Relationships Among Flag f -Vector Inequalities for Polytopes

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Abstract. We examine linear inequalities satisfied by the flag f -vectors of polytopes. One source of these inequalities is the toric g -vector; convolutions of its entries are non-negative for rational polytopes. We prove a conjecture of Meisinger about a redundancy in these inequalities. Another source of inequalities is the **cd**-index; among all d -polytopes, each **cd**-index coefficient is minimized on the d -simplex. We show that not all of the **cd**-index inequalities are implied by the toric g -vector inequalities, and that not all of the toric g -vector inequalities are implied by the **cd**-index inequalities. Finally, we show that some inequalities from convolutions of **cd**-index coefficients are implied by other **cd**-index inequalities.

1. Introduction

A *polytope* Q is the convex hull of a finite set of points in Euclidean space. It is d -dimensional, or a *d -polytope*, if its affine span is $\mathbf{R}^d$. Its boundary is a collection of lower-dimensional polytopes known as the *faces* of Q . Let $f_i(Q)$ be the number of i -faces of Q . We call $(f_0, f_1, \dots, f_{d-1})$ the *f -vector* of Q . More generally, let $f_{i_1 i_2 \dots i_k}$ be the number of chains of faces $F_{i_1} \subseteq F_{i_2} \subseteq \dots \subseteq F_{i_k}$ in Q with $\dim(F_{i_j}) = i_j$. These f_S , where S ranges over all subsets of $\{0, 1, \dots, d-1\}$, are the entries of the *flag f -vector* of Q . We often write f_S^d when we wish to emphasize that Q is d -dimensional.

The flag f -vectors of polytopes have not yet been completely characterized, but we do know some things about them. Bayer and Billera showed that the flag f -vectors of all polytopes satisfy the *generalized Dehn–Sommerville equations* [2]:

Theorem 1 [2]. *Let Q be a d -polytope and let $S \subseteq \{0, 1, \dots, d-1\}$. If $\{i, k\} \subseteq S \cup \{-1, d\}$, $i < k-1$, and S contains no j such that $i < j < k$, then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup j}(Q) = f_S(Q)(1 - (-1)^{k-i-1}). \quad (1)$$

All linear relations satisfied by flag f -vectors of polytopes are consequences of the generalized Dehn–Sommerville equations. The flag f -vector has length 2^d , but because of these equations the affine span of the flag f -vectors of d -polytopes has dimension $a_d - 1$, where a_d is the d th Fibonacci number ($a_0 = 1, a_1 = 1$). Let $Sp(d) = \{S \mid S \subset \{0, 1, \dots, d-2\}, \{i, i+1\} \not\subset S \text{ for all } 0 \leq i \leq d-3\}$. The $a_d - 1$ flag numbers $\{f_S \mid S \in Sp(d)\}$ are known as the *sparse basis*; every f_T can be expressed in terms of this basis via the generalized Dehn–Sommerville equations (see Appendix A of [15] for dimensions 3–8).

Since the linear relations are understood, it is natural to try to understand the linear inequalities satisfied by flag f -vectors of polytopes. In dimension 3 the linear inequalities actually tell the whole story. The flag f -vector is determined by the f -vector, and the f -vectors of 3-polytopes are characterized by the following theorem.

Theorem 2 (Steinitz). *An integer vector (f_0, f_1, f_2) is the f -vector of a 3-polytope if and only if it satisfies the Euler relation $f_2 - f_1 + f_0 = 2$ and*

$$\begin{aligned} \text{(i)} \quad & f_0 \leq 2f_2 - 4, \\ \text{(ii)} \quad & f_2 \leq 2f_0 - 4. \end{aligned} \tag{2}$$

In higher dimensions the situation is not as well understood. One source of linear inequalities is the *toric g -vector* introduced by Stanley [18]. This vector and the related toric h -vector depend only on the flag f -vector of the polytope, and can be defined recursively via two polynomials:

Definition 1. Let $h(Q, x) = \sum_{i=0}^d h_i(Q)x^{d-i}$ and $g(Q, x) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i(Q)x^i$ where

$$\begin{aligned} \text{(i)} \quad & h(\emptyset, x) = g(\emptyset, x) = 1, \\ \text{(ii)} \quad & g_i(Q) = h_i(Q) - h_{i-1}(Q) \quad \text{for } 0 \leq i \leq \lfloor d/2 \rfloor, \\ \text{(iii)} \quad & h(Q, x) = \sum_{\emptyset \subseteq F \subset Q} g(F, x)(x-1)^{\dim(Q) - \dim(F) - 1}, \end{aligned} \tag{3}$$

setting $h_{-1} = 0$. The coefficients of these polynomials are the entries of the toric h - and g -vectors.

The h_i 's and g_i 's are all linear combinations of entries of the flag f -vector. Notice that $h_0 = g_0 = 1$.

As a consequence of the generalized Dehn–Sommerville equations we have

Theorem 3 [17]. *If Q is a polytope (or, more generally, an Eulerian poset), then*

$$h_i(Q) = h_{d-i}(Q). \tag{4}$$

If Q is a rational polytope, $h_i(Q)$ is the rank of the $(2d - 2i)$ th intersection homology group of the toric variety associated with Q . This interpretation of h_i allowed Stanley to prove the following:

Theorem 4 [18]. *If Q is a rational polytope, then*

$$g_i(Q) \geq 0. \quad (5)$$

It is conjectured that $g_i \geq 0$ for all polytopes, not just rational ones. This has been proven in the cases $i = 1$ ($g_1 = f_0 - (d + 1) \geq 0$ since it takes $d + 1$ points to affinely span d -space) and $i = 2$ ($g_2 = f_{02} - 3f_2 + f_1 - (d + 1)f_0 + \binom{d+1}{2} \geq 0$ by a rigidity argument due to Kalai [13]), but remains an open question for $i \geq 3$. See Appendix B of [15] for a list of the g_i^d 's for $d \leq 10$.

We can use Kalai's convolution operation to combine the g_i 's to create more linear inequalities [14]. Given f_S^m and f_T^n , define their convolution to be

$$f_S^m * f_T^n = \sum_{\substack{F \subset Q \\ \dim(F)=m}} f_S^m(F) f_T^n(Q/F),$$

where Q/F is the quotient of Q by the face F (see [21] for a discussion of quotient polytopes). Note the convolution is just $f_{S \cup \{m\} \cup (T+m+1)}^{m+n+1}$, where $T+m+1 = \{i+m+1 \mid i \in T\}$. Now extend this to linear forms $M = \sum_S \alpha_S f_S^m$ and $N = \sum_T \beta_T f_T^n$ ($\alpha_S, \beta_T \in \mathbf{R}$) by

$$M * N = \sum_{\substack{F \subset Q \\ \dim(F)=m}} \sum_{S,T} \alpha_S \beta_T f_S^m(F) f_T^n(Q/F) = \sum_{S,T} \alpha_S \beta_T f_S^m * f_T^n. \quad (6)$$

If $M \geq 0$ for all m -polytopes and $N \geq 0$ for all n -polytopes, then $M * N \geq 0$ for all $(m + n + 1)$ -polytopes.

Although $f_\emptyset$ is 1 for any polytope, it is useful to think of it as a linear form in its own right. This convention allows us to convolve linear forms involving constants. For example, $g_1^2 * g_0^0 = (f_0^2 - 3f_\emptyset^2) * f_\emptyset^0 = f_{02}^3 - 3f_2^3 \geq 0$ for all 3-polytopes. Rewritten in terms of f_0 and f_2 this becomes the second of Steinitz's inequalities; the first is $g_0^0 * g_1^2 \geq 0$.

Every polytope Q has a *dual* polytope Q^* ; see [21] for a geometric construction. There is a bijection between the i -faces of Q and the $(d - i - 1)$ -faces of Q^* . So $f_i(Q) = f_{d-i-1}(Q^*)$ and more generally $f_{i_1, i_2, \dots, i_k}(Q) = f_{d-i_k-1, \dots, d-i_2-1, d-i_1-1}(Q^*)$. Thus a linear form M has a dual form $\overline{M}$ where $\overline{M}(Q)$ is defined to be $M(Q^*)$. In particular, the $\overline{g}_i$'s are also non-negative for all rational polytopes.

