Combinatorics of Polytopes with a Group of Linear Symmetries of Prime Power Order

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Abstract. Generalizing a result of Stanley [13] on centrally symmetric polytopes, Adin [2], [3] has derived tight lower bounds for the face numbers of a rational simplicial polytope equipped with a fixed-point-free linear action of a cyclic group G of prime power order. The main goal of this paper is to extend these results further by replacing Adin's fixed-point-free condition with the assumption that the action of G is proper. As corollaries, we obtain a generalization of Adin's equivariant lower bound theorem and of a condition by Stanley [13] implying combinatorial isomorphism with a minimal polytope. Finally, we prove sufficiency of an equivariant version of the McMullen and Walkup generalized lower bound conjecture.

Introduction

The combinatorics of a simplicial convex polytope is completely characterized in terms of its face numbers by *McMullen's conjecture* [10], whose sufficiency was proven by Billera and Lee [5], [6] and necessity by Stanley [12]. When the polytope has a group of linear symmetries, it is natural to expect even stronger constraints on the relations among its face numbers. Let P be a d -dimensional simplicial convex polytope in $\mathbb{R}^d$, and let f_i denote the number of i -dimensional faces of P . The h -vector $h(P) = (h_0, h_1, \dots, h_d)$ of P is defined by the formula

$$\sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{i=0}^d h_i x^{d-i},$$

where $f_{-1} = 1$ by convention. For simplicial convex polytopes P which are centrally symmetric about the origin, Stanley [13] proved the following relations:

$$h_i \geq \binom{d}{i}, \quad 0 \leq i \leq d, \quad (1)$$

and more strongly

$$h_i - h_{i-1} \geq \binom{d}{i} - \binom{d}{i-1}, \quad 1 \leq i \leq \lfloor d/2 \rfloor. \quad (2)$$

Inequalities (1) and (2) represent a $\mathbb{Z}_2$ -equivariant version of the nonnegativity conditions $h_i \geq 0$ and $h_i - h_{i-1} \geq 0$ of McMullen's conjecture. In [2] and [3] Adin extends (1) and (2) above to rational simplicial d -polytopes with a *fixed-point-free* (f.p.f.) linear symmetry of prime power order p^α . Moreover, he proves that these inequalities are sharp by constructing examples of stacked polytopes having the desired symmetry with the minimum possible number of faces. Finally, based on Stanley and Adin's results, A'Campo-Neuen (see [1]) obtains tight lower bounds for the coefficients of the generalized h -vector of a rational polytope with a f.p.f. linear symmetry of prime order.

The first goal of this paper is to generalize Adin's results by removing the f.p.f. condition on the group of symmetries of the polytope. Let $V_0, \dots, V_\alpha$ be the distinct rational irreducible representation spaces of the group $G = \mathbb{Z}_{p^\alpha}$, of dimensions 1, $\varphi(p)$, $\dots$, $\varphi(p^\alpha)$, and let $n_0, \dots, n_\alpha$ be their multiplicities in the given action of G on $\mathbb{R}^d$, where φ is the Euler phi-function with $\varphi(p^n) = p^n - p^{n-1}$, for any integer $n \geq 1$. Recall that we assume the polytope P to be rational. Fix a group generator $g \in G$ and consider the family of subgroups $G_\mu = \langle g^{p^\mu} \rangle$ of order $p^{\alpha-\mu}$ generated by g^{p^μ} , for $0 \leq \mu \leq \alpha$. Let $P_\mu = P \cap (\mathbb{R}^d)^{G_\mu}$ denote the corresponding subpolytopes of P , in particular $P_0 = P^G$ and $P_\alpha = P$, and for $1 \leq \mu \leq \alpha$ define the free sum¹ $\tilde{P}_{\min}^\mu = P_{\mu-1} * P_{\min}^\mu$, where $P_{\min}^\mu \subset n_\mu V_\mu$ is the free sum of $n_\mu p^{\mu-1}$ copies of the simplex Δ_{p-1} .

The action of a group G on a simplicial complex Δ is said to be *proper* if $\Delta^H = \{S \in \Delta : S^g = S, \forall g \in H\}$ is a simplicial complex for every subgroup H of G . Evidently, f.p.f. actions are proper, and the assumption made in the next theorem on the group action being proper, can always be achieved by taking an appropriate triangulation of the polytope P .

Theorem 1. *Let $P \subset \mathbb{R}^d$ be a convex rational simplicial d -polytope with a proper linear action of a cyclic group G of prime power order p^α . Define $d_\mu := \dim P_\mu$ ($= \dim \tilde{P}_{\min}^\mu$). Then the h -vectors $(h_0^\mu, \dots, h_{d_\mu}^\mu)$ and $(\tilde{h}_0^\mu, \dots, \tilde{h}_{d_\mu}^\mu)$ of the polytopes P_μ and $\tilde{P}_{\min}^\mu$ satisfy:*

- (1) $h_j^\mu - \tilde{h}_j^\mu = h_{d_\mu-j}^\mu - \tilde{h}_{d_\mu-j}^\mu$,
- (2) $h_j^\mu \geq \tilde{h}_j^\mu$,
- (3) $p^\mu | (h_j^\mu - \tilde{h}_j^\mu)$,
- (4) $h_j^\mu - h_{j-1}^\mu \geq \tilde{h}_j^\mu - \tilde{h}_{j-1}^\mu$

for all $1 \leq j \leq \lfloor d_\mu/2 \rfloor$ and $1 \leq \mu \leq \alpha$.

¹ For the definition of free sum we refer to Section 2.1 of [2].

