Cell Complexities in Hyperplane Arrangements*

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Abstract. We derive improved bounds on the complexity, i.e., the total number of faces of all dimensions, of many cells in arrangements of hyperplanes in higher dimensions, and use these bounds to obtain a very simple proof of an earlier bound, due to Aronov, Matoušek, and Sharir, on the sum of squares of cell complexities in such an arrangement.

1. Complexity of Many Cells

In this note we consider collections of cells in arrangements of hyperplanes in d -dimensional space. The *complexity* of a cell is the total number of faces of all dimensions appearing on its boundary. The complexity of a collection of cells is the sum of the complexities of its members. The main goal is to derive sharp bounds on the maximum complexity of any collection of m arbitrary distinct cells in an arrangement of n hyperplanes in $\mathbb{R}^d$. For more extensive background on the “many cells” problem considered in this paper, see [8] and [12]. A variety of bounds have been derived in [1], [4], [5],

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[7], [10], and [11]. The problem has also been extended to the case of many cells in arrangements of curves or surfaces; see, e.g., [3], [2], [7], and [9].

The main result of the paper improves upon the previous bound

$$O(m^{1/2}n^{d/2} \log^{(\lfloor d/2 \rfloor - 1)/2} n)$$

given in [4]:

Theorem 1.1. *The complexity of m distinct cells in an arrangement of n hyperplanes in d dimensions, for $d \geq 4$, is $O(m^{1/2}n^{d/2} \log^{(\lfloor d/2 \rfloor - 2)/2} n)$ with the implied constant of proportionality depending on d .*

Proof. The proof proceeds by induction on d . The base case $d = 4$ depends on a sharper bound known for $d = 3$ and cited below.

Let H be a collection of n hyperplanes in d -space. We assume that the hyperplanes are in *general position*, meaning that any k hyperplanes meet in a $(d-k)$ -flat, if $k = 1, \dots, d$, and not at all if $k > d$. It is not difficult to see that the worst-case complexity of the cells can always be achieved by hyperplanes in general position. Let P be a set of m points, not lying on any hyperplane. Denote by $K_k^{(d)}(P, H)$ the number of k -faces bounding the cells of $\mathcal{A}(H)$ that contain points of P . We mainly are concerned with the case $k = \lceil d/2 \rceil$, because, as follows from the Dehn–Sommerville relations (see, e.g., [8]), the total number of faces, of all dimensions, of a cell (which is a simple d -polytope) is at most proportional to the number of its $\lceil d/2 \rceil$ -faces, with the constants of proportionality depending only on d . We denote by $K_k^{(d)}(m, n)$ the maximum of $K_k^{(d)}(P, H)$ over all sets P, H as above.

Before continuing, we give a brief overview of our proof. It uses the “remove-and-reinsert” method that was introduced and applied, about a decade ago, to a variety of problems involving the complexity of various substructures in arrangements; see, e.g., [4]–[6] and [11]. Namely, we analyze what would happen if one hyperplane h was removed and reinserted into the arrangement. We consider each face counted in $K_k^{(d)}(P, H \setminus \{h\})$, and ask what becomes of it as h is reinserted: it could survive as a single face that is counted in $K_k^{(d)}(P, H)$, it could be split by h into two such faces, or it could disappear altogether from our count, if h separates it from any cell containing points of P . Note that in this manner we do not count features lying on h itself. Averaging this increment in our count over all choices of the hyperplane h being removed and reinserted yields a recurrence on $K_k^{(d)}(P, H)$, which relates $K_k^{(d)}(m, n)$ to quantities of the form $K_k^{(d)}(m', n-1)$ involving $n-1$ hyperplanes.

In this note we modify this standard analysis technique in the following manner: instead of treating this process as yielding a relation between the desired quantities for n and $n-1$ hyperplanes, we instead “unwind” the recurrence, all the way to a small number of hyperplanes. Viewed differently, we analyze what happens when we remove (and reinsert) essentially *all* hyperplanes in *all possible* sequences, and express $K_k^{(d)}(P, H)$ as an average over all such insertion orders. Since this analysis “amortizes” over the entire length of the insertion sequence, and averages over all such sequences, worst-case local effects are averaged out, and we are thus able to obtain (slightly) stronger bounds.

The proof is somewhat long since the base case ($d = 4$) and the general cases of odd and even dimensions have to be treated separately.

