deg(v_k) - 1).$$

If now $a_i(t)$ is the coefficient of λ^i in the characteristic polynomial of $\mathcal{E}_t(X)$, then since $a_i(t)$ is (up to a sign) the sum of all principal minors of size $n - i$, for $i > b$ each minor must have removed a leaf and hence have strictly lower leading exponent. So we get $\lim_{t \rightarrow \infty} (a_i(t)/a_0(t)) = 0$ for $i > b$, where b is the number of branch points of X , and for $i \leq b$ we get $\lim_{t \rightarrow \infty} (a_i(t)/a_0(t)) = b_i \neq 0$.

Let us try to determine the coefficients in this “limiting” polynomial $q(\lambda) = \sum_{i=0}^b b_i \lambda^i$. Clearly, $b_0 = 1$. For $i > 0$, it is easily seen that

$$b_i = \sum_{|J|=i} (-1)^i \prod_{k \in J} (\deg(v_k) - 1), \quad (44)$$

a sum over all $J \subseteq \{1, \dots, b\}$, with $|J| = i$. However, we see that the coefficients in

$$\prod_{k=1}^b (1 - [\deg(v_k) - 1]\lambda) \quad (45)$$

are exactly given by (44). Hence the roots of $q(\lambda)$ are $\lambda_i = -1/(\deg(v_i) - 1)$, $i = 1 \dots b$. The result now follows by Lemma 3. $\square$

Now we return to the $\mathbf{C}_\kappa(X)$ matrix. For $\kappa < 0$ we have

$$\mathbf{C}_\kappa(X) = \frac{1}{2}(\mathcal{E}_t(X) + \mathcal{E}_{-t}(X)), \quad (46)$$

where $t = \sqrt{|\kappa|}$. Since the second term $\mathcal{E}_{-t}(X)$ converges to the identity matrix for $t \rightarrow \infty$, the result follows directly from Lemma 4, taking the factor $\frac{1}{2}$ into account.

This finishes the proof of Theorem 6. $\square$

3.1. The Limiting Geometry

It has been shown that any leaf space X on n points can be isometrically embedded as a simplex in $\mathbb{H}(n-1, \kappa)$, when the curvature κ is negative enough. Intuition tells us that the simplex should get more “skinny”, resembling the tree that represents X more as $|\kappa|$ increases. A few descriptive results on the limiting geometry are presented in this final subsection.

As before we let h_i be the altitude from p_i onto the hypersurface spanned by $X \setminus \{p_i\}$. We have:

Proposition 11. *Let $X = \{p_1, \dots, p_n\}$ be a leaf space. Then $\lim_{\kappa \rightarrow -\infty} h_i = w_i$, where w_i is the weight of the edge in T that contains p_i .*

Proof. As for $\det(\mathcal{E}_t(X))$, see (37), we have an expansion

$$\det(\mathbf{C}_\kappa(X)) = \sum_{k \in I} c_k \exp(\omega_k t), \quad (47)$$

where $t = \sqrt{|\kappa|}$. Here the leading-order exponent must be as for $\det(\mathcal{E}_t(X))$, $\omega_{\max} = 2L(X)$. This can be seen by a noncombinatorial argument:

We have $2\mathbf{C}_\kappa(X) - \mathcal{E}_t(X) = \mathcal{E}_{-t}(X)$, multiplying by $\mathcal{E}_t(X)^{-1}$ and rearranging we get

$$2\mathbf{C}_\kappa(X)\mathcal{E}_t(X)^{-1} = \mathcal{E}_{-t}(X)\mathcal{E}_t(X)^{-1} + \mathbf{I}. \quad (48)$$

Now, the right-hand side will converge to $\mathbf{I}$ for $t = \sqrt{|\kappa|} \rightarrow \infty$. This is true because $\mathcal{E}_{-t}(X) \rightarrow \mathbf{I}$ and $\mathcal{E}_t(X)^{-1} \rightarrow \mathbf{0}$ for $t \rightarrow \infty$, since X does not contain branch points (see Theorem 6). Hence the claim follows by taking determinants.