Any simplicial polytope admitting a group action with a rational linear representation in $\mathbb{R}^d$ can always be perturbed to a rational simplicial polytope, preserving the equivariant combinatorics, i.e. the face numbers of different orbit-types (see Proposition 3.2 of [9]).

A simplicial d -polytope P is called k -stacked if it can be triangulated without introducing new j -faces, for all $j \leq d - k - 1$. The class of 1-stacked polytopes, referred to simply as *stacked polytopes*, can be alternatively described as the class containing every d -simplex and each simplicial polytope obtained from a stacked d -polytope with one fewer vertex by adding a pyramid over some facet. Therefore a stacked polytope P is the union of d -simplices, each $(d - 2)$ -face of which is a face of P . We say that a simplicial convex d -polytope P is k -stacked over a d -polytope Q if all the $(d - k - 1)$ -dimensional faces of Q are faces of P , and, moreover, P admits a triangulation consisting of the union of d -simplices together with a triangulation of Q , so that every simplex of dimension $d - k - 1$ is either in the triangulation of Q or is a face of P . In particular, P is stacked over Q if all the $(d - 2)$ -faces of Q are also faces of P , and P is obtained from Q by adding pyramids over the facets of Q .

From part (4) of Theorem 1 we obtain the following generalization of Adin's equivariant Lower Bound Theorem.

Corollary 2. *Let f_j^μ and $\tilde{f}_j^\mu$ be the number of j -dimensional faces of the polytopes P_μ and $\tilde{P}_{\min}^\mu$, respectively. Under the assumptions of Theorem 1, we have*

$$p^\mu | (f_0^\mu - f_0^{\mu-1}),$$

and for all $1 \leq \mu \leq \alpha$, $1 \leq j \leq d_\mu - 1$, the following relations hold:

$$\begin{aligned} f_j^\mu - \tilde{f}_j^\mu &\geq (f_0^\mu - f_0^{\mu-1} - n_\mu p^\mu) \left[\binom{d_\mu}{j} - \delta_j^{d_\mu-1} \right], \\ f_0^\mu - \tilde{f}_0^\mu &= f_0^\mu - f_0^{\mu-1} - n_\mu p^\mu \geq 0. \end{aligned}$$

Moreover, if P_μ is stacked over $\tilde{P}_{\min}^\mu$, then equality holds for all j . The f -vector of $\tilde{P}_{\min}^\mu$ can be computed explicitly from the generating function:

$$\begin{aligned} \sum_{j=0}^{d_\mu} \tilde{f}_{j-1}^\mu q^j &= h_{P_{\mu-1}} \left(\frac{1}{q} + 1 \right) [(1+q)^p - q^p]^{n_\mu p^{\mu-1}} q^{d_{\mu-1}} \\ &= \sum_{j=0}^{d_{\mu-1}} q^j f_{j-1}^{\mu-1} [(1+q)^p - q^p]^{n_\mu p^{\mu-1}} \end{aligned}$$

with the convention that $\tilde{f}_{-1}^\mu = f_{-1}^{\mu-1} := 1$.

Two polytopes are *combinatorially isomorphic* if there is a bijection between their faces which preserves the inclusion relation. Theorem 1 also provides the following extension of a result by Stanley [13, Proposition 4.3].

Corollary 3. *Under the assumptions of Theorem 1, if $h_i^\mu = \tilde{h}_i^\mu$ for some i , with $1 \leq i \leq d_\mu - 1$, then equality holds for all i , and P_μ is combinatorially isomorphic to the minimal polytope $\tilde{P}_{\min}^\mu$.*

McMullen and Walkup [11, Corollary of Theorem 2] have proven that the h -vector of a simplicial d -polytope P satisfies $h_k - h_{k-1} = 0$ for some $1 \leq k \leq \lfloor d/2 \rfloor$ provided that P is $(k-1)$ -stacked. Next, we adapt this result to the equivariant setting.

Theorem 4. *Let $P \subset \mathbb{R}^d$ be a convex rational simplicial d -polytope with a proper linear action of a cyclic group G of prime power order p^α . For all $1 \leq j \leq \lfloor d_\mu/2 \rfloor$ and $1 \leq \mu \leq \alpha$, if P_μ is $(j-1)$ -stacked over $\tilde{P}_{\min}^\mu$, then*

$$h_j^\mu - h_{j-1}^\mu = \tilde{h}_j^\mu - \tilde{h}_{j-1}^\mu.$$

1. The Isotypical Components of the Refined Hilbert–Poincaré Polynomial

Let P be a convex rational simplicial d -polytope and let G be a subgroup of the group of linear automorphisms of P .

Consider $V(P) = \{x_1, \dots, x_n\}$ the vertex set of P and form the abstract simplicial complex Δ built on the boundary of P . Define I_Δ as the ideal of the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ generated by all square free monomials $x_{i_1} \cdots x_{i_r}$ such that $\{x_{i_1}, \dots, x_{i_r}\} \notin \Delta$. The *face ring* $\mathbb{C}[\Delta]$ of Δ is the quotient

$$\mathbb{C}[\Delta] = \frac{\mathbb{C}[x_1, \dots, x_n]}{I_\Delta}.$$

The polytope P determines a complete fan Σ consisting of all the cones over the proper faces of P , together with 0. Denote by X the toric variety associated to the fan Σ (for the lattice $N = \mathbb{Z}^d$). Danilov [7, Theorem 10.8] proved that when P is simplicial there is an isomorphism:

$$H^*(X; \mathbb{C}) \cong \frac{\mathbb{C}[\Delta]}{(\Theta_1, \dots, \Theta_d)}$$

for a certain homogeneous system of parameters (h.s.o.p.) $\Theta_1, \dots, \Theta_d$ of degree 1 of $\mathbb{C}[\Delta]$. Define the quotient ring

$$A = \frac{\mathbb{C}[\Delta]}{(\Theta_1, \dots, \Theta_d)}$$

which inherits a grading $A = \bigoplus_{i \geq 0} A_i$, where $\dim_{\mathbb{C}} A_i = h_i(P)$. The linear action of G on the vector space induces an action on the quotient ring A (see Lemma 7 of [3] for details).