We now present the technical details of the proof, which derives a recurrence for $K_k^{(d)}(m, n)$. Pick $h \in H$, remove it and add it back. Consider the k -faces that are not contained in h and bound cells of the arrangement that contain points of P . Their number can increase when h is added to $\mathcal{A}(H \setminus \{h\})$ only when h splits a cell c containing points of P into two subcells, each containing points of P . In this case the “local” increase in the number of k -faces under consideration is equal to the number of $(k-1)$ -faces of the $(d-1)$ -face $c \cap h$ of $\mathcal{A}(H)$. Denote by H/h the set $\{h \cap h' \mid h' \in H \setminus \{h\}\}$ of $(d-2)$ -hyperplanes within h . Then the total increase in the number of k -faces under consideration that is caused by the reinsertion of h is equal to the number of $(k-1)$ -faces in the “splitting cells” of the $(d-1)$ -dimensional arrangement $\mathcal{A}(H/h)$ within h . If the number of cell splittings caused by the reinsertion of h is m_h , then the number of k -faces counted in $K_k^{(d)}(P, H)$ and not contained in h is at most $K_k^{(d)}(P_h, H \setminus \{h\}) + K_{k-1}^{(d-1)}(m_h, n-1)$, where P_h is a subset of P obtained by deleting m_h points, one from each cell of $\mathcal{A}(H \setminus \{h\})$ resulting from merging two cells of $\mathcal{A}(H)$ when h is removed. Repeating this analysis for all $h \in H$, and summing the respective bounds, we obtain

$$(n-d+k)K_k^{(d)}(P, H) \leq \sum_{h \in H} (K_k^{(d)}(P_h, H \setminus \{h\}) + K_{k-1}^{(d-1)}(m_h, n-1)), \quad (1)$$

where the factor $n-d+k$ comes from the observation that a k -face appears in the count for every $h \in H$, except for the $d-k$ hyperplanes containing it.

The case $d = 4$. We start with the base case $d = 4$ and $k = 2$. Equation (1) becomes

$$(n-2)K_2^{(4)}(P, H) \leq \sum_{h \in H} (K_2^{(4)}(P_h, H \setminus \{h\}) + K_1^{(3)}(m_h, n-1)). \quad (2)$$

It is known [1] that

$$K_1^{(3)}(m, n) = \begin{cases} \Theta(m^{2/3}n) & \text{for } m \geq n^{3/2}, \\ \Theta(n^2) & \text{for } n \leq m \leq n^{3/2}, \\ \Theta(mn) & \text{for } m \leq n. \end{cases} \quad (3)$$

Divide (2) by $n(n-1)(n-2)$, and put

$$F_2^{(4)}(P, H) := \frac{K_2^{(4)}(P, H)}{n(n-1)} \quad \text{and} \quad F_2^{(4)}(m, n) := \frac{K_2^{(4)}(m, n)}{n(n-1)}$$

to obtain

$$F_2^{(4)}(P, H) \leq \frac{1}{n} \sum_{h \in H} F_2^{(4)}(P_h, H \setminus \{h\}) + O\left(\frac{1}{n} \sum_{h \in H} \frac{K_1^{(3)}(m_h, n-1)}{n^2}\right). \quad (4)$$

We now unwind the recurrence in (4) all the way down to $n_0 = m^{1/4}$ remaining hyperplanes. We obtain a recurrence tree T . The j th level of T is the collection of all nodes whose corresponding substructure involves j hyperplanes of H ; thus the root of T is at

level n (it represents the whole set H) and the leaves are at level n_0 . Let π be a path in T , let $v_j(\pi)$ denote the node of π at level j , and let $h_j(\pi)$ denote the hyperplane removed and reinserted at $v_j(\pi)$, for $j = n, n-1, \dots, n_0+1$; in other words, $h_j(\pi)$ is the hyperplane that represents the edge of π between parent node $v_j(\pi)$ and child node $v_{j-1}(\pi)$. It is easily verified that the unwound recurrence can be rewritten as

$$F_2^{(4)}(P, H) \leq \frac{n_0!}{n!} \sum_{\pi} \left(F_2^{(4)}(P_{\pi}, H_{\pi}) + O \left(\sum_{j=n_0+1}^n \frac{K_1^{(3)}(m_j(\pi), j-1)}{j^2} \right) \right), \quad (5)$$

where π ranges over all paths in T , and where (i) $m_j(\pi)$ is the number of points removed from the current subset of P when $h_j(\pi)$ is removed from the subset of H associated with $v_j(\pi)$, (ii) P_{π} denotes the set of points remaining in P after all these removals, (iii) $m^*(\pi)$ denotes the size of P_{π} , and (iv) H_{π} denotes the subset of the n_0 hyperplanes of H that were not removed along π . We have $m^*(\pi) + \sum_{j=n_0+1}^n m_j(\pi) = m$. In other words, $F_2^{(4)}(P, H)$ is the average, over all paths of T , of the path-dependent expression in the brackets in (5). Denote this expression by $E(\pi) := F_2^{(4)}(P_{\pi}, H_{\pi}) + O(\sum_{j=n_0+1}^n E_j(\pi))$, where $E_j(\pi) := K_1^{(3)}(m_j(\pi), j-1)/j^2$.

