The Integer Hull of a Convex Rational Polytope

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Abstract. Given $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, we consider the integer program $\max\{c'x \mid Ax = b; x \in \mathbb{N}^n\}$ and provide an *equivalent* and *explicit* linear program $\max\{\widehat{c}'q \mid Mq = r; q \geq 0\}$, where $M, r, \widehat{c}$ are easily obtained from A, b, c with no calculation. We also provide an explicit algebraic characterization of the integer hull of the convex polytope $P = \{x \in \mathbb{R}^n \mid Ax = b; x \geq 0\}$. All strong valid inequalities can be obtained from the generators of a convex cone whose definition is explicit in terms of M .

1. Introduction

Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{R}^n$ and consider the integer program

$$\mathbb{P} \rightarrow := \max\{c'x \mid Ax = b; x \in \mathbb{N}^n\}, \quad (1.1)$$

where the convex polyhedron $P := \{x \in \mathbb{R}^n \mid Ax = b; x \geq 0\}$ is compact. If P_1 denotes the *integer hull* of P , then solving $\mathbb{P}$ is equivalent to solving the linear program $\max\{c'x \mid x \in P_1\}$.

However, finding the integer hull P_1 of P is a difficult problem. As mentioned on p. 15 in [8], and to the best of our knowledge, no *explicit* (or “simple”) characterization (or description) of P_1 has been provided so far. In the general *cutting plane* methods originated by Gomory and Chvátal in the early sixties, and the *lift-and-project* methods described in, e.g., [4], one obtains P_1 as the final iterate of a *finite* nested sequence $P \supseteq P' \supseteq P'' \supseteq \cdots \supseteq P_1$ of polyhedra. However, in all those procedures, P_1 has no explicit description in terms of the initial data A, b . On the other hand, for specific polytopes P , one is often able to provide some *strong valid inequalities* in explicit form, but very rarely all of them (as for the matching polytope of a graph). For more details the interested reader is referred to [1], [2], [4], [6], [7, Section 23], [8, Sections 8 and 9], and the many references therein.

Contribution. The main goal of this paper is to provide a *structural* result on the integer hull P_1 of a convex rational polytope P , in the sense that we obtain an explicit algebraic characterization of the defining hyperplanes of P_1 , in terms of generators of a convex cone C which is itself described directly from the initial data A , with *no* calculation.

We first show that the integer program $\mathbb{P}$ is equivalent to a linear program in the *explicit* form

$$\max_{q \in \mathbb{R}^s} \{ \widehat{c}'q \mid Mq = r; q \geq 0 \}. \quad (1.2)$$

By *explicit* we mean that the data $M, r, \widehat{c}$ of the linear program (1.2) are constructed *explicitly* and *easily* from the initial data A, b, c . In particular, *no* calculation is needed and M, r have all their entries in $\{0, \pm 1\}$. In addition M is very sparse. Of course, and as expected, the dimension of the matrix M is in general exponential in the problem size. However, for the class of problems where A has nonnegative integral entries, and b and the column sums of A are bounded, then (1.2) is solvable in time polynomial in the problem size.

There is a simple linear relation $x = Eq$ linking x and q , but q is not a *lifting* of x like in the lift-and-project procedures described in [4]. It is more appropriate to say that q is a *disaggregation* of x , as will become clear in what follows. Moreover, with *each* extreme point q of the convex polyhedron $\Omega := \{q \in \mathbb{R}^s \mid Mq = r, q \geq 0\}$ is associated an integral point $x = Eq$ of P (i.e., $x \in P \cap \mathbb{Z}^n$).

Using the latter result, and when P is compact, we provide the integer hull P_1 in the *explicit* form $\{x \in \mathbb{R}^n \mid Ux \geq u\}$ for some matrix U and vector u . By this we mean that U, u are obtained from the generators of a convex cone C which has a very simple and explicit description in terms of A (via M). Hence, all strong valid inequalities for P_1 can be obtained from the generators of the cone C . Of course, in view of the potentially large size of M , one cannot expect to get all generators of C in general. However, we hope that this structural result on the characterization of P_1 will be helpful in either deriving strong valid inequalities or validating some candidates inequalities, at least for some specific polytopes P .

2. Notation and Preliminary Results

Let $\mathbb{N}$ denote the natural numbers or, equivalently, $\mathbb{Z}_+$. For a vector $b \in \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{m \times n}$, denote by b' and $A' \in \mathbb{R}^{n \times m}$ their respective transpose. Denote by $e_m \in \mathbb{R}^m$ the vector with all entries equal to 1. Let $\mathbb{R}[x_1, \dots, x_n]$ be the ring of real-valued polynomials in the variables $x_1, \dots, x_n$. A polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is written

$$x \mapsto f(x) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

for *finitely many* real coefficients $\{f_\alpha\}$, in the (usual) basis of monomials.