Consider a graded $\mathbb{C}G$ -module $M = \bigoplus_{i=0}^{\infty} M_i$, with $\dim_{\mathbb{C}} M_i < \infty$, for all i . The *Hilbert–Poincaré series* of M , given by the formal power series

$$F_M(q) = \sum_{i=0}^{\infty} (\dim_{\mathbb{C}} M_i) q^i,$$

can be further decomposed into the *refined Hilbert–Poincaré series* of M :

$$F_M(q, \hat{G}) = \sum_{\chi \in \hat{G}} \sum_{i=0}^{\infty} (\dim_{\mathbb{C}} M_i^\chi) \chi q^i,$$

where $\hat{G}$ denotes the group formed by the complex irreducible characters of G , under pointwise multiplication, and M_i^χ is the χ -isotypical component of grade i . For $\chi \in \hat{G}$, we form the polynomial

$$\begin{aligned} P_\chi(q) &:= \sum_{i=0}^d (\dim_{\mathbb{C}} A_i^\chi) q^i \\ &= \frac{1}{p^\alpha} \sum_{h \in G} \overline{\chi(h)} \cdot F_A(q, \hat{G})(h). \end{aligned} \quad (3)$$

The *refined Hilbert–Poincaré polynomial* of P can be decomposed into its isotypical components as follows:

$$h_P(q, \hat{G}) := \sum_{\chi \in \hat{G}} \sum_{i=0}^d (\dim H^{2i}(X; \mathbb{C})^\chi) \chi q^i = \sum_{\chi \in \hat{G}} P_\chi(q) \chi.$$

In the next proposition we determine each isotypical polynomial $P_\chi(q)$.

Proposition 1.1. *Let $P \subset \mathbb{R}^d$ be a convex rational simplicial d -polytope with a proper linear action of a cyclic group of prime power order $G \cong \mathbb{Z}_{p^\alpha}$, and let $\tilde{P}_{\min}^\alpha$ denote the minimal polytope defined in the Introduction. Then*

$$P_\chi(q) = \frac{1}{p^\alpha} \left\{ h_P(q) - h_{\tilde{P}_{\min}^\alpha}(q) \right\}$$

for any generator χ of the group $\hat{G}$ of irreducible characters.

Remark 1.2. The polynomial P_χ is independent of the generator $\chi \in \hat{G}$ chosen.

The proof of Proposition 1.1 extends Adin’s approach for the f.p.f. case [2, Theorem 1]. We compute $F_A(q, \hat{G})$ with the use of the following relation [3, Lemma 9]:

$$F_{\mathbb{C}[\Delta]}(q, \hat{G}) = F_{\mathbb{C}[\Theta_1, \dots, \Theta_d]}(q, \hat{G}) \cdot F_A(q, \hat{G}). \quad (4)$$

The face ring $\mathbb{C}[\Delta]$ has an $\mathbb{N} \times \hat{G}$ grading given by decomposing each space of homogeneous polynomials of the same degree into isotypical components:

$$\mathbb{C}[\Delta] = \coprod_{i \geq 0} \coprod_{\chi \in \hat{G}} \mathbb{C}[\Delta]_i^\chi,$$

and refined Hilbert–Poincaré series:

$$F_{\mathbb{C}[\Delta]}(q, \hat{G}) = \sum_{i=0}^{\infty} \sum_{\chi \in \hat{G}} (\dim \mathbb{C}[\Delta]_i^\chi) \chi q^i \in (\mathbb{Z}\hat{G}) \otimes \mathbb{Z}[[q]].$$

In particular [4, Chapter 11],

$$F_{\mathbb{C}[\Delta]}(q, \hat{G})(e) = \sum_{i \geq 0} (\dim \mathbb{C}[\Delta]_i) q^i = \frac{h_P(q)}{(1-q)^d}.$$

Fix a group generator $g \in G$, and define the characters χ_m^j as follows:

$$\chi_m^j(g) = e^{2\pi i m / p^j}, \quad 1 \leq j \leq \alpha, \quad 0 \leq m \leq p^j - 1.$$

The complex vector space $\mathbb{C}^d$ has the following decomposition into a direct sum of complex irreducible representation spaces:

$$\mathbb{C}^d = n_0 V_0 \oplus n_1 \left(\bigoplus_{\substack{1 \leq m \leq p-1 \\ (p,m)=1}} W_m^1 \right) \oplus \cdots \oplus n_\alpha \left(\bigoplus_{\substack{1 \leq m \leq p^\alpha-1 \\ (p,m)=1}} W_m^\alpha \right),$$

where $n_0, \dots, n_\alpha$ are the multiplicities defined in the Introduction, V_0 is the irreducible representation space of G left (pointwise) fixed by its action, and W_m^j is the irreducible representation spaces defined by the relation

$$g W_m^j = \chi_m^j(g) W_m^j$$

for $1 \leq j \leq \alpha$, $1 \leq m \leq p^j - 1$, and $(p, m) = 1$.