We have

$$\begin{aligned} \log(|\det(\mathbf{C}_\kappa(X))|) &= \log \left| \sum_{k \in I} c_k \exp(\omega_k t) \right| \\ &= \omega_{\max} t + \log \left| c_{\max} + \sum_{\omega_k \neq \omega_{\max}} (c_k \exp((\omega_k - \omega_{\max})t)) \right|. \end{aligned} \quad (49)$$

Hence $\log(|\det(\mathbf{C}_\kappa(X))|)/t \rightarrow \omega_{\max} = 2L(X)$ for $t = \sqrt{|\kappa|} \rightarrow \infty$. Using Theorem 3 we get, assuming for the sake of notation that $i = n$,

$$\log(|\det(\mathbf{C}_\kappa(X))|) = \sum_{j=2}^n 2 \left(h_{<j} t + \log \left(\frac{1 - \exp(-2th_{<j})}{2} \right) \right). \quad (50)$$

Dividing by t and taking the limit $t \rightarrow \infty$ we get $2 \sum_{j=2}^n h_{<j} = 2L(X)$. The result follows by induction using $L(X \setminus \{p_n\}) = L(X) - w_n$. $\square$

Let now $\tilde{\theta}_{ij}$ denote the *interior* dihedral angle at the codimension 2 face $X_{ij} := X_i \cap X_j$. Let $T = (V, E, w)$ be the weighted tree that represents X , and for $p \in X$ let $v(p) \in V$ be the branch point of T such that $v(p) \sim p$. Then we have:

Proposition 12. *Let $X = \{p_1, \dots, p_n\}$ be a leaf space. Then for $i \neq j$,*

$$\lim_{\kappa \rightarrow -\infty} \tilde{\theta}_{ij} = \begin{cases} \arccos\left(\frac{1}{(\deg(v) - 2)}\right) & \text{if } v(p_i) = v(p_j), \\ \frac{\pi}{2} & \text{if } v(p_i) \neq v(p_j). \end{cases} \quad (51)$$

Proof. By (48) we determine that the leading-order exponent $\omega_{\max}$ and the corresponding coefficient $c_{\max}$ in the expansions of $\det(\mathbf{C}_{\kappa}(X))$ and $\det(\mathcal{E}_t(X))$ (47) and (37) satisfy

$$c_{\max, \mathbf{C}_{\kappa}}(X) = 2^{-n} c_{\max, \mathcal{E}}(X), \quad \omega_{\max, \mathbf{C}_{\kappa}}(X) = \omega_{\max, \mathcal{E}}(X) = 2L(X). \quad (52)$$

Also by the proof of Proposition 5:

$$1 - \frac{\det(\mathbf{C}_{\kappa}(X_{ij})) \det(\mathbf{C}_{\kappa}(X))}{\det(\mathbf{C}_{\kappa}(X_i)) \det(\mathbf{C}_{\kappa}(X_j))} = \cos(\tilde{\theta}_{ij})^2. \quad (53)$$

Assuming at first that $v(p_i) = v(p_j) := v$, then the total weights satisfy $L(X_i) = L(X) - w_i$, $L(X_j) = L(X) - w_j$ and $L(X_{ij}) = L(X) - w_i - w_j$, unless $\deg(v) = 3$ in which case $L(X_{ij}) < L(X) - w_i - w_j$, with notation as earlier. Inserting in (53), we see that

$$\cos(\tilde{\theta}_{ij})^2 \rightarrow 1 - \frac{c_{\max}(X_{ij})c_{\max}(X)}{c_{\max}(X_i)c_{\max}(X_j)}, \quad (54)$$

if $\deg(v) > 3$, and $\sin(\tilde{\theta}_{ij})^2 \rightarrow 0$ if $\deg(v) = 3$. Assuming $\deg(v) > 3$ we get by Proposition 10

$$\frac{c_{\max}(X)}{c_{\max}(X_i)} = -\frac{\deg(v) - 1}{\deg(v) - 2} \quad \text{and} \quad \frac{c_{\max}(X_{ij})}{c_{\max}(X_j)} = -\frac{\deg(v) - 3}{\deg(v) - 2}.$$

It is left to show that the interior dihedral angles are acute. For this we refer to Lemma 5 below.

The case $v(p_i) \neq v(p_j)$ is similar, but here the quotient in (54) is seen to be 1. $\square$

The result in the case where $\lim \tilde{\theta}_{ij} = \arccos(1/(\deg(v) - 2))$ seems a bit surprising, since $\lim h_i = \lim h_{ij} = w_i$ (with notation as in the proof of Proposition 4). However, the finer details of the convergence are needed, and these could be found from (49) and (50).