We fix a path π in T , and estimate $E(\pi)$. First we have

$$F_2^{(4)}(P_{\pi}, H_{\pi}) \leq F_2^{(4)}(m, m^{1/4}) = \frac{K_2^{(4)}(m, m^{1/4})}{m^{1/4}(m^{1/4}-1)} = O\left(\frac{m}{m^{1/2}}\right) = O(m^{1/2}),$$

where we have used the fact that an arrangement of $m^{1/4}$ hyperplanes has fewer than m cells and their total complexity is $O(m)$. Partition the nodes of π into three subsets:

$$\begin{aligned} J_1 &= \{j \mid m_j(\pi) > (j-1)^{3/2}\}, \\ J_2 &= \{j \mid j-1 < m_j(\pi) \leq (j-1)^{3/2}\}, \quad \text{and} \\ J_3 &= \{j \mid m_j(\pi) \leq j-1\}. \end{aligned}$$

Using (3) and Hölder's inequality, we obtain

$$\begin{aligned} \sum_{j \in J_1} E_j(\pi) &= O \left(\sum_{j \in J_1} \frac{m_j(\pi)^{2/3}}{j} \right) \\ &= O \left(\left(\sum_{j \in J_1} m_j(\pi) \right)^{2/3} \left(\sum_{j > n_0} \frac{1}{j^3} \right)^{1/3} \right) \\ &= O \left(\frac{m^{2/3}}{n_0^{2/3}} \right) = O(m^{1/2}). \end{aligned}$$

Next we have

$$\sum_{j \in J_3} E_j(\pi) = O \left(\sum_{j \in J_3} \frac{m_j(\pi)}{j} \right) = O \left(\sum_{\substack{j \in J_3 \\ j \leq m^{1/2}}} \frac{m_j(\pi)}{j} + \sum_{\substack{j \in J_3 \\ j > m^{1/2}}} \frac{m_j(\pi)}{j} \right).$$

In the first sum, we use the fact that $m_j(\pi) < j$ to conclude that the sum is $O(m^{1/2})$. As for the second sum, we have

$$\sum_{\substack{j \in J_3 \\ j > m^{1/2}}} \frac{m_j(\pi)}{j} < \frac{1}{m^{1/2}} \sum_{\substack{j \in J_3 \\ j > m^{1/2}}} m_j(\pi) \leq \frac{1}{m^{1/2}} \cdot m = m^{1/2}.$$

Finally, we have

$$\sum_{j \in J_2} E_j(\pi) = O\left(\sum_{j \in J_2} 1\right) = O\left(\sum_{\substack{j \in J_2 \\ j \leq m^{1/2}}} 1 + \sum_{\substack{j \in J_2 \\ j > m^{1/2}}} 1\right).$$

The first subsum is at most $m^{1/2}$, while the second is at most

$$\sum_{m_j \geq m^{1/2}} 1 = \frac{m}{m^{1/2}} = m^{1/2}.$$

To summarize, we have shown that $E(\pi) = O(m^{1/2})$ for each path π in T . Since $F_2^{(4)}(P, H)$ is the average of these expressions, we conclude that

$$F_2^{(4)}(m, n) = \max_{\substack{|P|=m \\ |H|=n}} F_2^{(4)}(P, H) = O(m^{1/2}),$$

and hence $K_2^{(4)}(m, n) = O(m^{1/2}n^2)$. This establishes the base case $d = 4$, since the Dehn–Sommerville relations imply that $K_k^{(4)}(m, n) = O(K_2^{(4)}(m, n))$, for $k = 0, 1, 3$, as already mentioned.

