Asymptotically Efficient Triangulations of the d -Cube*

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Abstract. Let P and Q be polytopes, the first of “low” dimension and the second of “high” dimension. We show how to triangulate the product $P \times Q$ efficiently (i.e., with few simplices) starting with a given triangulation of Q . Our method has a computational part, where we need to compute an efficient triangulation of $P \times \Delta^m$, for a (small) natural number m of our choice. Δ^m denotes the m -simplex. Our procedure can be applied to obtain (asymptotically) efficient triangulations of the cube I^n : We decompose $I^n = I^k \times I^{n-k}$, for a small k . Then we recursively assume we have obtained an efficient triangulation of the second factor and use our method to triangulate the product. The outcome is that using $k = 3$ and $m = 2$, we can triangulate I^n with $O(0.816^n n!)$ simplices, instead of the $O(0.840^n n!)$ achievable before.

1. Introduction

“Simple” triangulations of the regular d -cube $I^d = [0, 1]^d$ have several applications, such as solving differential equations by finite-element methods or calculating fixed points. See, for example, [10]. In particular, it has brought special attention both from a theoretical point of view and from an applied one to determine the smallest size of a triangulation of the d -cube (see Section 14.5.2 of [8] for a recent survey). We point up that the general problem of computing the smallest triangulation of an arbitrary polytope is NP-complete even when restricted to dimension 3, see [2].

When we speak about *triangulations* of a polytope P of dimension d we mean decompositions of P into d -simplices that (i) use as vertices only vertices of P , and (ii) intersect face to face (i.e., forming a geometric simplicial complex). Some authors do not require these two conditions in triangulations. We will always require the first one,

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and when the second condition is not fulfilled, we call the decompositions *simplicial dissections* of P . The *size* of a triangulation or dissection T is its number of d -simplices and we denote it $|T|$. It is an open question whether high-dimensional cubes admit dissections with less simplices than needed in a triangulation. Actually, the minimum size of dissections of I^7 is unknown, while the minimum triangulation is known (see below).

The paper [3] describes a general method to obtain the smallest triangulation of a polytope P as the optimal integer solution of a certain linear program. The linear program has as many variables as d -simplices with the vertex set contained in the vertices of P exist. That is, $\binom{\# \text{vertices}}{\dim(P)+1}$ if the vertices of P are in general position and less than that if not. For the d -cube, the direct application of this method is impossible in practice beyond dimension 4 or 5. With a somewhat similar method but simplifying the system of equations via the symmetries of the cube, Anderson and Hughes [1] have calculated the smallest size among triangulations of the 6-cube and the 7-cube, in a computational tour-de-force which involved a problem with 1,456,318 variables and ad hoc ways of decomposing the system into smaller subsystems. The smallest sizes up to dimension 7 are shown in Table 1.

In order to compare sizes of triangulations of cubes in different dimensions, Todd [10] defines the *efficiency* of a triangulation T of the d -cube to be the number $(|T|/d!)^{1/d}$. This number is at most 1, since every simplex with integer vertices has a multiple of $1/d!$ as Euclidean volume, so $|T| \leq d!$. Triangulations of efficiency 1 (i.e., unimodular) can be easily constructed in any dimension. On the other extreme, Hadamard's inequality for determinants of matrices with coefficients in $[-1, 1]$ implies that the volume of every d -simplex inscribed in the regular d -cube $I^d = [0, 1]^d$ is at most $(d+1)^{(d+1)/2}/2^d d!$. Hence, every triangulation has size at least $2^d d!/(d+1)^{(d+1)/2}$ and efficiency at least $2/(d+1)^{(d+1)/2d} \approx 2/\sqrt{d+1}$.

Following the notation in Section 14.5.2 of [8], let φ_d and ρ_d be the smallest size and efficiency, respectively, of all triangulations of the cube of dimension d . The number φ_d (or some variations in which one or both of the conditions (i) and (ii) are not required) is known as the *simplicity* of the d -cube. Obviously, $\rho_d = (\varphi_d/d!)^{1/d}$.

In [6] (see also pages 283–284 of [8]), Haiman observes that a triangulation of I^{k+l} with $t_k t_l \binom{k+l}{k}$ simplices can be constructed from given triangulations of I^k and I^l with t_k and t_l simplices, respectively. With this, one easily concludes:

Theorem 1.1 [6]. *For every k and l , $\rho_{k+l}^{k+l} \leq \rho_k^k \rho_l^l$.*

Corollary 1.2. *The sequence $(\rho_i)_{i \in \mathbb{N}}$ converges and*

$$\lim_{i \rightarrow \infty} \rho_i \leq \rho_d, \quad \forall d \in \mathbb{N}.$$

Proof. We fix $d \in \mathbb{N}$ and $k \in \{1, \dots, d\}$. Haiman's theorem implies that, for every $i \in \mathbb{N}$,

$$\rho_{k+id} \leq \rho_k^{k/(k+id)} \rho_d^{id/(k+id)}.$$

Since the right-hand side converges to ρ^d when i grows, the d subsequences of indices modulo d , and hence the whole sequence $(\rho_i)_{i \in \mathbb{N}}$, have an upper limit bounded by ρ_d .

Table 1. Smallest size and efficiency of triangulations of I^d .

	Dimension							
	1	2	3	4	5	6	7	8
Smallest size (φ_d)	1	2	5	16	67	308	1493	$\leq 11,944$
Smallest efficiency (ρ_d)	1	1	0.941	0.904	0.890	0.868	0.840	≤ 0.859
Lower bound Hadamard's inequality	1	0.877	0.794	0.731	0.683	0.643	0.610	0.581
Lower bound Smith [9]	1	1	0.941	0.889	0.833	0.789	0.751	0.718

An upper limit bounded by every term in the sequence must coincide with the lower limit. $\square$

What is the limit of this sequence? In particular, is it positive or is it zero? The known values of ρ_d (up to $d = 7$) form a strictly decreasing sequence, as shown in Table 1, but it is not even known whether this occurs in general.

Concerning lower bounds, the only significant improvement to Hadamard's inequality has been obtained in [9], where the same volume argument is used, but with respect to a hyperbolic metric. The last row of the Table 1 shows the lower bound obtained, translated into efficiency of triangulations. For small dimensions, it is an excellent approximation of the smallest efficiency. Asymptotically, it only increases the lower bound obtained using Hadamard's inequality by a constant factor $\sqrt{3/2}$.

In this paper we propose a method to obtain efficient triangulations of a product polytope $P \times Q$ starting from a triangulation of Q and another of $P \times \Delta^{m-1}$, where Δ^{m-1} denotes a simplex of dimension $m - 1$ and m is any relatively small number. We apply this with P being a small-dimensional cube and Q a high-dimensional one, iteratively. This allows us to obtain asymptotically efficient triangulations of arbitrarily high-dimensional cubes from any (efficient) triangulation of $I^l \times \Delta^{m-1}$. Sections 3–5 explain our method, which is first outlined in Section 2. If the reader is happy with dissections, we cannot offer better efficiencies for them than for triangulations. Then Section 5, which contains most of the technicalities in this paper, can be skipped.

The asymptotic efficiency of the triangulations obtained, clearly depends on how good our triangulation of $I^l \times \Delta^{m-1}$ is. Finding the triangulation of $I^l \times \Delta^{m-1}$ which is optimal for our purposes reduces to an integer programming problem, similar to finding the smallest triangulation of that polytope (actually, it is the same system of linear equations, with a different objective function). Using the linear programming software CPLEX, we have solved the system for some values of l and m . The best triangulation we have found is one of $I^3 \times \Delta^2$, with which we obtain

$$\lim_{i \rightarrow \infty} \rho_i \leq \sqrt[3]{\frac{44/3}{27}} \approx 0.8159.$$

The best bound existing before was $\lim_{i \rightarrow \infty} \rho_i \leq \rho_7 = 0.840$. In other words, we prove that (asymptotically) the d -cube can be triangulated with $0.8159^d d!$ simplices, instead of the $0.840^d d!$ achievable before.