Let $x \in \mathbb{C}[\Delta]_i$ be a monomial of positive degree i lying in a G -orbit of order p^j . The $\mathbb{C}$ -span of its G -orbit affords the regular representation of the subgroup $G_{\alpha-j} = \langle g^{p^{\alpha-j}} \rangle$ of order p^j . We define the set of integers $\{n_j^i, 0 \leq j \leq \alpha\}$ where n_j^i is the number of G -orbits of order p^j of monomials of degree i , thus left invariant by the action of G_μ , for all $\alpha - j \leq \mu \leq \alpha$. In particular, n_0^i is the number of monomials of degree i left invariant by G . Since each G -invariant face must contain a fixed point in its interior, and the G -action is proper, this implies that $n_0^i = \dim \mathbb{C}[\Delta^G]_i$.

Given $x \in \mathbb{C}[\Delta]_i$ is a monomial of positive degree i and order p^α , we can always construct the polynomial $x_j \in \mathbb{C}[\Delta]_i$ of order p^j , for all $j = 0, \dots, \alpha - 1$, as follows:

$$x_j = (1 + g^{p^j} + g^{2p^j} + \cdots + g^{(p^{\alpha-j}-1)p^j})x = \sum_{k=0}^{p^{\alpha-j}-1} g^{kp^j} x.$$

We therefore have the following decomposition:

$$\begin{cases} \dim \mathbb{C}[\Delta]_i^{\chi_m^\alpha} = n_\alpha^i, & 1 \leq m \leq p^\alpha - 1, \quad (m, p) = 1, \\ \dim \mathbb{C}[\Delta]_i^{\chi_m^{\alpha-1}} = n_{\alpha-1}^i + n_\alpha^i, & 1 \leq m \leq p^{\alpha-1} - 1, \quad (m, p) = 1, \\ \vdots & \vdots \\ \dim \mathbb{C}[\Delta]_i^{\chi_m^1} = n_1^i + n_2^i + \cdots + n_\alpha^i, & 1 \leq m \leq p - 1, \quad (m, p) = 1, \\ \dim \mathbb{C}[\Delta]_i^{\chi_0} = n_0^i + n_1^i + \cdots + n_\alpha^i \end{cases} \quad (5)$$

with χ_0 the trivial character. Adding the relations in (5) with respect to all the generators of $\hat{G}$, we obtain

$$\begin{aligned} \dim \mathbb{C}[\Delta]_i &= n_\alpha^i \varphi(p^\alpha) + (n_{\alpha-1}^i + n_\alpha^i) \varphi(p^{\alpha-1}) + \cdots + (n_0^i + n_1^i + \cdots + n_\alpha^i) \\ &= n_0^i + p n_1^i + \cdots + p^\alpha n_\alpha^i. \end{aligned}$$

Note that $\mathbb{C}[\Delta]_0 \cong \mathbb{C}$, $n_0^0 = 1$, and $n_j^0 = 0$ for all $j > 0$.

Lemma 1.3. *The refined Hilbert–Poincaré series has the following expression:*

$$F_{\mathbb{C}[\Delta]}(q, \hat{G}) = \sum_{i \geq 0} n_0^i q^i + \sum_{k=1}^{\alpha} p^k \mu_k \sum_{i \geq 0} n_k^i q^i,$$

where $\mu_k = (1/p^k) \sum_{m=0}^{p^k-1} \chi_m^k$.

Proof. Follows directly from (5). $\square$

Corollary 5. *The refined Hilbert–Poincaré series satisfies the recurrence formula*

$$F_{\mathbb{C}[\Delta]}(q, \hat{G}) = \mu_{\alpha} \left(\frac{h_P(q)}{(1-q)^d} - \frac{h_{P_{\alpha-1}}(q)}{(1-q)^{d_{\alpha-1}}} \right) + F_{\mathbb{C}[\Delta_{\alpha-1}]}(q, \hat{G})$$

with the closed form

$$F_{\mathbb{C}[\Delta]}(q, \hat{G}) = \mu_{\alpha} \frac{h_P(q)}{(1-q)^d} + \sum_{i=1}^{\alpha} (\mu_{\alpha-i} - \mu_{\alpha-i+1}) \frac{h_{P_{\alpha-i}}(q)}{(1-q)^{d_{\alpha-i}}},$$

where $\Delta_{\alpha-1}$ is the simplicial complex built on the boundary of the polytope $P_{\alpha-1}$, μ_0 is the trivial character of $\hat{G}$, and for $k = 1, \dots, \alpha$, the μ_k are defined in Lemma 1.3.

Lemma 1.4. *For any $h \in G$,*

$$F_{\mathbb{C}[\Delta]}(q, \hat{G})(h) = \begin{cases} \frac{h_P(q)}{(1-q)^d}, & h = e, \\ \frac{h_{P_k}(q)}{(1-q)^{d_k}}, & h \text{ has order } p^{\alpha-k}, \quad 0 \leq k < \alpha. \end{cases}$$

Proof. Observe that if $h \in G$ has order $p^{\alpha-k}$, then

$$\mu_i(h) = \begin{cases} 1, & 0 \leq i \leq k, \\ 0, & \text{else,} \end{cases}$$

and apply Corollary 5. $\square$

Lemma 1.5 [3, Proposition 10]. *Let $\chi_1, \dots, \chi_d$ be the irreducible G -characters that compose the G -action on $W = \text{span}\{\Theta_1, \dots, \Theta_d\}$, and denote by $\pi: G \rightarrow \text{LAut}(P)$ the linear action of G on P . The refined Hilbert–Poincaré series of $\mathbb{C}[\Theta_1, \dots, \Theta_d]$ is then*

$$\begin{aligned} F_{\mathbb{C}[\Theta_1, \dots, \Theta_d]}(q, \hat{G})(h) &= \prod_{i=1}^d (1 - \chi_i(h)q)^{-1} \\ &= [\det_W(I - q\pi(h))]^{-1}, \quad \text{for all } h \in G. \end{aligned}$$