The case of odd d . Next assume that $d > 4$ is odd, say $d = 2q + 1$. In this case we focus on $k = \lceil d/2 \rceil = q + 1$ and (1) becomes

$$(n - q)K_{q+1}^{(2q+1)}(P, H) \leq \sum_{h \in H} (K_{q+1}^{(2q+1)}(P_h, H \setminus \{h\}) + K_q^{(2q)}(m_h, n - 1)). \quad (6)$$

By the induction hypothesis, we have

$$K_q^{(2q)}(m, n) = O(m^{1/2}n^q \log^{(q-2)/2} n).$$

We substitute this bound in (6), divide (6) by $n(n - 1) \cdots (n - q)$, and put

$$F_{q+1}^{(2q+1)}(P, H) := \frac{K_{q+1}^{(2q+1)}(P, H)}{n(n - 1) \cdots (n - q + 1)}$$

and

$$F_{q+1}^{(2q+1)}(m, n) := \frac{K_{q+1}^{(2q+1)}(m, n)}{n(n - 1) \cdots (n - q + 1)}$$

to obtain

$$F_{q+1}^{(2q+1)}(P, H) \leq \frac{1}{n} \sum_{h \in H} F_{q+1}^{(2q+1)}(P_h, H \setminus \{h\}) + O\left(\frac{1}{n} \sum_{h \in H} m_h^{1/2} \log^{(q-2)/2} n\right). \quad (7)$$

We now unwind the recurrence in (7) until only one hyperplane remains. We obtain a recurrence tree T , and continue to use the same notations as in the case $d = 4$. It is easily verified that the unwound recurrence can be rewritten as

$$F_{q+1}^{(2q+1)}(P, H) \leq \frac{1}{n!} \sum_{\pi} \left(\sum_{j=1}^n O(m_j(\pi)^{1/2} \log^{(q-2)/2} j) \right), \quad (8)$$

where π ranges over all paths in T . In other words, as above, $F_{q+1}^{(2q+1)}(P, H)$ is the average, over all paths of T , of the path-dependent expression in the brackets in (8). By the Cauchy–Schwarz inequality, we have

$$\sum_{j=1}^n m_j(\pi)^{1/2} \leq \left(\sum_{j=1}^n m_j(\pi) \right)^{1/2} n^{1/2} \leq m^{1/2} n^{1/2}.$$

Hence $F_{q+1}^{(2q+1)}(m, n) = O(m^{1/2} n^{1/2} \log^{(q-2)/2} n)$, and thus

$$K_{q+1}^{(2q+1)}(m, n) = O(m^{1/2} n^{q+1/2} \log^{(q-2)/2} n),$$

which is the asserted bound for $d = 2q + 1$.

The case of even d . Finally consider the case where d is even, say $d = 2q > 4$. Here we take $k = \lceil d/2 \rceil = q$. In this case, (1) becomes

$$(n - q)K_q^{(2q)}(P, H) \leq \sum_{h \in H} (K_q^{(2q)}(P_h, H \setminus \{h\}) + K_{q-1}^{(2q-1)}(m_h, n - 1)). \quad (9)$$

By the induction hypothesis, we have

$$K_{q-1}^{(2q-1)}(m, n) = O(m^{1/2} n^{q-1/2} \log^{(q-3)/2} n).$$

We substitute this bound in (9), divide (9) by $n(n - 1) \cdots (n - q)$, and put

$$F_q^{(2q)}(P, H) := \frac{K_q^{(2q)}(P, H)}{n(n - 1) \cdots (n - q + 1)}$$

and

$$F_q^{(2q)}(m, n) := \frac{K_q^{(2q)}(m, n)}{n(n - 1) \cdots (n - q + 1)}$$

to obtain

$$F_q^{(2q)}(P, H) \leq \frac{1}{n} \sum_{h \in H} F_q^{(2q)}(P_h, H \setminus \{h\}) + O\left(\frac{1}{n} \sum_{h \in H} \frac{m_h^{1/2}}{n^{1/2}} \log^{(q-3)/2} n\right). \quad (10)$$

Once again unwind the recurrence in (10) until only one hyperplane remains. We obtain a recurrence tree T , and express the result as

$$F_q^{(2q)}(P, H) \leq \frac{1}{n!} \sum_{\pi} \left(\sum_{j=1}^n O\left(\frac{m_j(\pi)^{1/2}}{j^{1/2}} \log^{(q-3)/2} j\right) \right), \quad (11)$$

where π ranges over all paths in T . In other words, as above, $F_q^{(2q)}(P, H)$ is the average, over all paths of T , of the path-dependent expression in the brackets in (11). By the Cauchy–Schwarz inequality, we have

$$\sum_{j=1}^n \frac{m_j(\pi)^{1/2}}{j^{1/2}} \leq \left(\sum_{j=1}^n m_j(\pi) \right)^{1/2} \left(\sum_{j=1}^n \frac{1}{j} \right)^{1/2} = O(m^{1/2} \log^{1/2} n). \quad (12)$$

Hence $F_q^{(2q)}(m, n) = O(m^{1/2} \log^{(q-2)/2} n)$, and thus

$$K_q^{(2q)}(m, n) = O(m^{1/2} n^q \log^{(q-2)/2} n),$$

which is the asserted bound for $d = 2q$.