It has to be observed that, even if the particular triangulation of $I^3 \times \Delta^2$ that we use was obtained by an intensive computer calculation, once the triangulation is found it

is a simple task to check that it is indeed a triangulation and compute the asymptotic efficiency obtained from it. Actually, in Section 7 we use the so-called Cayley Trick [7] to do this checking with no need of computers at all. We also use the Cayley Trick to explore the minimum efficiency that can be obtained from the product $I^2 \times \Delta^k$ for any k . We briefly explain the trick in Section 6, where we also interpret our whole construction in terms of it.

2. Overview of the Method and Results

We start by describing Haiman's proof of Theorem 1.1, which is related to our method. Let T_k and T_l be triangulations of the regular cubes I^k and I^l , respectively. The product $T_k \times T_l$ of the two triangulations gives a decomposition of the cube $I^k \times I^l = I^{k+l}$ into $|T_k| \cdot |T_l|$ subpolytopes, each of them isomorphic to the product of simplices $\Delta^k \times \Delta^l$. It is well known that every triangulation of $\Delta^k \times \Delta^l$ has size $\binom{k+l}{k}$ (see, e.g., Chapter 7 of [5]). Hence, refining $T_k \times T_l$ in an arbitrary way one gets a triangulation of I^{k+l} of size $|T_k| \cdot |T_l| \cdot \binom{k+l}{k}$. The implication of this is that starting with a triangulation of a cube I^k of a certain efficiency ρ , one can construct a sequence of triangulations of I^{nk} for $n \in \mathbb{N}$ whose asymptotic efficiencies converge to ρ .

Our method is, in a way, similar. Starting from a triangulation of I^{n-1} and another one of $I^l \times \Delta^{m-1}$, we get one of I^{l+n-1} with the following general method to triangulate $P \times Q$ starting from a triangulation of Q and another one of $P \times \Delta^{m-1}$:

- (1) We first show (Sections 3 and 4) how to obtain triangulations of $P \times \Delta^{n-1}$ from triangulations of $P \times \Delta^{m-1}$, where $n - 1 = \dim(Q)$ is supposed to be much bigger than $m - 1$. We call our triangulations *multi-staircase triangulations*.
- (2) A triangulation of Q induces, as in Haiman's method, a decomposition of $P \times Q$ into polytopes isomorphic to $P \times \Delta^{n-1}$. Each of them can be triangulated using the previous paragraph, although this in principle only gives a dissection of $P \times Q$; if we want a triangulation, we have to apply (1) to all the polytopes $P \times \Delta^{n-1}$ in a compatible way. In Section 5 we show how to do this using an m -coloring of the vertices of Q .
- (3) The analysis of the size of the triangulation obtained is also carried out in Section 5.

In step (2) the final size of the triangulation is just the sum of the individual sizes of the triangulations used for the different subpolytopes $P \times \Delta^{n-1}$. In particular, if we are just interested in obtaining dissections, we can take an efficient triangulation of $P \times \Delta^{n-1}$ and repeat it in every subpolytope.

In step (1) the computation of the size is more complicated and to state it in a simple way we introduce the following definitions. When we speak about simplices in a polytope P we implicitly suppose that its vertices are vertices of P , and we identify the simplex with its vertex set.

Definition 2.1. Let P be a polytope of dimension l and let τ be a simplex of dimension $l + m - 1$ in $P \times \Delta^{m-1}$. Let $\{v_1, \dots, v_m\}$ be the vertices of Δ^{m-1} . Then τ , understood as a vertex set, decomposes as $\tau = \tau_1 \cup \dots \cup \tau_m$ with $\tau_i \subset P \times \{v_i\}$.

- (i) We define the sequence $(|\tau_1| - 1, \dots, |\tau_m| - 1)$ to be the *type* of the simplex τ . The *weight* of a simplex τ of type $(t_1, \dots, t_m)$ is the number $1 / \prod_{i=1}^m t_i!$.
- (ii) The *weighted size* of a triangulation T of $P \times \Delta^{m-1}$ is $\sum_{\tau \in T} \text{weight}(\tau)$, and the *weighted efficiency* is

$$\sqrt[l]{\frac{\sum_{\tau \in T} \text{weight}(\tau)}{m^l}}.$$

With this, our main result can be stated as:

Theorem 2.2. *Consider polytopes P of dimension l and Q of dimension $n - 1$. Let m be such that $m \leq n$. Given a triangulation T_0 of $P \times \Delta^{m-1}$ of weighted size t_0 and a triangulation T_Q of Q , then there are triangulations of $P \times Q$ with size at most*

$$|T_Q| t_0 \left(\frac{n}{m} + l \right)^l.$$

Using this method to triangulate $I^{l+n-1} = I^l \times I^{n-1}$ starting from a triangulation of I^{n-1} and another one of $I^l \times \Delta^{m-1}$ we conclude:

Theorem 2.3. *If there is a triangulation T_0 of $I^l \times \Delta^{m-1}$ with weighted efficiency ρ_0 , then for every $\varepsilon > 0$ and all n bigger than $l m \rho_0 / \varepsilon$ we have*

$$\rho_{n+l-1}^{n+l-1} \leq \rho_{n-1}^{n-1} (\rho_0 + \varepsilon)^l.$$

As a consequence,

$$\lim_{i \rightarrow \infty} \rho_i \leq \rho_0.$$

Observe that the definition of weighted efficiency makes sense even in the case $m = 1$, where all the simplices of $P \times \Delta^0$ have the same type, equal to (l) if $\dim(P) = l$, and the same weight, equal to $1/l!$. In particular, the weighted efficiency of a triangulation of $I^l \times \Delta^0$ is the same as the usual “non-weighted” one. Another common point between efficiency and weighted efficiency is that the weighted efficiency of a triangulation of $I^l \times \Delta^{m-1}$ is always less than or equal to 1, and it is 1 if and only if the triangulation is unimodular; i.e., if every simplex has volume $1/(l+m-1)!$ (this is proved in Section 6).

We now describe the practical results obtained. The last theorem leads us to study the smallest weighted efficiency of triangulations of $I^l \times \Delta^{m-1}$; we denote it by $\rho_{l,m}$. This number can be calculated by minimizing a linear form over the so-called *universal polytope* of all the triangulations of $I^l \times \Delta^{m-1}$.

The definition of this universal polytope for triangulations of an arbitrary polytope P is as follows (see [3] for further details): Let $\Sigma(P)$ be the set of all the simplices of maximal dimension which use as vertices only vertices of P . Given a triangulation T of P , its *incidence vector* $V_T \in \mathbb{R}^{\Sigma(P)}$ has a 1 in the coordinates corresponding to simplices of T and a 0 in the others. The *universal polytope* of P is $\text{conv}\{V_T : T \text{ is a triangulation of } P\}$.