Lemmas 1.4 and 1.5 combined with (4) lead to the next lemma:

Lemma 1.6. *For all $h \in G$,*

$$F_A(q, \hat{G})(h) = \begin{cases} h_P(q), & h = e, \\ \frac{h_{P_k(q)}}{(1-q)^{d_k}} \det_W(I - q\pi(h)), & h \text{ has order } p^{\alpha-k}, \quad 0 \leq k < \alpha. \end{cases}$$

Let $g \in G$ be the generator fixed previously, and let $n_0, n_1, \dots, n_\alpha$ be the multiplicities defined in the Introduction. The rational representation matrix $\pi(g)$ has eigenvalue 1 with multiplicity n_0 , and the eigenvalues that are primitive p^j -roots of unity, i.e. roots of the cyclotomic polynomial $\Phi_{p^j}(q)$ (which is irreducible over $\mathbb{Q}$), with multiplicity n_j , for all $1 \leq j \leq \alpha$. Its characteristic polynomial:

$$\det(I - q\pi(g)) = (1-q)^{n_0} \prod_{i=1}^{\alpha} [\Phi_{p^i}(q)]^{n_i}.$$

To determine the characteristic polynomial of $\pi(g^{mp^\rho})$, where m is an integer with $\gcd(m, p) = 1$, we need the next two lemmas.

Lemma 1.7 [2, Lemma 10]. *Let $n = p^\alpha$ and assume that $\det_W(I - q\pi(g)) = [\Phi_n(q)]^l$. Then for any integer m ,*

$$\det_W(I - q\pi(g^m)) = [\Phi_{p^{\alpha-\rho}}(q)]^{p^\rho l},$$

where $p^\rho = \gcd(m, n)$.

Lemma 1.8 [2, Lemma 11]. *Let $n = p^\alpha$ and let l be any integer. Denote $p^\lambda = \gcd(l, n)$, $w = e^{2\pi i/n}$, and choose $0 \leq \rho \leq \alpha$. Then*

$$\sum_{\substack{1 \leq m \leq n \\ \gcd(m, n) = p^\rho}} w^{lm} = \begin{cases} 1, & \text{when } \alpha - \rho = 0, \\ p^{\alpha-\rho-1}(p-1), & \text{when } 1 \leq \alpha - \rho \leq \lambda, \\ -p^\lambda, & \text{when } \alpha - \rho = \lambda + 1, \\ 0, & \text{otherwise.} \end{cases}$$

We write $w = e^{2\pi i/p^\alpha}$ and $m = rp^\rho$, for $(r, p) = 1$. Then

$$\begin{aligned} \det(I - q\pi(g^{rp^\rho})) &= (1-q)^{n_0} \left(\prod_{\substack{1 \leq j \leq p-1 \\ (j, p)=1}} (1 - qw^{jrp^{\alpha+\rho-1}}) \right)^{n_1} \\ &\quad \dots \left(\prod_{\substack{1 \leq j \leq p^{\rho-1}-1 \\ (j, p)=1}} (1 - qw^{jrp^\rho}) \right)^{n_\alpha}. \end{aligned}$$

Applying Lemma 1.7 for all $1 \leq l \leq \alpha$, we obtain

$$\prod_{\substack{1 \leq j \leq p^l-1 \\ (j, p)=1}} (1 - qw^{jrp^{\alpha+\rho-l}}) = \begin{cases} (1-q)^{\varphi(p^l)} & \text{for } 1 \leq l \leq \rho, \\ (\Phi_{p^{l-\rho}}(q))^{p^\rho} & \text{for } \rho < l \leq \alpha, \end{cases}$$

from which we derive that

$$\det_W(I - q\pi(g^{rp^\rho})) = (1 - q)^{\sum_{i=0}^{\rho} n_i \varphi(p^i)} \left[\prod_{i=1}^{\alpha-\rho} [\Phi_{p^i}(q)]^{n_{\rho+i}} \right]^{p^\rho}. \quad (6)$$

Proof of Proposition 1.1. Let $g \in G$ be a fixed group generator. From (3) and Lemma 1.6, it follows that

$$P_\chi(q) = \frac{1}{p^\alpha} \left\{ F_A(q, \hat{G})(e) \overline{\chi(e)} + \sum_{k=1}^{p^\alpha-1} F_{\mathbb{C}[\Delta]}(q, \hat{G})(g^k) \det_W(I - q\pi(g^k)) \overline{\chi(g^k)} \right\}.$$

Write $k = rp^\rho$, with $(r, p) = 1$, and observe that by Lemma 1.4 $F_{\mathbb{C}[\Delta]}(q, \hat{G})(g^k)$ does not depend on r , and by (6) the same holds for $\det_W(I - q\pi(g^k))$, therefore,

$$P_\chi(q) = \frac{1}{p^\alpha} \left\{ h_P(q) + \sum_{\rho=0}^{\alpha-1} F_{\mathbb{C}[\Delta]}(q, \hat{G})(g^{p^\rho}) \det_W(I - q\pi(g^{p^\rho})) \cdot \left[\sum_{\substack{k=1 \\ \gcd(k, p^\alpha)=p^\rho}}^{p^\alpha-1} \overline{\chi(g^k)} \right] \right\}.$$