This completes the proof of the theorem. $\square$

Remarks. (1) The harmonic series that appears in (12) is the only source of the poly-logarithmic factors in our bounds. A major open problem is thus to obtain a better upper bound for the left-hand side of (12).

(2) In four dimensions our bound is $O(m^{1/2} n^2)$. We do not know whether this bound is tight for the whole range of m . It is clearly tight for $m = \Theta(1)$ and for $m = \Theta(n^4)$. It is also tight for $m = \Theta(n^2)$. This has been noted in Theorem 3.3(b) of [4]. For the sake of completeness, here is a sketch of the construction. Take two orthogonal 2-planes π, π' in 4-space. Construct in π an arbitrary arrangement of $n/2$ lines in general position, and construct in π' an arrangement of $n/2$ lines that has a cell c so that all lines appear on its boundary. Now extend each of these n lines to a hyperplane in 4-space by taking its Cartesian product with the complementary 2-plane. The cells under consideration in the resulting four-dimensional arrangement are the Cartesian products of each cell of the arrangement in π with c . We obtain a collection of $m = \Theta(n^2)$ cells whose overall complexity is $\Theta(n^2 \cdot n) = \Theta(n^3) = \Theta(m^{1/2} n^2)$.

(3) The method of proof employed above can also be used to derive the known (and tight!) bound of $O(m^{2/3} n^{d/3} + n^{d-1})$, $d \geq 4$, on $K_{d-1}^{(d)}(m, n)$, from the corresponding bound in three dimensions. We omit the details.

2. Sum of Squares of Cell Complexities

We next apply Theorem 1.1 to obtain a simple proof of the following result, originally established in [4].

Theorem 2.1. *The sum of squares of the cell complexities in an arrangement of n hyperplanes in d dimensions, for $d \geq 4$, is $O(n^d \log^{\lfloor d/2 \rfloor - 1} n)$.*

Proof. Let H be a set of n hyperplanes in d -space, and let $|C|$ denote the combinatorial complexity of a cell C in $\mathcal{A}(H)$. We wish to bound the quantity $\Sigma(H) = \sum_C |C|^2$, where the sum ranges over all cells C of $\mathcal{A}(H)$.

Let $\mathcal{C}_k$ denote the subset of cells whose complexity is exactly k , for any $k = O(n^{\lfloor d/2 \rfloor})$. Let $\mathcal{C}_{\geq k}$ denote the subset of cells whose complexity is at least k , and let m_k denote the

cardinality of $\mathcal{C}_{\geq k}$. Apply the bound of Theorem 1.1 to $\mathcal{C}_{\geq k}$, to obtain

$$km_k = O(m_k^{1/2} n^{d/2} \log^{(\lfloor d/2 \rfloor - 2)/2} n),$$

which implies that

$$m_k = O\left(\frac{n^d \log^{\lfloor d/2 \rfloor - 2} n}{k^2}\right).$$

We thus have

$$\begin{aligned} \Sigma(H) &= \sum_k k^2 |\mathcal{C}_k| = \sum_k k^2 (|\mathcal{C}_{\geq k}| - |\mathcal{C}_{\geq k+1}|) \leq O\left(n^d + \sum_k km_k\right) \\ &= O(n^d) + O\left(\sum_{k \leq \Theta(n^{\lfloor d/2 \rfloor})} \frac{n^d \log^{\lfloor d/2 \rfloor - 2} n}{k}\right) = O(n^d \log^{\lfloor d/2 \rfloor - 1} n), \end{aligned}$$

as asserted. $\square$

Remarks. (1) Lemma 3.4 of [4] provides an alternative derivation of this bound from the many-cell bound of Theorem 1.1.

(2) This proof shows that for any $\beta < 2$ we have

$$\sum_C |C|^\beta = O(n^d \log^{\lfloor d/2 \rfloor - 2} n).$$

In other words, the sum for $\beta < 2$ yields a slightly smaller bound than the one given in Theorem 2.1 (for $\beta = 2$), and, for the cases $d = 4$ and $d = 5$, settles in the affirmative a conjecture in [4].

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