Given a matrix $A \in \mathbb{Z}^{m \times n}$, let $A_j \in \mathbb{Z}^m$ denote its j th column (equivalently, the j th row of A'); then z^{A_j} stands for

$$z^{A_j} := z_1^{A_{1j}} \cdots z_m^{A_{mj}} = e^{\langle A_j, \ln z \rangle} = e^{(A' \ln z)_j},$$

and if $A_j \in \mathbb{N}^m$, then z^{A_j} is a monomial of $\mathbb{R}[z_1, \dots, z_m]$.

2.1. Preliminary Result

We first recall the following result:

Theorem 2.1 (A Discrete Farkas Lemma). *Let $A \in \mathbb{N}^{m \times n}$, $b \in \mathbb{N}^m$. Then the following two statements are equivalent:*

- (i) *The linear system $Ax = b$ has a solution $x \in \mathbb{N}^n$.*
- (ii) *The real-valued polynomial $z \mapsto z^b - 1 := z_1^{b_1} \cdots z_m^{b_m} - 1$ can be written*

$$z^b - 1 = \sum_{j=1}^n Q_j(z)(z^{A_j} - 1) \quad (2.1)$$

for some real-valued polynomials $Q_j \in \mathbb{R}[z_1, \dots, z_m]$, $j = 1, \dots, n$, all of them with nonnegative coefficients.

In addition, the degree of the Q_j 's in (2.1) is bounded by

$$b^* := \sum_{j=1}^m b_j - \min_k \sum_{j=1}^m A_{jk}. \quad (2.2)$$

A proof based on counting techniques via generating functions and inverse $\mathbb{Z}$ -transform can be found in [3]. However, thanks to an anonymous referee's remark, a self-contained and simpler proof is provided in Section 2.3 below. First, we make some useful remarks and introduce some additional material.

2.2. Discussion

(a) With b^* as in (2.2) denote by

$$s := s(b^*) := \binom{m + b^*}{b^*}$$

the dimension of the vector space of polynomials of degree b^* in m variables. In view of Theorem 2.1, and given $b \in \mathbb{N}^m$, checking the existence of a solution $x \in \mathbb{N}^n$ to $Ax = b$ reduces to checking whether or not there exists a nonnegative solution q to a system of linear equations

$$Mq = r, \quad q \geq 0, \quad (2.3)$$

for some matrix $M \in \mathbb{Z}^{p \times ns}$, and vector $r \in \mathbb{Z}^p$, with

- ns variables $\{q_{j\alpha}\}$, the nonnegative coefficients of the Q_j 's,
- p equations to identify the terms of the same power in both sides of (2.1); obviously one has

$$p \leq s(b^* + a) := \binom{m + b^* + a}{b^* + a} \quad \left(\text{with } a := \max_k \sum_{j=1}^m A_{jk} \right).$$

In fact we may and will take $p = s(b^* + a)$.

This in turn reduces to solving a linear programming (LP) problem. Observe that in view of (2.1), the matrix of constraints $M \in \mathbb{Z}^{p \times ns}$, which has only 0 and ± 1 coefficients, is *easily* deduced from A with *no* calculation (and is very sparse). The same is true for $r \in \mathbb{Z}^p$ which has only two nonzero entries (equal to -1 and 1).

(b) In fact, from the proof of Theorem 2.1, it follows that one may even enforce the weights Q_j in (2.1) to be polynomials in $\mathbb{Z}[z_1, \dots, z_m]$ (instead of $\mathbb{R}[z_1, \dots, z_m]$) with nonnegative coefficients (and even with coefficients in $\{0, 1\}$). However, (a) above shows that the strength of Theorem 2.1 is precisely in allowing $Q_j \in \mathbb{R}[z_1, \dots, z_m]$ as it permits to check feasibility by solving a (continuous) linear program. Enforcing $Q_j \in \mathbb{Z}[z_1, \dots, z_m]$ would result in an *integer* program of size larger than that of the original problem.

(c) Theorem 2.1 reduces the issue of the existence of a solution $x \in \mathbb{N}^n$ to a particular *ideal membership problem*, that is, $Ax = b$ has a solution $x \in \mathbb{N}^n$ if and only if the polynomial $z^b - 1$ belongs to the *binomial ideal* $I = \langle z^{A_j} - 1 \rangle_{j=1, \dots, n} \subset \mathbb{R}[z_1, \dots, z_m]$ and for some weights Q_j all with *nonnegative coefficients*. In fact, one could prove Theorem 2.1 by an appropriate reduction of the initial problem of the existence of a solution $x \in \mathbb{N}^n$ to $Ax = b$, to a polynomial ideal membership problem (with special features) in the framework developed in Section 3 of [5], another alternative to the proof in [3].