In our case, to calculate the smallest weighted size (and efficiency), we have to minimize over the universal polytope of $I^l \times \Delta^{m-1}$ the linear form having as coefficient of each simplex its weight $1/\prod_{i=1}^m t_i!$. (To minimize non-weighted efficiency and size

Table 2. Values of $\rho_{l,m}$ for $l \leq 2$ or for $l = 3$ and $m \leq 3$.

l	m			
	1	2	3	≥ 3
1	1	1	1	1
2	1	$\sqrt{\frac{3}{4}} \approx 0.866$	$\sqrt{\frac{7}{9}} \approx 0.8819$	$\geq \sqrt{\frac{3}{4}}$
3	$\sqrt[3]{\frac{5/6}{1}} \approx 0.941$	$\sqrt[3]{\frac{14/3}{8}} \approx 0.8355$	$\sqrt[3]{\frac{44/3}{27}} \approx 0.8159$	

one just uses the functional with all coefficients equal to 1.) One of the results in [3] is a description of the vertices of the universal polytope as integer solutions of a certain system of linear inequalities derived from the oriented matroid of P . Thus, our minimization problem is restated as an integer linear programming problem.

In order to apply our method, we have used the program `UNIVERSAL BUILDER` by de Loera and Peterson [4]. Given as input the vertices of P , the program generates the linear system of equations defining the universal polytope of P . The output is a file readable by the linear programming software `CPLEX`. We have created a routine that generates our particular objective function. Table 2 shows the results obtained in the cases we have been able to solve. Note that $\rho_{l,1} = \rho_l$ and so the column $m = 1$ in Table 2 is the same as the second row of Table 1.

The fact that $\rho_{1,m} = 1$ for every m reflects that every triangulation of the prism $I \times \Delta^{m-1}$ is unimodular. In the case of $\rho_{2,m}$ we prove (Section 7.1) that the smallest weighted efficiency is always $\sqrt{\lceil 3m^2/4 \rceil}/m^2$. That is to say, $\sqrt{3}/4$ for even m and $\sqrt{3}/4 + \Theta(m^{-2})$ for odd m .

The computation of $\rho_{3,3}$ involved a system with 74,400 variables, whose resolution by `CPLEX` took 37 hours of CPU on a SUN UltraSparc.

3. Polyhedral Subdivision of $P \times \Delta^{k_1+\dots+k_m-1}$ Induced by a Polyhedral Subdivision of $P \times \Delta^{m-1}$

We call *polyhedral subdivisions* of a polytope P its face-to-face partitions into subpolytopes which only use vertices of P as vertices.

Let P be a polytope of dimension l . Let m and $k_1, \dots, k_m$ be natural numbers and let $n := k_1 + \dots + k_m$. Let $v_1, \dots, v_m$ be the m vertices of the standard simplex Δ^{m-1} and let $v_1^1, \dots, v_{k_1}^1, \dots, v_1^m, \dots, v_{k_m}^m$ be the vertices of Δ^{n-1} . Observe that, implicitly, we have the following surjective map:

$$\begin{array}{ccc} \text{vert}(\Delta^{n-1}) & \rightarrow & \text{vert}(\Delta^{m-1}), \\ v_j^i & \mapsto & v_i. \end{array}$$

This map uniquely extends to an affine projection $\pi_0: \Delta^{n-1} \rightarrow \Delta^{m-1}$. In turn, this induces a projection

$$\begin{array}{ccc} \pi = 1 \times \pi_0: P \times \Delta^{n-1} & \rightarrow & P \times \Delta^{m-1}, \\ (p, a) & \mapsto & (p, \pi_0(a)). \end{array}$$

Given the projection π and a polyhedral subdivision S of the target polytope $P \times \Delta^{m-1}$, it is obvious that the inverse images $\pi^{-1}(B)$ of the subpolytopes of S form a subdivision of $P \times \Delta^{n-1}$ into subpolytopes matching face to face. In a more general projection it would not be true that those subpolytopes use as vertices only vertices of $P \times \Delta^{n-1}$. However, it is true in our case:

Lemma 3.1. *Let $B \subset P \times \Delta^{m-1}$ be a subpolytope with $\text{vert}(B) \subset \text{vert}(P \times \Delta^{m-1})$. Let π be the projection considered before. Let $\tilde{B} = \{(p, v_j^i) : (p, v_i) \in \text{vert}(B)\}$. Then*

$$\pi^{-1}(B) \subset \text{conv}(\tilde{B}).$$

Proof. Let (p, a) be a point of $\pi^{-1}(B)$, so $(p, \pi_0(a)) \in B$. Let us write a as a convex combination of the vertices of Δ^{n-1} , that is, $a = \sum_{i=1}^m \sum_{j=1}^{k_i} \lambda_j^i v_j^i$, with $\lambda_j^i \geq 0, \forall i, j$ and $\sum_i \sum_j \lambda_j^i = 1$. Without loss of generality, we can suppose that none of the sums $\sum_{j=1}^{k_i} \lambda_j^i$ is zero, otherwise everything “happens” on a face of Δ^{m-1} and we can restrict the statement to that face.

Let $(p_1^1, v_1), \dots, (p_{l_1}^1, v_1), (p_1^2, v_2), \dots, (p_{l_2}^2, v_2), \dots, (p_1^m, v_m), \dots, (p_{l_m}^m, v_m)$ be the vertices of B , so we have $\tilde{B} = \{(p_h^i, v_j^i) : i = 1, \dots, m, j = 1, \dots, k_i, h = 1, \dots, l_i\}$. We now write $(p, \pi_0(a))$ as a convex combination of the vertices of B , that is,

$$(p, \pi_0(a)) = \sum_{i=1}^m \sum_{h=1}^{l_i} \mu_h^i (p_h^i, v_i),$$

with $\mu_h^i \geq 0, \forall i, h$, and $\sum_i \sum_h \mu_h^i = 1$. Observe that $\sum_{j=1}^{k_i} \lambda_j^i = \sum_{h=1}^{l_i} \mu_h^i$ for every i , because $\pi_0(a) = \sum_{i=1}^m \sum_{j=1}^{k_i} \lambda_j^i v_i$. Then it is easy to check that

$$(p, a) = \sum_{i=1}^m \sum_{j=1}^{k_i} \sum_{h=1}^{l_i} \frac{\lambda_j^i \mu_h^i}{\sum_{j=1}^{k_i} \lambda_j^i} (p_h^i, v_j^i) \quad \text{and} \quad 1 = \sum_{i=1}^m \sum_{j=1}^{k_i} \sum_{h=1}^{l_i} \frac{\lambda_j^i \mu_h^i}{\sum_{j=1}^{k_i} \lambda_j^i}.$$

That is, (p, a) is a convex combination of points of $\tilde{B}$, as we wanted to prove. $\square$

Corollary 3.2. *Under the previous conditions:*

- (i) $\pi^{-1}(B) = \text{conv}(\pi^{-1}(\text{vert}(B)))$.
- (ii) $\text{vert}(\pi^{-1}(B)) = \tilde{B}$.

Proof. In both equalities, the inclusion from right to left is trivial. In the second one, observe that if (p, v_j^i) is in $\tilde{B}$, then (p, v_i) is a vertex of B . Therefore, $(p, v_j^i) \in \pi^{-1}(B)$ and it is a vertex of $P \times \Delta^{n-1}$, which implies that it is a vertex of $\pi^{-1}(B)$.