Applying Lemma 1.8 with $w^k = \chi(g^k) = e^{2\pi i k/p^\alpha}$ we obtain

$$\sum_{\substack{k=1 \\ \gcd(k, p^\alpha)=p^\rho}}^{p^\alpha} w^{-k} = \begin{cases} -1 & \text{if } \alpha - \rho = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the only term of the sum in ρ that survives is the last one, $\rho = \alpha - 1$. We now substitute (6) in $P_\chi(q)$ and apply Lemma 1.4 to obtain

$$P_\chi(q) = \frac{1}{p^\alpha} \{ h_P(q) - h_{P_{\alpha-1}}(q) [\Phi_p(q)]^{n_\alpha p^{\alpha-1}} \}.$$

Finally we observe that the second polynomial on the right is, up to the multiplicative factor p^α , the h -polynomial of $\tilde{P}_{\min}^\alpha$. $\square$

2. Proof of Theorem 1

We prove Theorem 1 for $\mu = \alpha$, the other cases follow by applying Proposition 1.1 to each of the polytopes P_μ . Part (1) of Theorem 1 follows from the fact that since the h -vectors of P and $P_{\min}^\alpha$ satisfy the Dehn–Sommerville equations, their difference does as well. Meanwhile, parts (2) and (3) are a direct consequence of the relation $\dim H^{2i}(X; \mathbb{C})^\chi = [h_i(P) - h_i(\tilde{P}_{\min}^\alpha)]/p^\alpha$, that follows from Proposition 1.1. To prove part (4), we note that the unimodality of the coefficients $h_i(P) - h_i(\tilde{P}_{\min}^\alpha)$, $1 \leq i \leq \lfloor d/2 \rfloor$, follows

from the hard Lefschetz theorem applied to each isotypical component $H^{2i}(X; \mathbb{C})^\chi$. The theorem holds for projective V -varieties and states the existence of the class of a hyperplane section $w \in H^2(X; \mathbb{C})$ such that the map $w^i: H^{d-i}(X; \mathbb{C}) \mapsto H^{d+i}(X; \mathbb{C})$ given by multiplication by w^i is an isomorphism for all $1 \leq i \leq d$. In particular, the map $w: H^{2i}(X; \mathbb{C}) \mapsto H^{2(i+1)}(X; \mathbb{C})$ that is the multiplication by w , is injective for $0 \leq i \leq \lfloor (d-1)/2 \rfloor$ (and surjective for $\lfloor (d-1)/2 \rfloor \leq i \leq d-1$). Since the toric variety X is complete, there is an embedding of X into a projective space $\mathbb{C}P^r$, for some dimension r (we refer to page 69 of [8]) and the action of G on X extends to a linear action on $\mathbb{C}P^r$. By an argument of S. Kleiman described in [13], it is proved that the action of any element of the group on $H^*(X, \mathbb{C})$ commutes with multiplication by w , and so the subspaces $H^*(X; \mathbb{C})^\chi$ are w -invariant and the map $w: H^{2i}(X; \mathbb{C})^\chi \mapsto H^{2(i+1)}(X; \mathbb{C})^\chi$ is injective for $0 \leq i \leq \lfloor (d-1)/2 \rfloor$.

3. Proof of Corollary 2

Part (4) of Theorem 1 implies that for $1 \leq j \leq d_\mu - 1$,

$$\begin{aligned} f_{j-1}^\mu - \tilde{f}_{j-1}^\mu &= \sum_{i=1}^j \binom{d_\mu - i}{d_\mu - j} (h_i^\mu - \tilde{h}_i^\mu) \\ &\geq \sum_{i=1}^j \binom{d_\mu - i}{d_\mu - j} (h_1^\mu - \tilde{h}_1^\mu) \\ &= \binom{d_\mu}{d_\mu - j + 1} (f_0^\mu - f_0^{\mu-1} - n_\mu p^\mu) \end{aligned}$$

and

$$f_{d_\mu-1}^\mu - \tilde{f}_{d_\mu-1}^\mu = \sum_{i=1}^{d_\mu-1} (h_i^\mu - \tilde{h}_i^\mu) \geq (d_\mu - 1)(f_0^\mu - f_0^{\mu-1} - n_\mu p^\mu).$$

The optimal polytope $\tilde{P}_{\text{opt}}^\mu$ that realizes the equalities and that presents the desired symmetry properties is constructed by taking a stellar subdivision of each p^μ -orbit of facets of $\tilde{P}_{\text{min}}^\mu$ where each vertex is added according to the *symmetric stacking step* described in [3]. The contribution to the h -polynomial is $p^\mu(q + \dots + q^{d_\mu-1})$. By performing the stellar subdivision to $(f_0^\mu - f_0^{\mu-1} - n_\mu p^\mu)/p^\mu$ p^μ -orbits of facets, we obtain a polytope P_{opt}^μ that is stacked over $\tilde{P}_{\text{min}}^\mu$ with the desired symmetries and with h -polynomial:

$$h_{P_{\text{opt}}^\mu}(q) = h_{\tilde{P}_{\text{min}}^\mu}(q) + (f_0^\mu - f_0^{\mu-1} - n_\mu p^\mu)(q + \dots + q^{d_\mu-1}). \quad \square$$

Since the vector $f(P^G)$ satisfies McMullen's g -conjecture, Billera and Lee's argument [5], [6] for the proof of sufficiency of this conjecture provides a way to construct a simplicial polytope P_{opt}^0 that realizes this f -vector. Given the numbers $\{f_0^\mu, \mu = 1, \dots, \alpha\}$ and the polytope P^G , Corollary 2 illustrates how to construct inductively, in α steps, an optimal polytope with the desired symmetry properties, and fulfilling the

face number relations, for all $\mu = 1, \dots, \alpha$ as equalities. In general, P_{opt}^μ is obtained from $P_{\text{opt}}^{\mu-1}$ by forming the free sum $P_{\text{opt}}^{\mu-1} * P_{\text{min}}^\mu$, and then applying successive stellar subdivisions to the orbits of facets of order p^μ , until $(f_0^\mu - f_0^{\mu-1} - n_\mu p^\mu)$ vertices have been added.