Next, with $A \in \mathbb{N}^{m \times n}$, $b \in \mathbb{N}^m$, let $P \subset \mathbb{R}^n$ be the convex polyhedron

$$P := \{x \in \mathbb{R}^n \mid Ax = b; x \geq 0\}. \quad (2.4)$$

Similarly, with $M \in \mathbb{Z}^{p \times ns}$, $r \in \mathbb{Z}^p$ as in (2.3), let

$$\Omega := \{q \in \mathbb{R}^{ns} \mid Mq = r; q \geq 0\} \quad (2.5)$$

be the convex polyhedron of feasible solutions $q \in \mathbb{R}^{ns}$ of (2.3). So, obviously, (2.1) holds if and only if $\Omega \neq \emptyset$.

Define the row vector $e_s := (1, \dots, 1) \in \mathbb{R}^s$ and let $E \in \mathbb{N}^{n \times ns}$ be the block diagonal matrix, whose each diagonal block is the row vector e_s , that is,

$$E := \begin{bmatrix} e_s & 0 & \dots & 0 \\ 0 & e_s & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & e_s \end{bmatrix}. \quad (2.6)$$

Proposition 2.2. *Let $A \in \mathbb{N}^{m \times n}$, $b \in \mathbb{N}^m$ be given and let M be as in (1.2). Let P, Ω be the convex polyhedra defined in (2.4) and (2.5).*

- (a) *Let $q \in \Omega$. Then $x := Eq \in P$. In particular, if $q \in \Omega \cap \mathbb{Z}^{ns}$, then $x \in P \cap \mathbb{Z}^n$.*
- (b) *Let $x \in P \cap \mathbb{Z}^n$. Then $x = Eq$ for some $q \in \Omega \cap \mathbb{Z}^{ns}$.*
- (c) *The matrix M is totally unimodular.*
- (d) *Whenever $\Omega \neq \emptyset$, each vertex of Ω is integral.*

Proof. (a) With $q \in \Omega$, let $\{Q_j\}_{j=1}^n \subset \mathbb{R}[z_1, \dots, z_m]$ be the set of polynomials (with vector of nonnegative coefficients q) which satisfy (2.1). Taking the derivative of both

sides of (2.1) with respect to z_k , at the point $z = (1, \dots, 1)$, yields

$$b_k = \sum_{j=1}^n Q_j(1, \dots, 1) A_{kj} = \sum_{j=1}^n A_{kj} x_j, \quad k = 1, \dots, n,$$

with $x_j := Q_j(1, \dots, 1)$ for all $j = 1, \dots, n$. Next, use the facts that (i) all the Q_j 's have nonnegative coefficients $\{q_{j\alpha}\}$, and (ii) $Q_j(1, \dots, 1) = \sum_{\alpha \in \mathbb{N}^m} q_{j\alpha} = (Eq)_j$ for all $j = 1, \dots, n$, to obtain $x := Eq \in P$. Moreover, if $q \in \Omega \cap \mathbb{Z}^{ns}$, then obviously $x \in P \cap \mathbb{Z}^n$.

(b) Let $x \in P \cap \mathbb{Z}^n$ so that $x \in \mathbb{N}^n$ and $Ax = b$; write

$$z^b - 1 = z^{A_1 x_1} - 1 + z^{A_1 x_1} (z^{A_2 x_2} - 1) + \dots + z^{\sum_{j=1}^{n-1} A_j x_j} (z^{A_n x_n} - 1),$$

and, whenever $x_j \neq 0$,

$$z^{A_j x_j} - 1 = (z^{A_j} - 1)[1 + z^{A_j} + \dots + z^{A_j(x_j-1)}], \quad j = 1, \dots, n,$$

to obtain (2.1) with

$$z \mapsto Q_j(z) := z^{\sum_{k=1}^{j-1} A_k x_k} [1 + z^{A_j} + \dots + z^{A_j(x_j-1)}],$$

and $Q_j \equiv 0$ if $x_j = 0$, $j = 1, \dots, n$. We immediately see that each Q_j has all its coefficients $\{q_{j\alpha}\}$ nonnegative (and even in $\{0, 1\}$). Moreover, $Q_j(1, \dots, 1) = x_j$ for all $j = 1, \dots, n$, or, equivalently, $x = Eq$ with $q \in \Omega \cap \mathbb{Z}^{ns}$.