Inclusions from left to right follow from Lemma 3.1, because:

- (1) $\tilde{B} \subset \pi^{-1}(\text{vert}(B)) \Rightarrow \pi^{-1}(B) \subset \text{conv}(\tilde{B}) \subset \text{conv}(\pi^{-1}(\text{vert}(B)))$.
- (2) $\pi^{-1}(B) \subset \text{conv}(\tilde{B}) \Rightarrow \text{vert}(\pi^{-1}(B)) \subset \text{vert}(\text{conv}(\tilde{B})) = \tilde{B}$, where the last equality follows from the fact that all the elements of $\tilde{B}$ are vertices of $P \times \Delta^{n-1}$. $\square$

Corollary 3.3. *Every polyhedral subdivision S of $P \times \Delta^{m-1}$ induces a polyhedral subdivision $\tilde{S} := \pi^{-1}(S) = \{\pi^{-1}(B) : B \in S\}$ of $P \times \Delta^{n-1}$. Furthermore, the vertices of each subpolytope $\pi^{-1}(B)$ in this subdivision are $\tilde{B} := \{(p, v_i^j) : (p, v_i) \in \text{vert}(B)\}$.*

4. Triangulation of $P \times \Delta^{k_1+\dots+k_m-1}$ Induced by a Triangulation of $P \times \Delta^{m-1}$

We suppose now that the polyhedral subdivision S of $P \times \Delta^{m-1}$ is a triangulation. A convenient graphic representation of the vertices of the polytope $P \times \Delta^{m-1}$ is as a grid whose rows represent vertices of P and whose m columns represent the vertices $v_1, \dots, v_m$ of Δ^{m-1} . In order to represent a subset of vertices of $P \times \Delta^{m-1}$ we just mark the corresponding squares. In the same way we can represent the vertices of $P \times \Delta^{k_1+\dots+k_m-1}$, but now it is convenient to divide the grid horizontally in blocks, each of them corresponding to each k_i and containing the vertices $v_1^i, \dots, v_{k_i}^i$ of $\Delta^{k_1+\dots+k_m-1}$.

Figure 1 shows how to obtain, with the notation of the previous section, the set $\tilde{B}$ associated to a simplex $B \in S$ in this graphic representation. In the i th block, rows corresponding to vertices (p, v_i) in B have all their squares marked. Restricting $\tilde{B}$ to that block gives precisely $\text{conv}(\{p_1^i, \dots, p_{l_i}^i\}) \times \text{conv}(\{v_1^i, \dots, v_{k_i}^i\})$, with the notation used for the vertices of B in the proof of Lemma 3.1.

Since B is a simplex, any subset of its vertices also forms a simplex; thus, the restriction of $\tilde{B}$ to each block is a product of two simplices. Recall that the *staircase triangulation* of the product of two simplices $\Delta^k \times \Delta^l$ is the one whose $\binom{k+l}{k}$ simplices are all the possible monotone staircases in a grid of size $(k+1) \times (l+1)$. Following this analogy, we define *multi-staircases* as follows:

Definition 4.1. Let $\tilde{B} = \pi^{-1}(B)$ be a subpolytope of $P \times \Delta^{k_1+\dots+k_m-1}$ of the kind obtained in Section 3.

- (i) A *multi-staircase* in $\tilde{B}$ is any subset of vertices which, restricted to every block, forms a monotone staircase.
- (ii) The *multi-staircase triangulation* of $\tilde{B}$ is the one which has as simplices the different multi-staircases (see Fig. 2).

Lemma 4.2. *The multi-staircases indeed form a triangulation of $\tilde{B}$ and taking the multi-staircase triangulations of the different $\tilde{B}$'s obtained from a triangulation of*

Fig. 1. How to obtain $\tilde{B}$ from B .

Fig. 2. The four multi-staircases (right) forming the multi-staircase triangulation of the polytope $\tilde{B}$ on the left.

$P \times \Delta^{m-1}$ we get a triangulation of $P \times \Delta^{k_1+\dots+k_m-1}$, which we call *multi-staircase triangulation*.

Proof. It is clear that multi-staircases form full-dimensional simplices in $\tilde{B}$. A way to prove that a collection T of full-dimensional simplices in a polytope $\tilde{B}$ is a triangulation is to show that:

- (1) They induce a triangulation on one face of $\tilde{B}$.
- (2) For each full-dimensional simplex in the collection, the removal of any single vertex produces a codimension 1 simplex either
 - lying on a facet of $\tilde{B}$ and not contained in any other simplex of T , or
 - contained in exactly another full-dimensional simplex of T which is separated from the first one by their common facet.

In our case, the first condition follows by induction on $k_1 + \dots + k_m$, the base case being $k_1 + \dots + k_m = m$.

For the second condition, let σ be a multi-staircase, and let (p, v) be a vertex in it. Then one of the following three things happens (Fig. 3 gives an example of a multi-staircase):

- If (p, v) is the only point of σ in its column, then no other multi-staircase contains $\sigma \setminus \{(p, v)\}$. Removing (p, v) indeed produces a codimension 1 simplex contained in a facet $P \times \Delta^{k_1+\dots+k_m-2}$ of $P \times \Delta^{k_1+\dots+k_m-1}$.
- If (p, v) is the only point of σ in a row within a block, then no other multi-staircase contains $\sigma \setminus \{(p, v)\}$. Removing (p, v) indeed produces a codimension 1 simplex in a facet of $\tilde{B}$ of the form $\pi^{-1}(B \setminus \{(p, \pi_0(v))\})$ (remember that $B = \pi(\tilde{B})$ is a simplex).

Fig. 3. An example of a multi-staircase.

- If (p, v) is an elbow in the multi-staircase, then removing it leads to a uniquely different way of completing the multi-staircase. More precisely, let (p', v) and (p, v') be the points of the multi-staircase adjacent to (p, v) . Then removing (p, v) and inserting (p', v') produces the other possible multi-staircase. The obvious affine dependency $(p, v) + (p', v') = (p, v') + (p', v)$ implies that the two multi-staircases lie in opposite sides of their common facet. $\square$

Lemma 4.3. *Let l_i be the number of vertices of B in the i th column. Then the multi-staircase triangulation of $\tilde{B}$ has exactly $\prod_{i=1}^m \binom{k_i-1+l_i-1}{k_i-1}$ simplices.*

Proof. In each sub-block there are $\binom{k_i-1+l_i-1}{k_i-1}$ possible monotone staircases. $\square$

5. A Triangulation of $P \times Q$

We consider polytopes P of dimension l and Q of dimension $n - 1$. We assume we are given a triangulation T_Q of Q , which induces a decomposition of $P \times Q$ into cells isomorphic to $P \times \Delta^{n-1}$. Then any decomposition $n := k_1 + \dots + k_m$ allows us to apply the procedure of Sections 3 and 4 to triangulate the cells $P \times \Delta^{n-1}$ starting from a triangulation T_0 of $P \times \Delta^{m-1}$.

There are two important tasks remaining: first, show that the triangulations of the different $P \times \Delta^{n-1}$ can be achieved in a coordinated way to obtain a real triangulation of $P \times Q$; second, analyze the efficiency of that triangulation. Both of them will be done in this section, using the following trick:

Consider a partition of the vertices of Q into m “colors”. Then in each subpolytope $P \times \Delta^{n-1}$ of $P \times Q$ the vertices of the factor Δ^{n-1} are colored as well. We use this coloring to construct the projections $\Delta^{n-1} \rightarrow \Delta^{m-1}$ we need for each of them. Then on each common face (isomorphic to $P \times \Delta^k$, $k < n - 1$) of two of the cells $P \times \Delta^{n-1}$ we get the multi-staircase triangulation induced by the m -coloring of Q restricted to that face. Hence, the triangulations of the different cells $P \times \Delta^{n-1}$ intersect face-to-face, and we obtain a triangulation $T_{P \times Q}$ of $P \times Q$, which we call the *multi-staircase triangulation* of $P \times Q$.

In order to analyze the size of $T_{P \times Q}$, we suppose that the m -coloring of the vertices of Q is chosen at random with a uniform distribution.