4. Proof of Corollary 3

We begin with the following proposition.

Proposition 4.1. *Under the assumptions of Theorem 1, if $f_i^\mu = \tilde{f}_i^\mu$ for some i , where $0 \leq i \leq d_\mu - 1$, then equality holds for all i , and P_μ is combinatorially isomorphic to the minimal polytope $\tilde{P}_{\text{min}}^\mu$.*

Proof. Suppose that equality holds for some i . Then it follows from Corollary 2 that $f_0^\mu = f_0^{\mu-1} + n_\mu p^\mu$, with consequent equality between the f -vectors of P_μ and $\tilde{P}_{\text{min}}^\mu$. Let $V(P_\mu)$ and $V(P_{\mu-1})$ be the sets of vertices of the polytopes P_μ and $P_{\mu-1}$ respectively. Fix a generator g of G and consider n_μ vertices $v_1, \dots, v_{n_\mu} \in V(P_\mu) \setminus V(P_{\mu-1})$ with respective G -orbits $Gv_1, \dots, Gv_{n_\mu}$, lying in mutually independent subspaces of dimension $\varphi(p^\mu)$ each, and generating a space of dimension $n_\mu \varphi(p^\mu)$. We describe each G -orbit Gv_k :

$$v_{i,j}^k = g^{(i-1)p^{\mu-1}+j-1} v_k, \quad 1 \leq i \leq p, \quad 1 \leq j \leq p^{\mu-1}, \quad 1 \leq k \leq n_\mu.$$

In this manner we enumerate $n_\mu p^\mu$, and therefore all the vertices of P_μ not in $P_{\mu-1}$. Let $x_k = v_{1,1}^k + \dots + v_{p,1}^k$. The following linearly independent relations hold for all k , with $1 \leq k \leq n_\mu$,

$$\begin{aligned} g^{j-1} x_k &= v_{1,j}^k + \dots + v_{p,j}^k, & 1 \leq j \leq p^{\mu-1}, \\ g^{p^{\mu-1}} x_k &= x_k. \end{aligned}$$

Thus the orbit Gx_k is left fixed by the action of $G_{\mu-1}$. Consider the following sets of vertices:

$$S_j^k = \{v_{1,j}^k, \dots, v_{p,j}^k\}, \quad 1 \leq j \leq p^{\mu-1}, \quad 1 \leq k \leq n_\mu.$$

Each facet F of P_μ has dimension $d_\mu - 1 = d_{\mu-1} - 1 + n_\mu \varphi(p^\mu)$, and is the convex hull of d_μ linearly independent vertices. None of the sets S_j^k can be contained in $V(F)$, otherwise $g^{j-1} x_k / p$, which has order $p^{\mu-1}$, would be an interior point of F . However, this contradicts the assumption that the G -action is proper. Thus F must contain at most $p - 1$ vertices from each S_j^k . Also F cannot contain more than $d_{\mu-1}$ vertices from $P_{\mu-1}$ since they must all be linearly independent. Moreover their convex hull must form a face of $P_{\mu-1}$ or else $F^{G_{\mu-1}}$ would not be a simplicial subcomplex of F , which contradicts the hypothesis that the G -action is proper. Therefore by a counting argument on the number of its vertices, F must be the convex hull of a facet of $P_{\mu-1}$ together with exactly $p - 1$ vertices from each S_j^k , for $j = 1, \dots, p^{\mu-1}$ and $k = 1, \dots, n_\mu$.

We now construct a bijection between P_μ and $\tilde{P}_{\min}^\mu = P_{\mu-1} * (\Delta_{p-1})^{n_\mu p^{\mu-1}}$ that is equivariant by the group action. The stacked polytope $(\Delta_{p-1})^{n_\mu p^{\mu-1}}$ is the free sum of n_μ copies of a polytope $P_{\min}(G, \varphi(p^\mu)) \cong (\Delta_{p-1})^{p^{\mu-1}}$ defined in the following way. We decompose each copy of the vector space $\mathbb{R}^{\varphi(p^\mu)}$ into a direct sum of copies of $\mathbb{R}^{p-1}$:

$$\mathbb{R}^{\varphi(p^\mu)} = R_1^k \oplus \cdots \oplus R_{p^{\mu-1}}^k,$$

where $R_i^k \cong \mathbb{R}^{p-1}$, for $i = 1, \dots, p^{\mu-1}$ and $k = 1, \dots, n_\mu$. Choose a rational basis $r_{1,i}^k, \dots, r_{p-1,i}^k$ for each R_i^k , and define $r_{p,i}^k = -(r_{1,i}^k + \cdots + r_{p-1,i}^k)$. Their convex hull is a $(p-1)$ -simplex. The group generator g acts freely on $R_1^k \oplus \cdots \oplus R_{p^{\mu-1}}^k$ as the cyclic permutation

$$(r_{1,1}^k, \dots, r_{1,p^{\mu-1}}^k, \dots, r_{p,1}^k, \dots, r_{p,p^{\mu-1}}^k)$$

and $P_{\min}(G, \varphi(p^\mu))$ is the convex hull of this G -orbit.