(c) That M is totally unimodular follows from the fact that M is a *network matrix*, that is, a matrix with $\{0, \pm 1\}$ entries and with *exactly two* nonzero entries 1 and -1 in each column (see p. 274 of [7]). Indeed, from the identity (2.1), and the definition of M , each row of M is associated with a monomial z^α , with $\sum_j \alpha_j \leq b^* + a$. Thus, consider a particular column of M associated with the variable $q_{k\alpha}$ (the coefficient of the monomial z^α of the polynomial Q_k in (2.1), with $\sum_j \alpha_j \leq b^*$). From (2.1), the variable $q_{k\alpha}$ is involved only:

- in the row (or equation) associated with the monomial z^α (with coefficient -1), and
- in the row (or equation) associated with the monomial $z^{\alpha+A_k}$ (with coefficient $+1$).

(d) The right-hand side r in the definition of Ω is integral. Therefore, as M is totally unimodular, whenever $\Omega \neq \emptyset$ each vertex of Ω is integral. $\square$

2.3. Proof of Theorem 2.1

(i) $\Rightarrow$ (ii) This follows directly from Proposition 2.2(b) and the fact that (2.1) holds if and only if $\Omega \neq \emptyset$.

(ii) $\Rightarrow$ (i) Suppose (2.1) holds for some polynomials $\{Q_j\} \subset \mathbb{R}[z_1, \dots, z_m]$. Then $\Omega \neq \emptyset$ and so pick any vertex $\hat{q}$ of Ω . By Proposition 2.2(d), $\hat{q} \in \Omega \cap \mathbb{Z}^{ns}$ and, by Proposition 2.2(a), $x := E\hat{q} \in P \cap \mathbb{Z}^n$, that is, $Ax = b$ and $x \in \mathbb{N}^n$. $\square$

From Proposition 2.2(b) and its proof, one sees that q is a disaggregation of $x \in \mathbb{N}^n$. Indeed, if we write $q = (q_1, \dots, q_n)$, then each q_j has exactly x_j nontrivial entries, all equal to 1. So q is *not* a *lifting* of x as in the lift-and-project procedures described in [4]. In the latter, x is part of the vector q in the augmented space, and is obtained by *projection* of q .

3. Main Result

We first prove our results in the case $A \in \mathbb{N}^{m \times n}$ and then in Section 3.3, we show that the general case $A \in \mathbb{Z}^{m \times n}$ reduces to the former by adding one variable and one constraint to the original problem.

So let $A \in \mathbb{N}^{m \times n}$, $b \in \mathbb{N}^m$, and with no loss of generality we may and will assume that every column of A has at least one nonzero entry, in which case P in (2.4) is a polytope.

Recall that with every solution $0 \leq q \in \mathbb{R}^{ns}$ of the linear system $Mq = r$ in (2.3) we may associate a set of polynomials $\{Q_j\} \subset \mathbb{R}[z_1, \dots, z_m]$, with nonnegative coefficients, such that (2.1) is satisfied, and conversely to such a set of polynomials $\{Q_j\}$ with nonnegative coefficients, is associated a vector $0 \leq q \in \mathbb{R}^{ns}$ that satisfies (2.3). In fact, $q = \{q_{j\alpha}\}$ is the vector of coefficients of the polynomials Q_j 's in the (usual) basis of monomials.

3.1. An Equivalent Linear Program

We now consider the integer program $\mathbb{P}$. For every $c \in \mathbb{R}^n$ let $\hat{c} \in \mathbb{R}^{ns}$ be defined as

$$\hat{c}' = (\hat{c}'_1, \dots, \hat{c}'_n) \quad \text{with} \quad \hat{c}'_j = c_j(1, \dots, 1) \in \mathbb{R}^s, \quad \forall j = 1, \dots, n. \quad (3.1)$$

Equivalently, $\hat{c}' = c'E$ with E as in (2.6). It also follows that $\hat{c}'q = c'x$ whenever $x = Eq$. As a consequence of Theorem 2.1 we immediately obtain

Corollary 3.1. *Let $A \in \mathbb{N}^{m \times n}$, $b \in \mathbb{N}^m$, $c \in \mathbb{R}^n$ be given. Let $M \in \mathbb{Z}^{p \times ns}$, $r \in \mathbb{Z}^p$, and $E \in \mathbb{N}^{n \times ns}$ be as in (2.3) and (2.6), respectively.*

(a) *The integer program*

$$\mathbb{P} \rightarrow \max_x \{ c'x \mid Ax = b; x \in \mathbb{N}^n \} \quad (3.2)$$

has same optimal value as the linear program

$$\mathbb{Q} \rightarrow \max_{q \in \mathbb{R}^{ns}} \{ \hat{c}'q \mid Mq = r; q \geq 0 \} \quad (3.3)$$

(including the case $-\infty$).