For each $\sigma \in T_Q$ let $\sigma_i := \{\text{vertices of } \sigma \text{ colored } i\}$, and for each $\tau \in T_0$, $\tau_i := \{\text{vertices of } \tau \text{ over the } i\text{th vertex of } \Delta^{m-1}\}$. By Lemma 4.3, the triangulation $T_{P \times Q}$ we obtain has

$$\sum_{\sigma \in T_Q} \sum_{\tau \in T_0} \prod_{i=1}^m \binom{|\sigma_i| - 1 + |\tau_i| - 1}{|\tau_i| - 1}$$

simplices. The expected value of this sum, when the coloring is random, equals the sum of the expected values. So let us fix a pair of simplices $\tau \in T_0$ and $\sigma \in T_Q$ and calculate the expected value of

$$\prod_{i=1}^m \binom{|\sigma_i| - 1 + |\tau_i| - 1}{|\tau_i| - 1}.$$

We say $l_i := |\tau_i|$ and $k_i := |\sigma_i|$. The l_i 's are considered constants, while the k_i 's are random variables depending on the coloring. They follow a multinomial distribution, in which the probability of the m -tuple $(k_1, \dots, k_m)$ is $P(k_1, \dots, k_m) = n! / (k_1! \cdots k_m! m^n)$.

Since $\binom{x}{n} = x^n / n!$, where $x^n := x(x-1) \cdots (x-n+1)$ is the n th falling power of x , we can write

$$E \left(\prod_{i=1}^m \binom{k_i - 1 + l_i - 1}{l_i - 1} \right) = \frac{E(\prod_{i=1}^m (k_i - 1 + l_i - 1)^{\underline{l_i-1}})}{\prod_{i=1}^m (l_i - 1)!}.$$

For the numerator in the right-hand side of this equation we use the following result from [11] (Theorem 4.4.4):

Theorem 5.1. *Let $X_1, \dots, X_m$ be scalar random variables. Then the expected value of $\prod_{i=1}^m X_i^{v_i}$ is given by the formal calculus*

$$E \left(\prod_{i=1}^m X_i^{v_i} \right) = \left[\frac{\partial^{v_1 + \dots + v_m}}{\partial z_1^{v_1} \cdots \partial z_m^{v_m}} E \left(\prod_{i=1}^m z_i^{X_i} \right) \right]_{z_1 = \dots = z_m = 1}$$

in which extra variables z_i , $i = 1, \dots, m$, appear with formal purposes.

Lemma 5.2. *Let $l_1, \dots, l_m$ be positive integers with $\sum l_i = l + m$ and let $k_1, \dots, k_m$ be random variables obeying a multinomial distribution with $\sum k_i = n$. Then*

$$E \left(\prod_{i=1}^m (k_i - 1 + l_i - 1)^{\underline{l_i-1}} \right) \leq \left(l + \frac{n}{m} \right)^l.$$

Proof. If some l_i equals 1 we can neglect it in the statement: we remove it and will still have $\sum (l_i - 1) = l$. Hence we assume $l_i \geq 2$ for every i .

We use the following formula for expectations under a multinomial distribution, again taken from [11] (p. 53, last formula in the proof of Theorem 4.2.1):

$$E \left(\prod_{i=1}^m z_i^{k_i} \right) = \left(\sum_{i=1}^m \frac{1}{m} z_i \right)^n.$$

Since $\sum_{i=1}^m (l_i - 1) = l$ and only the k_i depend on the random process, applying Theorem 5.1 to the random variables $X_i := k_i + l_i - 2$ and using the equality above we get

$$\begin{aligned} E \left(\prod_{i=1}^m (k_i - 1 + l_i - 1)^{\underline{l_i-1}} \right) &= \left[\frac{\partial^l}{\partial z_1^{l_1-1} \cdots \partial z_m^{l_m-1}} E \left(\prod_{i=1}^m z_i^{k_i-1+l_i-1} \right) \right]_{\vec{z}=1} \\ &= \left[\frac{\partial^l}{\partial z_1^{l_1-1} \cdots \partial z_m^{l_m-1}} \left(\prod_{i=1}^m z_i^{l_i-2} E \left(\prod_{i=1}^m z_i^{k_i} \right) \right) \right]_{\vec{z}=1} \\ &= \left[\frac{\partial^l}{\partial z_1^{l_1-1} \cdots \partial z_m^{l_m-1}} \left(\prod_{i=1}^m z_i^{l_i-2} \left(\sum_{i=1}^m \frac{1}{m} z_i \right)^n \right) \right]_{\vec{z}=1} \end{aligned}$$

$$\begin{aligned}
&\stackrel{(*)}{\leq} \prod_{i=1}^m \left(l_i - 2 + \frac{n}{m} \right)^{l_i-1} \\
&\leq \prod_{i=1}^m \left(l + \frac{n}{m} \right)^{l_i-1} = \left(l + \frac{n}{m} \right)^l,
\end{aligned}$$

where the last inequality comes from $l_i \leq l+1$ (since $\sum (l_i - 1) = l$ and $l_i - 1 \geq 1$), and the one marked with an asterisk needs to be proved. For this we say $F_{l_1, \dots, l_m}(\bar{z}) := \prod_{i=1}^m z_i^{l_i-2}$ and $G_n(\bar{z}) := \left(\sum_{i=1}^m \frac{1}{m} z_i \right)^n$. Using that

$$\frac{\partial^k}{\partial x^k} f(x) g(x) = \sum_{j=0}^k \binom{k}{j} \frac{\partial^j}{\partial x^j} f(x) \frac{\partial^{k-j}}{\partial x^{k-j}} g(x)$$

we come up to

$$\frac{\partial^{l_i-1}}{\partial z_i^{l_i-1}} (F_{l_1, \dots, l_m}(\bar{z}) G_n(\bar{z})) = \sum_{j=0}^{l_i-1} \binom{l_i-1}{j} (l_i-2) \frac{\partial^{l_i-1-j}}{\partial z_i^{l_i-1-j}} \left(\frac{1}{m} \right)^j n^j F_{l_1, \dots, j+1, \dots, l_m}(\bar{z}) G_{n-j}(\bar{z}).$$

Therefore, since

$$\frac{\partial^l}{\partial z_1^{l_1-1} \dots \partial z_m^{l_m-1}} h(\bar{z}) = \frac{\partial^{l_m-1}}{\partial z_m^{l_m-1}} \left(\dots \left(\frac{\partial^{l_1-1}}{\partial z_1^{l_1-1}} (h(\bar{z})) \right) \dots \right)$$

and

$$[F_{l_1, \dots, l_m}(\bar{z})]_{\bar{z}=1} = [G_n(\bar{z})]_{\bar{z}=1} = 1,$$

we get

$$\begin{aligned}
&\left[\frac{\partial^l}{\partial z_1^{l_1-1} \dots \partial z_m^{l_m-1}} F_{l_1, \dots, l_m}(\bar{z}) G_n(\bar{z}) \right]_{\bar{z}=1} \\
&= \sum_{j_1=0}^{l_1-1} \dots \sum_{j_m=0}^{l_m-1} \left(\prod_{i=1}^m \binom{l_i-1}{j_i} (l_i-2) \frac{\partial^{l_i-1-j_i}}{\partial z_i^{l_i-1-j_i}} \left(\frac{1}{m} \right)^{j_i} (n - j_1 - \dots - j_{i-1})^{j_i} \right) \\
&= \sum_{j_1=0}^{l_1-1} \dots \sum_{j_m=0}^{l_m-1} \left(\prod_{i=1}^m \binom{l_i-1}{j_i} (l_i-2) \frac{\partial^{l_i-1-j_i}}{\partial z_i^{l_i-1-j_i}} \left(\frac{1}{m} \right)^{j_i} \right) n^{\sum j_i} \\
&\leq \sum_{j_1=0}^{l_1-1} \dots \sum_{j_m=0}^{l_m-1} \left(\prod_{i=1}^m \binom{l_i-1}{j_i} (l_i-2) \frac{\partial^{l_i-1-j_i}}{\partial z_i^{l_i-1-j_i}} \left(\frac{1}{m} \right)^{j_i} \right) n^{\sum j_i} \\
&= \prod_{i=1}^m \left(\sum_{j_i=0}^{l_i-1} \binom{l_i-1}{j_i} (l_i-2) \frac{\partial^{l_i-1-j_i}}{\partial z_i^{l_i-1-j_i}} \left(\frac{1}{m} \right)^{j_i} n^{j_i} \right) = \prod_{i=1}^m \left(l_i - 2 + \frac{n}{m} \right)^{l_i-1},
\end{aligned}$$

as wanted. $\square$

This lemma is crucial to prove the two theorems announced in Section 2. Indeed, Theorem 2.2 is just a version of the following more precise statement:

Lemma 5.3. *Consider polytopes P of dimension l and Q of dimension $n - 1$. Let m be such that $m \leq n$. Given a triangulation T_0 of $P \times \Delta^{m-1}$ of weighted size t_0 and a triangulation T_Q of Q , the expected size of the multi-staircase triangulation $T_{P \times Q}$ of $P \times Q$ is bounded above by*

$$|T_Q| t_0 \left(\frac{n}{m} + l \right)^l.$$

Proof. Lemma 5.2 implies that

$$\begin{aligned} E \left(\prod_{i=1}^m \binom{k_i - 1 + l_i - 1}{l_i - 1} \right) &= E \left(\prod_{i=1}^m \frac{(k_i - 1 + l_i - 1)^{l_i - 1}}{(l_i - 1)!} \right) \\ &\leq \frac{1}{\prod_{i=1}^m (l_i - 1)!} \left(l + \frac{n}{m} \right)^l. \end{aligned}$$

Hence,

$$\begin{aligned} E \left(\sum_{\sigma \in T_Q} \sum_{\tau \in T_0} \prod_{i=1}^m \binom{k_i - 1 + l_i - 1}{l_i - 1} \right) \\ \leq \sum_{\sigma \in T_Q} \sum_{\tau \in T_0} \frac{1}{\prod_{i=1}^m (l_i - 1)!} \left(\frac{n}{m} + l \right)^l \\ = |T_Q| \left(\sum_{\tau \in T_0} \frac{1}{\prod_{i=1}^m (l_i - 1)!} \right) \left(\frac{n}{m} + l \right)^l = |T_Q| t_0 \left(\frac{n}{m} + l \right)^l. \quad \square \end{aligned}$$

Proof of Theorem 2.3. Let $t_0 = \rho_0^l m^l$ be the weighted size of the triangulation in the statement. For any $n \geq lm\rho_0/\varepsilon$,

$$\begin{aligned} \rho_0 + \varepsilon &\geq \rho_0 \frac{m}{n} \left(\frac{n}{m} + l \right), \\ n^l (\rho_0 + \varepsilon)^l &\geq \rho_0^l m^l \left(\frac{n}{m} + l \right)^l = t_0 \left(\frac{n}{m} + l \right)^l, \\ \frac{(n + l - 1)!}{(n - 1)!} (\rho_0 + \varepsilon)^l &\geq t_0 \left(\frac{n}{m} + l \right)^l. \end{aligned}$$

Theorem 2.2, with $Q = I^{n-1}$ and $P = I^l$, tells us that

$$\varphi_{n+l-1} \leq \varphi_{n-1} t_0 \left(\frac{n}{m} + l \right)^l,$$

or, in other words,

$$\frac{(n + l - 1)! \rho_{n+l-1}^{n+l-1}}{(n - 1)! \rho_{n-1}^{n-1}} \leq t_0 \left(\frac{n}{m} + l \right)^l.$$

So, we have proved the first part of the theorem:

$$\forall \varepsilon > 0, \quad \forall n \geq \frac{lm\rho_0}{\varepsilon}, \quad \rho_{n+l-1}^{n+l-1} \leq \rho_{n-1}^{n-1} (\rho_0 + \varepsilon)^l.$$

The second part of the statement follows from the first one with arguments similar to those of Corollary 1.2. Recursively, we have that

$$\forall \varepsilon > 0, \quad \forall i \in \mathbb{N}, \quad \forall n \geq \frac{lm\rho_0}{\varepsilon}, \quad \rho_{n+il} \leq \rho_n^{n/(n+il)}(\rho_0 + \varepsilon)^{il/(n+il)}.$$

This implies that for the given l and any fixed n , taking $\varepsilon = lm\rho_0/n$ we get

$$\lim_{i \rightarrow \infty} \rho_{n+il} \leq \rho_0 + \frac{lm\rho_0}{n}.$$

That is, the sequence of indices congruent to n modulo l has limit bounded by the right-hand side. Since we can make n as big as we want and the rest of the right-hand side are constants, the l subsequences of indices modulo l have limit bounded by ρ_0 , hence the limit of the whole sequence has this bound. $\square$

6. Interpretation of Our Method via the Cayley Trick

The Cayley Trick allows us to study triangulations of a product $P \times \Delta^{n-1}$ as mixed subdivisions of the Minkowski sum $P + \dots + P$ (n summands). We overview this method here, but the reader should look at [7] for more details.

Let $Q_1, \dots, Q_m \subset \mathbb{R}^d$ be convex polytopes of vertex sets $\mathcal{A}_i$. Consider their Minkowski sum, defined as

$$\sum_{i=1}^m Q_i = \{x_1 + \dots + x_m : x_i \in Q_i\}.$$

We understand $\sum_{i=1}^m Q_i$ as a *marked polytope*, whose associated point configuration is $\sum_{i=1}^m \mathcal{A}_i := \{q_1 + \dots + q_m : q_i \in \mathcal{A}_i\}$. Here a *marked polytope* is a pair $(P, \mathcal{A})$ where P is a polytope and $\mathcal{A}$ is a finite set of points of P including all the vertices. *Subdivisions* of a marked polytope are defined in Chapter 7 of [5] (sometimes they are called subdivisions of $\mathcal{A}$). Roughly speaking, they are the polyhedral subdivisions of P which use only elements of $\mathcal{A}$ as vertices (but perhaps not all of them). They form a poset under the refinement relation. The minimal elements are the triangulations of $\mathcal{A}$.

A subset B of $\sum_{i=1}^m \mathcal{A}_i$ is called *mixed* if $B = B_1 + \dots + B_m$ for some non-empty subsets $B_i \subset \mathcal{A}_i$, $i = 1, \dots, m$. A *mixed subdivision* of $\sum_{i=1}^m Q_i$ is a subdivision of it whose cells are all mixed. Mixed subdivisions form a subposet of the poset of all subdivisions, whose minimal elements are called *fine mixed*, in which every mixed cell is *fine*, i.e., does not properly contain any other mixed cell.

We call *Cayley embedding* of $\{Q_1, \dots, Q_m\}$ the marked polytope

$$(\mathcal{C}(Q_1, \dots, Q_m), \mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_m)) \text{ in } \mathbb{R}^d \times \mathbb{R}^{m-1}$$

defined as follows: let $e_1, \dots, e_m$ denote an affine basis in $\mathbb{R}^{m-1}$ and let $\mu_i: \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^{m-1}$ be the inclusion given by $\mu_i(x) = (x, e_i)$. Then we define

$$\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_m) := \bigcup_{i=1}^m \mu_i(\mathcal{A}_i), \quad \mathcal{C}(Q_1, \dots, Q_m) := \text{conv}(\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_m)).$$

Each Q_i is naturally embedded as a face in $\mathcal{C}(Q_1, \dots, Q_m)$. Moreover, the vertex set of $\mathcal{C}(Q_1, \dots, Q_m)$ is the disjoint union of the vertices of all the Q_i 's. This induces the following bijection between cells in $\mathcal{C}(Q_1, \dots, Q_m)$ and mixed cells in $Q_1 + \dots + Q_m$: To each mixed cell $B_1 + \dots + B_m$ we associate the disjoint union $B_1 \cup \dots \cup B_m$. To a cell B in $\mathcal{C}(Q_1, \dots, Q_m)$, we associate the Minkowski sum $B_1 + \dots + B_m$, where $B_i = B \cap Q_i$.