Lemma 5. *Let X be a leaf space. There is a $\kappa_0 < 0$ such that $\kappa < \kappa_0$ implies: $X \xrightarrow{\text{isom}} \mathbb{H}(|X| - 1, \kappa)$ and all interior dihedral angles $\tilde{\theta}_{ij}$ corresponding to leaves with $v(p_i) = v(p_j)$ are acute.*

Proof. Again, because of the easier algebra, we shall work with $\mathcal{E}_t(X)$. Let $p_i, p_j \in X$ have $v(p_i) = v(p_j)$. Then for every $q \in X_{ij} = X \setminus \{p_i, p_j\}$,

$$d(q, p_i) = d(q, p_j) - w_j + w_i. \quad (55)$$

Let $[c_{kl}]$ be the components of $\text{cof}(\mathcal{E}_t(X))$. $c_{ij} = (-1)^{i+j} |\mathcal{E}_t(X)_j^i|$ is the signed minor obtained by deleting column i and row j . Expanding the minor after column j of $\mathcal{E}_t(X)$, the obtained minors can be interpreted as minors of the principal submatrix $\mathcal{E}_t(X_j) = \mathcal{E}_t(X)_j^j$, where $X_j := X \setminus \{p_j\}$. Assume that $i < j$, then we get, since column j of $\mathcal{E}_t(X)$ is column $j - 1$ of $\mathcal{E}_t(X_j)_j^i$,

$$\begin{aligned} c_{ij} = (-1)^{i+j} & \left(\sum_{k=1}^{j-1} (-1)^{j-1+k} \exp(td_{kj}) |\mathcal{E}_t(X_j)_k^i| \right. \\ & \left. + \sum_{k=j}^{|X|-1} (-1)^{j-1+k} \exp(td_{k+1,j}) |\mathcal{E}_t(X_j)_k^i| \right). \end{aligned}$$

Multiplying by $\exp(t(w_i - w_j))$ and using (55), it is seen that we almost get the expansion of $\det(\mathcal{E}_t(X_j))$, except for a sign and the coefficient in the term with $k = i$, i.e. the coefficient of $|\mathcal{E}_t(X_j)_i^i| = \det(\mathcal{E}_t(X_{ij}))$:

$$c_{ij} \exp(t(w_i - w_j)) = -\det(\mathcal{E}_t(X_i)) - (\exp(2w_i) - 1) \det(\mathcal{E}_t(X_{ij})). \quad (56)$$

Using Proposition 10, we see that both terms on the right-hand side have the same leading-order exponent, but that the first term has the largest coefficient and is dominant as $t \rightarrow \infty$. Hence the sign of c_{ij} will be as for $\det(\mathcal{E}_t(X))$ for t large.

Now using $2\mathbf{C}_\kappa(X) = \mathcal{E}_t(X)(\mathbf{I} + \mathcal{E}_t(X)^{-1}\mathcal{E}_{-t}(X))$, the result follows for $\mathbf{C}_\kappa(X)$ by taking cofactors, since $\mathcal{E}_t(X)^{-1}\mathcal{E}_{-t}(X) \rightarrow \mathbf{0}$ for $t = \sqrt{|\kappa|} \rightarrow \infty$. It follows from (21) that the exterior dihedral angle θ_{ij} will have negative cosine, and hence the interior dihedral angle will be acute. $\square$

Claim. *In fact with a little more combinatorial insight it is possible to show that, for a leaf space X , all cofactors will have the same sign as $\det(\mathbf{C}_\kappa(X))$ in the limit $\kappa \rightarrow -\infty$. Hence the interior angles converging to $\pi/2$ will also be acute.*

The vertices of a regular simplex form the leaf space of a *star*, a tree with one interior vertex of degree n connected to n leaves by edges of similar length. In this case Proposition 12 can be interpreted as a result about the regular, *ideal simplex* (Fig. 3), a simplex with all vertices on the sphere at infinity, see [13].

By increasing $|\kappa|$ and considering the realization of $X \in \mathbb{H}(n, \kappa)$ as a realization of $\sqrt{|\kappa|}X$ in $\mathbb{H}(n, -1)$, which is the same thing, the vertices of $\sqrt{|\kappa|}X$ will converge to a subset of the sphere at infinity. This regular, ideal simplex is the unique simplex of maximal volume in $\mathbb{H}(n, -1)$, see [7].