(b) *In addition, let $q^* \in \mathbb{R}^{ns}$ be a vertex of Ω in (2.5), an optimal solution of the linear program $\mathbb{Q}$. Then $x^* := Eq^* \in \mathbb{N}^n$ and x^* is an optimal solution of the integer program $\mathbb{P}$.*

Proof. Let $\max \mathbb{P}$ and $\max \mathbb{Q}$ denote the respective optimal values of $\mathbb{P}$ and $\mathbb{Q}$. We first treat the case $-\infty$. $\max \mathbb{P} = -\infty$ only if $P \cap \mathbb{Z}^n = \emptyset$. However, then $\Omega = \emptyset$ as well, which in turn implies $\max \mathbb{Q} = -\infty$. Indeed, by Theorem 2.1, if $P \cap \mathbb{Z}^n = \emptyset$, i.e., if $Ax = b$ has no solution $x \in \mathbb{N}^n$, then one cannot find polynomials $\{\mathbb{Q}_j\} \subset \mathbb{R}[z_1, \dots, z_m]$ with nonnegative coefficients that satisfy (2.1). Therefore, from the definition of Ω , if $\Omega \neq \emptyset$ one would have a contradiction.

Conversely, if $\Omega = \emptyset$ (so that $\max \mathbb{Q} = -\infty$), then by definition of Ω , one cannot find polynomials $\{\mathbb{Q}_j\} \subset \mathbb{R}[z_1, \dots, z_m]$ with nonnegative coefficients that satisfy (2.1). Therefore, by Theorem 2.1, $Ax = b$ has no solution $x \in \mathbb{N}^n$ which in turn implies $\max \mathbb{P} = -\infty$, i.e., $P \cap \mathbb{Z}^n = \emptyset$.

In the case when $\max \mathbb{P} \neq -\infty$, we necessarily have $\max \mathbb{P} < \infty$ because the convex polyhedron P is compact. Next, consider a feasible solution $q \in \Omega$ of $\mathbb{Q}$. From Proposition 2.2(a) $x := Eq \in P$. Therefore, as x is bounded then so is Eq , which, in view of definition (2.6) of E , also implies that q is bounded. Hence Ω is compact which in turn implies that the optimal value of $\mathbb{Q}$ is finite and attained at some vertex q^* of Ω .

Now, let $x^* \in \mathbb{N}^n$ be an optimal solution of $\mathbb{P}$. By Proposition 2.2(b) there exists some $q \in \Omega$ with $Eq = x^*$. From definition (3.1) of the vector $\hat{c}$ we have

$$\hat{c}'q = c'Eq = c'x^*,$$

which implies $\max \mathbb{Q} \geq \max \mathbb{P}$.

On the other hand, let $q^* \in \Omega$ be a vertex of Ω , an optimal solution of $\mathbb{Q}$. By Proposition 2.2(d), $q^* \in \Omega \cap \mathbb{Z}^{ns}$ and by Proposition 2.2(a), $x := Eq^* \in P \cap \mathbb{Z}^n$, that is, $x \in \mathbb{N}^n$ is a feasible solution of $\mathbb{P}$. Again, from definition (3.1) of the vector $\hat{c}$ we have

$$c'x = c'Eq^* = \hat{c}'q^*,$$

which, in view of $\max \mathbb{P} \leq \max \mathbb{Q}$, implies $\max \mathbb{P} = \max \mathbb{Q}$, and $x \in \mathbb{N}^n$ is an optimal solution of $\mathbb{P}$. This completes the proof of (a) and (b). $\square$

Remark 3.2. Let $b^* := \sum_j b_j - \min_k \sum_{j=1}^m A_{jk}$ and $a := \max_k \sum_{j=1}^m A_{jk}$. From the discussion immediately after Theorem 2.1, $M \in \mathbb{Z}^{p \times ns}$ where

$$p = \binom{m + b^* + a}{b^* + a} = p_1(m) \quad \text{and} \quad s = \binom{m + b^*}{b^*} = p_2(m).$$

The polynomial $m \mapsto p_1(m)$ has degree $b^* + a$ whereas the polynomial $m \mapsto p_2(m)$ has degree b^* . Moreover, all the entries of M , r are $0, \pm 1$. Let $\mathcal{M}$ be the class of integer programs $\mathbb{P}$ with $A \in \mathbb{N}^{m \times n}$, $b \in \mathbb{N}^m$, and where, uniformly in $\mathbb{P} \in \mathcal{M}$,