Theorem 6.1 (The Cayley Trick [7]). *Let $Q_1, \dots, Q_m \subset \mathbb{R}^d$ be convex polytopes. The bijection just exhibited induces an isomorphism between the poset of all subdivisions of $\mathcal{C}(Q_1, \dots, Q_m)$ and the poset of mixed subdivisions of $\sum_{i=1}^m Q_i$. In this isomorphism triangulations correspond to fine mixed subdivisions.*

Remark 6.2. With the previous definitions, for any polytope P :

$$P \times \Delta^{m-1} = \mathcal{C}(P, \dots, P).$$

In particular, in our context the Cayley Trick provides the following bijections between triangulations and mixed subdivisions:

Our interest in the Cayley Trick is twofold. On the one hand, it provides a way to visualize our candidate triangulations of $I^l \times \Delta^{m-1}$ as objects in dimension l , instead of $l + m - 1$. We use this in Section 7. On the other hand, the construction of the previous sections has a simple geometric interpretation in terms of the Cayley Trick. More precisely, the polyhedral subdivision of Corollary 3.3 can be obtained as follows: Let S be a polyhedral subdivision of $P \times \Delta^{m-1}$, and let S_M be the corresponding mixed subdivision of $P + \dots + P$. Each cell in S decomposes uniquely as

$$\bigcup_{i=1}^m B_i \times \{v_i\},$$

where the B_i are subsets of vertices of P . The corresponding cell in S_M is just $B_1 + \dots + B_m$. To construct our polyhedral subdivision of $P \times \Delta^{n-1}$ we just need to scale each summand B_i by the integer k_i which tells us how many vertices of Δ^{n-1} correspond to the vertex v_i of Δ^{m-1} .

That is to say, from S_M we construct a mixed subdivision of $P + \dots + P$ by the formula

$$\widetilde{S}_M = \{B_1 + \dots + B_1 + \dots + B_m + \dots + B_m : B_1 + \dots + B_m \in S_M\}.$$

The polyhedral subdivision $\tilde{S}$ of $P \times \Delta^{n-1}$ stated in Corollary 3.3 is the one corresponding via the Cayley Trick to the mixed subdivision $\tilde{S}_M$ of $P + \cdots + P$.

Also the type and weight of a simplex in $P \times \Delta^{m-1}$ have a simple interpretation via the Cayley Trick. With the notation of Definition 2.1, let $\tau = \tau_1 \cup \cdots \cup \tau_m$ be a simplex. The corresponding cell τ_M in $P + \cdots + P$ is the Minkowski sum of the simplices $\tau_1, \dots, \tau_m$, which lie in complementary affine subspaces. Hence, τ_M is combinatorially a product of m simplices, of dimensions $t_1, \dots, t_m$ where $(t_1, \dots, t_m)$ is the type of τ . Then the weight of τ represents the volume of τ_M , normalized with respect to the unit parallelepiped in the lattice spanned by the vertices of τ_M . With this we can prove:

Proposition 6.3. *Let P be a lattice polytope of dimension l . Let V be its volume, normalized to the unit parallelepiped in the lattice. Then the weighted size of a triangulation of $P \times \Delta^{m-1}$ is at most $m^l V$, with equality if and only if the triangulation is unimodular (with respect to the lattice). In particular, the weighted efficiency of a triangulation of $I^l \times \Delta^{m-1}$ is at most one, with equality for unimodular triangulations.*

Proof. If P is a lattice polytope (e.g., a cube), then the lattice spanned by τ_M is a sublattice of the one spanned by the point configuration $P + \cdots + P$, and coincides with it if and only if τ is unimodular. Then, for unimodular triangulations, the weighted size is just the volume of $P + \cdots + P$, normalized to the unit parallelepiped, which equals $m^l V$. For non-unimodular triangulations the weighted size is smaller than that. $\square$

7. More on $\rho_{l,m}$

In this section we obtain the value of $\rho_{2,m}$ for any m and we also show triangulations of $I^3 \times \Delta^1$ and $I^3 \times \Delta^2$ providing the values of $\rho_{3,2}$ and $\rho_{3,3}$ stated in Section 2.

7.1. Smallest Weighted Efficiency of Triangulations of $I^2 \times \Delta^{m-1}$

Here we prove:

Theorem 7.1. *The smallest weighted efficiency $\rho_{2,m}$ of triangulations of $I^2 \times \Delta^{m-1}$ is*

$$\rho_{2,m} = \sqrt{\frac{\lceil 3m^2/4 \rceil}{m^2}}.$$

That is, $\sqrt{3/4}$ for even m and $\sqrt{3/4} + \Theta(m^{-2})$ for odd m . A fine mixed subdivision of $I^2 + \cdots + I^2$ corresponding to a triangulation of $I^2 \times \Delta^{m-1}$ with that weighted efficiency is given in Fig. 4.

Let $B_M = B_1 + \cdots + B_m$ be a cell in a fine mixed subdivision of $I^2 + \cdots + I^2$ (a square of size m). Each B_i must be a simplex coming from the i th copy of I^2 , and in order to have a fine mixed subdivision, the different B_i 's must lie in complementary

Fig. 4. Fine mixed subdivisions corresponding to triangulations of $I^2 \times \Delta^{m-1}$ with smallest weighted efficiency for m even (left) or odd (right).

affine subspaces. Then there are the following possibilities:

- B_M is a triangle, obtained as the sum of a triangle in one of the I^l 's and a single point in the others. The weight of B_M is $\frac{1}{2}$.
- B_M is a quadrangle, obtained as the sum of two (non-parallel) segments from two of the B_i 's and a point in the rest of them. Three types of quadrangles can appear, depending on whether none, one, or both of the two segments involved is a diagonal of I^2 : a square parallel to the axes, a rhomboid (both with area 1), and a diagonal square (of area 2). The weight of B_M is 1.

In particular, the weighted size of the mixed subdivision equals $T/2 + S_1 + S_2 + R$, where T , S_1 , S_2 , and R denote, respectively, the numbers of triangles, squares of area 1, squares of area 2, and rhombi in the subdivision. Since the total area of $I^2 + \dots + I^2$ is $m^2 = T/2 + S_1 + 2S_2 + R$, we conclude that:

Proposition 7.2. *The weighted size of a mixed subdivision of $I^2 + \dots + I^2$ equals $m^2 - S_2$, where S_2 is the number of squares of size 2 in the subdivision.*

With this we can already conclude that the fine mixed subdivisions shown in Fig. 4 (one for m even and one for m odd) have weighted size equal to $\lceil 3m^2/4 \rceil$, since they have exactly $\lfloor m^2/4 \rfloor$ squares of area 2.