Define $\bar{S}_i^k = \{r_{1,i}^k, \dots, r_{p,i}^k\}$. Any set of vertices of $P_{\min}(G, \varphi(p^\mu))$ not containing any of the sets $\bar{S}_i^k$ is a face of this polytope. The bijection map is defined by sending each vertex $v_{i,j}^k$ to $r_{i,j}^k$. $\square$

To prove Corollary 3, we note that since the vector $h(P_\mu) - h(\tilde{P}_{\min}^\mu)$ is nonnegative and unimodal, and, moreover, satisfies the Dehn–Sommerville equations, if $h_i^\mu = \tilde{h}_i^\mu$ for some i , then $h_1^\mu - \tilde{h}_1^\mu = 0$. Consequently, $f_0^\mu = \tilde{f}_0^\mu = f_0^{\mu-1} + n_\mu p^\mu$. From Proposition 4.1 it follows that the f -vectors of P_μ and $\tilde{P}_{\min}^\mu$ are equal, and therefore so are their h -vectors, as well as the combinatorial isomorphism. $\square$

5. Proof of Theorem 4

Let P be a k -stacked polytope over $\tilde{P}_{\min}^\alpha$, where $1 \leq k \leq \lfloor d/2 \rfloor$, and let K and $\tilde{K}_{\min}^\alpha$ be the triangulations of P and $\tilde{P}_{\min}^\alpha$, respectively, described as in the definition of a polytope k -stacked over another polytope. We denote by ∂K (resp. $\partial \tilde{K}_{\min}^\alpha$) the triangulation induced on the boundary of K (resp. $\tilde{K}_{\min}^\alpha$), and by K° (resp. $(\tilde{K}_{\min}^\alpha)^\circ$) the open complex $K - \partial K$ (resp. $\tilde{K}_{\min}^\alpha - \partial \tilde{K}_{\min}^\alpha$). We adopt the convention that $f_{-1}(K^\circ) = 0$ for the open complex K° (resp. $(\tilde{K}_{\min}^\alpha)^\circ$) only, to ensure that $f_j(K) = f_j(\partial K) + f_j(K^\circ)$, for all j (resp. $\tilde{K}_{\min}^\alpha$). For integers $k \geq -1$ and $e \geq d$, define

$$g_k^{(e)}(P) = \sum_{j=-1}^k (-1)^{k-j} \binom{e-j-1}{e-k-1} f_j(P)$$

with the conventions $f_{-1}(P) = 1$ and $f_j(P) = 0$, if $j < -1$ or $j \geq d$.

Observe that $g_k^{(e)}(P) = 0$, if $k \leq -2$ or $k \geq e$, and $g_k^{(d+1)}(P) = g_{k+1} = h_{k+1} - h_k$, for $k = 0, \dots, \lfloor d/2 \rfloor - 1$, are the g -numbers of McMullen's g -conjecture. The following is a result of McMullen and Walkup [11, Theorem 2]. For all $e \geq d+1$ and $k = -1, \dots, e-1$,

$$g_k^{(e)}(\partial K) = -g_k^{(e)}(K^\circ) + (-1)^{e-d-1} g_{e-k-2}^{(e)}(K^\circ). \quad (7)$$

Since P is k -stacked over $\tilde{P}_{\min}^\alpha$ it follows that

$$f_j(K^\circ) = f_j((\tilde{K}_{\min}^\alpha)^\circ)$$

for all $0 \leq j \leq d - k - 1$, and by applying the symmetry relation [11, Theorem 1]

$$g_k^{(d+1)}(P) = -g_{d-k-1}^{(d+1)}(P), \quad k = -1, \dots, \lfloor (d-1)/2 \rfloor$$

we obtain

$$\begin{aligned} g_k^{(d+1)}(K^\circ) &= - \sum_{j=-1}^{d-k-1} (-1)^{d-k-1-j} \binom{d-j}{k+1} f_j(K^\circ) \\ &= - \sum_{j=-1}^{d-k-1} (-1)^{d-k-1-j} \binom{d-j}{k+1} f_j((\tilde{K}_{\min}^\alpha)^\circ) \\ &= g_k^{(d+1)}((\tilde{K}_{\min}^\alpha)^\circ). \end{aligned}$$

By (7) with $e = d + 1$ it follows that

$$\begin{aligned} g_k^{(d+1)}(\partial K) &= -g_k^{(d+1)}(K^\circ) + g_{d-k-1}^{(d+1)}(K^\circ) \\ &= -g_k^{(d+1)}((\tilde{K}_{\min}^\alpha)^\circ) + g_{d-k-1}^{(d+1)}((\tilde{K}_{\min}^\alpha)^\circ) \\ &= g_k^{(d+1)}(\partial \tilde{K}_{\min}^\alpha). \end{aligned}$$

Observe that $f_j(\partial K) = f_j(P) - f_j(\tilde{P}_{\min}^\alpha) + f_j(\partial \tilde{K}_{\min}^\alpha)$, for all $0 \leq j \leq d - k - 1$, and thus

$$g_k^{(d+1)}(\partial K) - g_k^{(d+1)}(\partial \tilde{K}_{\min}^\alpha) = g_k^{(d+1)}(P) - g_k^{(d+1)}(\tilde{P}_{\min}^\alpha) = 0.$$

Therefore $g_{k+1}(P) = g_{k+1}(\tilde{P}_{\min}^\alpha)$. $\square$

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