- the column sums of A are bounded (i.e., $\sup_k \sum_j A_{jk}$ is bounded), and
- $\sum_j b_j$ is bounded,

so that a and b^* above are bounded, uniformly in $\mathbb{P} \in \mathcal{M}$. Then one may solve the integer programs $\mathbb{P}$ of the class $\mathcal{M}$ in time polynomial in the problem size, because it suffices to solve the linear program $\mathbb{Q}$ which has $p_1(m)$ constraints and $np_2(m)$ variables. One may consider this result as a dual counterpart of the known result which states that integer programs are solvable in time polynomial in the problem size when the dimension n

is fixed. (A dual counterpart would not be that integer programs are solvable in time polynomial in the problem size when the number of constraints m is fixed. Just think of the knapsack problem where $m = 1$.)

3.2. The Integer Hull

We are now interested in describing the integer hull P_1 of P , i.e., the convex hull of $P \cap \mathbb{Z}^n$.

Theorem 3.3. *Let $A \in \mathbb{N}^{m \times n}$, $b \in \mathbb{N}^m$, and let $E \in \mathbb{N}^{n \times ns}$, $M \in \mathbb{Z}^{p \times ns}$, $r \in \mathbb{Z}^p$ be as in (2.6) and (2.3), respectively. Let $\{(u^k, v^k)\}_{k=1}^t \subset \mathbb{R}^{n \times p}$ be a (finite) set of generators of the convex cone $C \subset \mathbb{R}^{n \times p}$ defined by*

$$C := \{(u, v) \in \mathbb{R}^{n \times p} \mid E'u + M'v \geq 0\}. \quad (3.4)$$

(a) *The integer hull P_1 of P is the convex polyhedron defined by the linear constraints*

$$\langle u^k, x \rangle + \langle v^k, r \rangle \geq 0, \quad \forall k = 1, \dots, t, \quad (3.5)$$

or, equivalently,

$$P_1 := \{x \in \mathbb{R}^n \mid Ux \geq u\}, \quad (3.6)$$

where the matrix $U \in \mathbb{R}^{t \times n}$ has row vectors $\{u^k\}$, and the vector $u \in \mathbb{R}^t$ has coordinates $u_k = \langle -v^k, r \rangle$, $k = 1, \dots, t$.

(b) *Equivalently $P_1 = E(\Omega)$.*

Proof. (a) Given $x \in \mathbb{R}^n$, consider the following linear system:

$$\begin{cases} Eq = x, \\ Mq = r, \\ q \geq 0, \end{cases} \quad (3.7)$$

where M, E are defined in (2.3) and (2.6), respectively. Invoking the celebrated Farkas lemma (see, e.g., [7]), system (3.7) has a solution $q \in \mathbb{R}^{ns}$ if and only if (3.5) holds.

Therefore, let $x \in \mathbb{R}^n$ satisfy $Ux \geq u$ with U, u as in (3.6). By the Farkas lemma, system (3.7) has a solution $q \in \mathbb{R}^{ns}$, that is, $Mq = r, q \geq 0$, and $x = Eq$. As $q \in \Omega$ and Ω is compact, q is a convex combination $\sum_k \gamma_k \hat{q}^k$ of the vertices $\{\hat{q}^k\}$ of Ω . By Proposition 2.2(d), (a), for each vertex $\hat{q}^k$ of Ω we have $\hat{x}^k := E\hat{q}^k \in P \cap \mathbb{Z}^n$. Therefore,

$$x = Eq = \sum_k \gamma_k E\hat{q}^k = \sum_k \gamma_k \hat{x}^k, \quad (3.8)$$

that is, x is a convex combination of points $\hat{x}^k \in P \cap \mathbb{Z}^n$, i.e., $x \in P_1$; hence $\{x \in \mathbb{R}^n \mid Ux \geq u\} \subseteq P_1$.

Conversely, let $x \in P_1$, i.e., $x \in \mathbb{R}^n$ is a convex combination $\sum_k \gamma_k \hat{x}^k$ of points $\hat{x}^k \in P \cap \mathbb{Z}^n$. By Proposition 2.2(b), for each k , $\hat{x}^k = Eq^k$ for some vector $q^k \in \Omega \cap \mathbb{Z}^{ns}$. Therefore, as each $(\hat{x}^k, q^k)$ satisfies (3.7), then so does their convex combination

$(x, q) := \sum_k \gamma_k(\hat{x}^k, q^k)$. By the Farkas lemma again, we must have $Ux \geq u$, and so $P_1 \subseteq \{x \in \mathbb{R}^n \mid Ux \geq u\}$, which completes the proof.