Our task is now to prove that no mixed subdivision can have more than $\lfloor m^2/4 \rfloor$ squares of area 2. For this we use:

Lemma 7.3. *Taking the i th summands of all the mixed cells $B_1 + \dots + B_m$ in a mixed subdivision of $P_1 + \dots + P_m$ produces a polyhedral subdivision of P_i . If the mixed subdivision was fine, the polyhedral subdivision is a triangulation.*

Proof. By the Cayley Trick, the mixed subdivision of $P_1 + \dots + P_m$ induces a polyhedral subdivision of $\mathcal{C}(P_1, \dots, P_m)$, and a triangulation if the mixed subdivision is fine. Since the polytope P_i appears as a face in $\mathcal{C}(P_1, \dots, P_m)$, any subdivision (resp. triangulation) of $\mathcal{C}(P_1, \dots, P_m)$ induces a subdivision (resp. triangulation) of P_i . That this subdivision is the one obtained taking the i th summands of all the mixed cells follows from Theorem 6.1. $\square$

Proposition 7.4. *Let S_M be a fine mixed subdivision of $I^2 + \dots + I^2$. Let a and b be the number of summands I^2 's which are triangulated in one and the other possible triangulations of I^2 (i.e., using one or the other diagonal). Then the number of squares of area 2 in S_M is at most ab and the weighted size of S_M is at least $m^2 - ab$.*

Proof. Each square of area 2 is the Minkowski sum of two opposite diagonals of two copies of I^2 (say, the i th and j th copies), and a point in the other $m - 2$ of the copies of I^2 . Clearly, the two copies which contribute diagonals have to be triangulated in opposite ways. The only thing which remains to be shown is that the same pair of copies of I^2 cannot contribute two different squares of size 2. For this, observe that “contracting” in every mixed cell of a mixed subdivision all the summands other than the i th and j th should give a mixed subdivision of $P_i + P_j$. In a mixed subdivision of two squares there is no room to put two different diagonal squares of area 2. $\square$

In the statement of Proposition 7.4, we have that $a + b = m$. In particular, the maximum possible value of ab is $\lfloor m^2/4 \rfloor$. This finishes the proof of Theorem 7.1.

7.2. Smallest Weighted Efficiency Triangulations of $I^3 \times \Delta^{m-1}$, $m = 2, 3$

Here we try to visualize triangulations of $I^3 \times \Delta^1$ and $I^3 \times \Delta^2$ which minimize the weighted efficiency, which we computed using the integer programming approach sketched in Section 2.

Of course, we use the Cayley Trick to decrease the dimension, so we show the corresponding fine mixed subdivision of a Minkowski sum instead of the triangulation itself.

$$\bullet \quad \boxed{I^3 \times \Delta^1} \quad I^3 \times \Delta^1 = \mathcal{C}(I^3, I^3) \longleftrightarrow I^3 + I^3.$$

We have to give a subdivision of a 3-cube of size 2. For this we first cut the eight corners of the cube, producing a *cubeoctahedron*, a semiregular 3-polytope with six square and eight triangular facets. The edges of the cubeoctahedron can be distributed in four “equatorial hexagons,” each of which cuts the polytope into two halves. Figure 5 depicts these halves for one of the equatorial hexagons. The labels in the vertices are heights, interpreted as follows: our point configuration is $\{0, 1, 2\}^3$ and the height of point (i, j, k) is just $i + j + k$.

It turns out that performing three of these four possible halvings, the cubeoctahedron is decomposed into six triangular prisms and two tetrahedra. In Fig. 5 each half is actually decomposed into three prisms and one tetrahedron. These, together with the eight tetrahedra we have cut from corners, form a fine mixed subdivision of $\{0, 1, 2\}^3$ with ten tetrahedra and six triangular prisms. Its weighted size is then

$$10 \cdot \frac{1}{6} + 6 \cdot \frac{1}{2} = \frac{14}{3}$$

and its weighted efficiency $\sqrt[3]{(14/3)/8}$.

Fig. 5. Fine mixed subdivision of $I^3 + I^3$ corresponding to a triangulation of $I^3 \times \Delta^1$ with smallest weighted efficiency.

• $I^3 \times \Delta^2$ $I^3 \times \Delta^2 = \mathcal{C}(I^3, I^3, I^3) \longleftrightarrow I^3 + I^3 + I^3.$

The fine mixed subdivision of $I^3 + I^3 + I^3$ is given in Fig. 6. It consists of 20 triangular prisms, 16 tetrahedra, and 2 parallelepipeds.

Thus, the weighted size of the triangulation is $20\frac{1}{2} + 16\frac{1}{6} + 2\frac{1}{1} = \frac{44}{3}$, and then the smallest weighted efficiency is

$$\rho_{3,3} = \sqrt[3]{\frac{44/3}{3^3}}.$$

Let us explain how to interpret Fig. 6. Again, each point (i, j, k) in $\{0, 1, 2, 3\}^3$ has been given a height $i + j + k$, which is written next to it. The subdivision is displayed in five parts. The left-top portion in the figure is a half cubeoctahedron exactly as the

Fig. 6. Fine mixed subdivision of $I^3 + I^3 + I^3$ corresponding to a triangulation of $I^3 \times \Delta^2$ with smallest weighted efficiency.

one in Fig. 5. The reader has to assume it divided into a tetrahedron and three triangular prisms, as before. The left-bottom portion consists of a corner tetrahedron and three triangular prisms located at corners of the cube, each joined to the half cubeoctahedron by a tetrahedron. So far we have six prisms and five tetrahedra, and the same is got from the right-top and right-bottom portions of the figure.

In between the two half cubeoctahedra, however, we now have a hexagonal prism decomposed into two triangular prisms and two quadrilateral prisms, all of height $\sqrt{3}$. The hexagonal prism is surrounded by a belt formed by six triangular prisms and six tetrahedra.

The thick edges in Fig. 6 represent edges of the big 3-cube of side 3, whose three visible facets are drawn by dotted lines. We have also shaded the facets of the subdivision which are contained in those facets of the big 3-cube.

References

1. R. Anderson, B. Hughes, Simplexity of the cube, *Discrete Math.*, **158** (1996), 99–150.
2. A. Below, J. A. de Loera, J. Richter-Gebert, Finding minimal triangulations of convex 3-polytopes is NP-hard, in *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms 2000, San Francisco, California, United States*, ACM Press, New York, 2000, pp. 65–66.
3. J. A. de Loera, S. Hoşten, F. Santos, B. Sturmfels, The polytope of all triangulations of a point configuration, *Doc. Math.*, **1** (1996), 103–119.
4. J. A. de Loera, S. Peterson, *universalbuilder*. Software available at http://www.math.ucdavis.edu/~deloera/RECENT_WORK/universalbuilder.tar.gz
5. I. M. Gel'fand, M. M. Kapranov, A. V. Zelevinsky, *Multidimensional Determinants, Discriminants and Resultants*, Birkhäuser, Boston, 1994.
6. H. Haiman, A simple and relatively efficient triangulation of the n -cube, *Discrete Comput. Geom.*, **6** (1991), 287–289.
7. B. Huber, J. Rambau, F. Santos, The Cayley Trick, lifting subdivisions and the Bohné–Dress Theorem on zonotopal tilings, *J. European Math. Soc. (JEMS)*, **2**(2) (2000), 179–198.
8. C. W. Lee, Subdivisions and triangulations of polytopes, in *Handbook of Discrete and Computational Geometry* (J.E. Goodman and J. O'Rourke, eds.), CRC Press, New York, 1997, pp. 271–290.
9. W. D. Smith, A lower bound for the simplexity of the N -cube via hyperbolic volumes, *European J. Combin.*, **21**(1) (2000), 131–137.
10. M. J. Todd, *The Computation of Fixed Points and Applications*, Lecture Notes in Economics and Mathematical Systems, Vol. 124, Springer-Verlag, Berlin, 1976.
11. P. Whittle, *Probability via Expectation*, 3rd edition, Texts in Statistics, Springer-Verlag, Berlin, 1992.

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