In the limit we find for the regular, ideal simplex in $\mathbb{H}(n, -1)$,

$$\cos(\tilde{\theta}_{ij}) = \frac{1}{n-1}, \quad \text{for } i \neq j. \quad (57)$$

Fig. 3. An ideal simplex with vertices at the sphere at infinity.

This is then interpreted as a description of the *dual ideal simplex*. Hence we have that $\mathbf{C}_\kappa(X^*) = [x_{ij}]$ with $x_{ii} = 1$ and $x_{ij} = -1/(n-1)$ (exterior dihedral angles). It is easily checked that $\mathbf{C}_\kappa(X^*)$ is regular with all codimension 1 principal minors vanishing. *X is a simplex with degenerate/lightlike codimension 1 faces.*

What about a leaf space which is less symmetric, will the vertices converge to the vertices of a simplex at the sphere at infinity?

Proposition 13. *Let X be the leaf space of a tree containing more than one branch point, then the limit of $\mathbf{C}_\kappa(X^*)$ is singular, hence the realizations of $\sqrt{|\kappa|}X$ cannot be made to converge to a simplex on the sphere at infinity for $\kappa \rightarrow -\infty$.*

Proof. Define a vector $x \in \mathbb{R}^{|X|}$ such that $x_i = 1$ whenever the corresponding leaf l_i is adjacent to an interior vertex $v(l_i)$, which again is adjacent to exactly one other interior point; and put $x_i = 0$ otherwise. It is easy to see that we do not get the zero vector when X is not the leaf space of a star: simply take any vertex in T and take a point which is furthest away with the combinatorial distance. Then this leaf and all other leaves adjacent to the same interior vertex will satisfy the requirement and get weight 1.

From Proposition 12 it is then seen that this vector is in the kernel of $\mathbf{C}_\kappa(X^*)$. $\square$

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References

1. M. Berger, *Geometry I*, Springer-Verlag, Berlin, 1987.
2. L. M. Blumenthal, *Theory and Applications of Distance Geometry*, 2nd edn., Chelsea, New York, 1970.
3. M. Deza and M. Laurent, *Geometry of Cuts and Metrics*, Algorithms and Combinatorics, Vol. 15, Springer-Verlag, Berlin, 1997.
4. R. Diaz, A characterization of Gram matrices of polytopes, *Discrete Comput. Geom.* **21** (1999), 581–601.
5. F. R. Gantmacher, *The Theory of Matrices*, Vol. 2, Chelsea, New York, 1959.
6. E. P. Ghys and P. de la Harpe, Espaces métriques hyperboliques, in *Sur les groupes hyperboliques d'après Mikhael Gromov*, pp. 27–45, Progress in Mathematics, Vol. 83, Birkhäuser, Boston, MA, 1990.
7. U. Haagerup and H. J. Munkholm, Simplices of maximal volume in hyperbolic n -space, *Acta. Math.* **147** (1981), 1–11.
8. P. G. Hjorth, S. L. Kokkendorff and S. Markvorsen, Hyperbolic spaces are of strictly negative type, *Proc. Amer. Math. Soc.* **130** (2002), 175–181.
9. S. L. Kokkendorff, *Geometry & Combinatorics*, Ph.D. thesis, Department of Mathematics, Technical University of Denmark, available at <http://www.mat.dtu.dk/people/S.L.Kokkendorff/>.
10. J. Milnor, The Schläfli differential equality, in *Collected Papers. Vol. 1. Geometry*, Publish or Perish, Houston, TX, 1994.
11. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
12. I. Rivin and C. D. Hodgson, A characterization of compact convex polyhedra in hyperbolic 3-space, *Invent. Math.* **111** (1993), 77–111.
13. I. Rivin, A characterization of ideal polyhedra in hyperbolic 3-space, *Ann. of Math. (2)* **143** (1996), 51–70.
14. I. J. Schoenberg, A remark to Maurice Fréchet's article, *Ann. of Math.* **36** (1935), 724–732.
15. J.-M. Schlenker, Métriques sur les polyèdres hyperboliques convexes, *J. Differential Geom.* **48** (1998), 323–405.
16. J.-M. Schlenker, Convex polyhedra in Lorentzian space-forms, *Asian J. Math.* **5** (2001), 1–35.
17. E. Suárez-Peirò, A Schläfli differential formula for simplices in semi-Riemannian hyperquadrics, *Pacific J. Math.* **194** (2000), 229–255.

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