(b) This follows directly from (a) and

$$E(\Omega) = \{x \in \mathbb{R}^n \mid x = Eq; Mq = r; q \geq 0\}. \quad \square$$

Observe that the convex cone C in (3.4) of Theorem 3.3 is defined *explicitly* in terms of the initial data A , and with *no* calculation. Indeed, the matrix M in (2.3) is easily obtained from A and E is explicitly given in (2.6). Thus, the interest of Theorem 3.3 is that we obtain an algebraic characterization (3.6) of P_1 via generators of a cone C simply related to A .

From the proof of Theorem 3.3, every element (u, v) of the cone C produces a valid inequality for P_1 , and, clearly, all strong valid inequalities can be obtained from *generators* of C .

Next suppose that for some $a \in \mathbb{R}^n$, $w \in \mathbb{R}$, we want to test whether $a'x \geq w$ is a valid inequality. If there is some $v \in \mathbb{R}^p$ such that $M'v \geq -E'a$ and $-v'r \geq w$, then, indeed, $a'x \geq w$ is a valid inequality. In fact, w can be improved to $\tilde{w}$ with

$$\tilde{w} := \max_v \{-v'r \mid M'v \geq -E'a\}.$$

3.3. The General Case $A \in \mathbb{Z}^{m \times n}$

In this section we consider the case where $A \in \mathbb{Z}^{m \times n}$, that is, A may have negative entries. We assume that the convex polyhedron $P \subset \mathbb{R}^n$, defined in (2.4), is compact.

Let $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}$ be such that for all $j = 1, \dots, m$,

$$\hat{b}_j := b_j + \beta \geq 0, \quad \hat{A}_{jk} := A_{jk} + \alpha_k \geq 0, \quad k = 1, \dots, n. \quad (3.9)$$

As P is compact we have

$$\max_{x \in \mathbb{N}^n} \left\{ \sum_{j=1}^n \alpha_j x_j \mid Ax = b \right\} \leq \max_{x \in \mathbb{R}^n, x \geq 0} \left\{ \sum_{j=1}^n \alpha_j x_j \mid Ax = b \right\} =: \rho^*(\alpha) < \infty.$$

Given $\alpha \in \mathbb{N}^n$, the scalar $\rho^*(\alpha)$ is easily calculated by solving an LP problem. Note that we can choose $\beta \in \mathbb{N}$ as large as desired. Therefore, choose $\rho^*(\alpha) \leq \beta \in \mathbb{N}$. Let $\hat{A} \in \mathbb{N}^{m \times n}$, $\hat{b} \in \mathbb{N}^m$ be as in (3.9) with $\beta \geq \rho^*(\alpha)$.

The feasible solutions $x \in \mathbb{N}^n$ of $Ax = b$, i.e., the points of $P \cap \mathbb{Z}^n$, are in one-to-one correspondence with the solutions $(x, u) \in \hat{P} \cap \mathbb{Z}^{n+1}$ where $\hat{P} \subset \mathbb{R}^{n+1}$ is the convex polytope

$$\hat{P} := \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{R} \mid \begin{cases} \hat{A}x + e_m u = \hat{b}; \\ \alpha'x + u = \beta; \end{cases} \quad x, u \geq 0 \right\}. \quad (3.10)$$

Indeed, if $x \in P \cap \mathbb{Z}^n$, i.e., $Ax = b$ with $x \in \mathbb{N}^n$, then

$$Ax + e_m \sum_{j=1}^n \alpha_j x_j - e_m \sum_{j=1}^n \alpha_j x_j = b + (\beta - \rho^*(\alpha))e_m,$$

or, equivalently,

$$\widehat{A}x + \left(\beta - \sum_{j=1}^n \alpha_j x_j \right) e_m = \widehat{b},$$

and thus, as $\beta \geq \rho^*(\alpha) \geq \alpha'x$, letting $u := \beta - \alpha'x \in \mathbb{N}$, yields $(x, u) \in \widehat{\mathbf{P}} \cap \mathbb{Z}^{n+1}$. Conversely, let $(x, u) \in \widehat{\mathbf{P}} \cap \mathbb{Z}^{n+1}$. Using the definitions of $\widehat{A}$ and $\widehat{b}$, it then follows immediately that

$$Ax + e_m \sum_{j=1}^n \alpha_j x_j + u e_m = b + \beta e_m, \quad \sum_{j=1}^n \alpha_j x_j + u = \beta,$$

so that $Ax = b$ with $x \in \mathbb{N}^n$, i.e., $x \in \mathbf{P} \cap \mathbb{Z}^n$. In other words,

$$x \in \mathbf{P} \cap \mathbb{Z}^n \Leftrightarrow (x, \beta - \alpha'x) \in \widehat{\mathbf{P}} \cap \mathbb{Z}^{n+1}. \quad (3.11)$$

The convex polytope $\widehat{\mathbf{P}}$ can be written

$$\widehat{\mathbf{P}} := \left\{ (x, u) \in \mathbb{R}^{n+1} \mid B \begin{bmatrix} x \\ u \end{bmatrix} = (\widehat{b}, \beta); x, u \geq 0 \right\}, \quad (3.12)$$

with

$$B := \begin{bmatrix} \widehat{A} & | & e_m \\ - & | & - \\ \alpha' & | & 1 \end{bmatrix}.$$

As $B \in \mathbb{N}^{(m+1) \times (n+1)}$, we are back to the case analyzed in Sections 3.1 and 3.2.

In particular, the integer program $\mathbb{P} \rightarrow \max\{c'x \mid Ax = b; x \in \mathbb{N}^n\}$ is equivalent to the integer program

$$\widehat{\mathbb{P}} \rightarrow \max \left\{ c'x \mid B \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \widehat{b} \\ \beta \end{bmatrix}; (x, u) \in \mathbb{N}^n \times \mathbb{N} \right\}. \quad (3.13)$$

Hence, Theorem 2.1, Proposition 2.2, Corollary 3.1, and Theorem 3.3 are still valid with $B \in \mathbb{N}^{(m+1) \times (n+1)}$ in lieu of $A \in \mathbb{N}^{m \times n}$, $(\widehat{b}, \beta) \in \mathbb{N}^m \times \mathbb{N}$ in lieu of $b \in \mathbb{N}^m$, and $\widehat{\mathbf{P}} \subset \mathbb{R}^{n+1}$ in lieu of $\mathbf{P} \subset \mathbb{R}^n$.

So again, as in previous sections, the polytope $\widehat{\Omega}$ associated with $\widehat{\mathbf{P}}$ is explicitly defined from the initial data A , because $\widehat{A}$ is simply defined from A and α . In turn, as the convex cone C in Theorem 3.3 is also defined explicitly from A via M , again one obtains a simple characterization of the integer hull $\widehat{\mathbf{P}}_1$ of $\widehat{\mathbf{P}}$ via the generators of C .

If we are now back to the initial data A, b , then $\mathbf{P}_1$ is easily obtained from $\widehat{\mathbf{P}}_1$. Indeed, by Theorem 3.3, let

$$\widehat{\mathbf{P}}_1 = \{(x, u) \in \mathbb{R}^{n+1} \mid \langle w^k, x \rangle + \delta^k u \geq \rho^k; k = 1, \dots, t\},$$

for some $\{(w^k, \delta^k) \in \mathbb{R}^n \times \mathbb{R}\}_{k=1}^t$, and some $t \in \mathbb{N}$. Then from (3.11) it immediately follows that

$$\mathbf{P}_1 = \{x \in \mathbb{R}^n \mid \langle w^k - \delta^k \alpha, x \rangle \geq \rho^k - \beta \delta^k; k = 1, \dots, t\}.$$

3.4. 0–1 Integer Programs

The extension to 0–1 integer programs

$$\max_x \{c'x \mid Ax = b; x \in \{0, 1\}^n\}$$

is straightforward by considering the equivalent integer program

$$\max_{x,u} \{c'x \mid Ax = b; x_j + u_j = 1, \forall j = 1, \dots, n; (x, u) \in \mathbb{N}^n \times \mathbb{N}^n\},$$

which is an integer program in the form (1.1). However, the resulting linear equivalent program $\mathbb{Q}$ of Corollary 3.1 is now more complicated. For instance, if $A \in \mathbb{N}^{m \times n}$, then $q \in \mathbb{R}^{2ns}$ and s is now the dimension of the vector space of polynomials in $n + m$ variables and of degree at most $n + \sum_j b_j$.

4. Conclusion

We have presented an explicit algebraic characterization of the integer hull P_I of a convex polytope $P \subset \mathbb{R}^n$. Indeed, the defining hyperplanes of P_I are obtained from the generators of a convex cone whose description is obtained from the data A, b with no calculation. Of course, and as expected, this convex cone is in a space of large dimension (exponential in the problem size). However, this structural result shows that all *strong valid inequalities* can be obtained in this manner. Therefore, we hope this result is helpful in deriving strong valid inequalities, or in validating some candidate inequalities, at least for some specific polytopes P .

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