This paper examines the relationships among the currently known flag f -vector inequalities. In Section 2 we show that some of the toric g inequalities are implied by other toric g inequalities. In Section 3 we discuss the **cd**-index and show that some inequalities derived from the **cd**-index are not implied by the toric g inequalities. In Section 4 we show that the toric g inequalities are not all implied by the **cd**-index inequalities, and in Section 5 we show that the convolution of some **cd**-index inequalities yields inequalities implied by other **cd**-index inequalities.

2. Redundancy of Some Inequalities From the Toric g -Vector

By convolving the g_i 's and $\overline{g}_i$'s we can generate many linear inequalities satisfied by flag f -vectors of rational polytopes. Are any of these inequalities implied by the others?

Meisinger [15] showed that $g_0^0 * g_0^0$ and $g_0^0 * g_0^1$ are redundant. We have $g_0^0 * g_0^0 = f_0^0 * f_0^0 = f_0^1 = 2g_0^1$ since an edge has two vertices and $g_0^0 * g_0^1 = f_0^2 = f_1^2 = g_0^1 * g_0^0$ since a polygon has equal numbers of vertices and edges. He also proved that if a linear form is redundant, then every convolution containing that form is redundant.

Kalai (unpublished) and Bayer and Klapper [4] proved that if d is even, $g_{d/2}^d = \bar{g}_{d/2}^d$. We also have $g_0^d = 1 = \bar{g}_0^d$ for all d . Bayer and Ehrenborg [3] showed that these are the only redundancies within the set $\{g_0^d, g_1^d, \dots, g_{\lfloor d/2 \rfloor}^d, \bar{g}_0^d, \bar{g}_1^d, \dots, \bar{g}_{\lfloor d/2 \rfloor}^d\}$. Meisinger [15] conjectured that there are two additional redundancies involving convolutions. We prove this conjecture in the form of

Theorem 5. *For $i \geq 0$,*

$$(i+2)g_i^{2i+1} = \sum_{j=0}^i (j+1)(g_j^{2j} * g_{i-j}^{2(i-j)}), \quad (7)$$

$$(i+2)\bar{g}_i^{2i+1} = \sum_{j=0}^i (i-j+1)(g_j^{2j} * g_{i-j}^{2(i-j)}). \quad (8)$$

The second equation is the dual of the first, so it is enough to prove (7). We prove it by showing this equation holds for a basis of $(2i+1)$ -polytopes, and therefore for all $(2i+1)$ -polytopes. In [1] Bayer and Billera described a basis created by successive application of two operations: *pyramid*, denoted P , and *bipyramid*, denoted B . Let Q be a d -polytope contained in H , a hyperplane in $\mathbf{R}^{d+1}$. The pyramid over Q is the convex hull of Q and a point not in H . The bipyramid over Q is the convex hull of Q and a line segment that intersects the interior of Q and has one endpoint on each side of H . The choices of point and line segment affect the geometry of the pyramid and bipyramid, but do not change their combinatorial types.

A word in B and P represents the polytope given by applying these operations from right to left, where the rightmost P stands for the initial point (or pyramid over the empty polytope). For example, $PP = BP$ is a line segment, PPP is a triangle, BPP is a square, and $PBPP$ is a pyramid over a square. Words of length $2i+2$ with no two adjacent B 's and ending in PP are a basis of $(2i+1)$ polytopes.

Before we begin, we need three lemmas. Kalai notes that $g(PQ, x) = g(Q, x)$ [14, Remark 3], where Q is any polytope. So we have

Lemma 1. *If $d \geq 1$, then*

$$g_i^d(PQ) = g_i^{d-1}(Q). \quad (9)$$

If $\dim(PQ) = 2j$, $j \geq 1$, then $g(Q)$ has degree at most $j-1$, so the coefficient of x^j in $g(PQ)$ must be 0. This gives us

Lemma 2. *If $j \geq 1$, then*

$$g_j^{2j}(PQ) = 0. \quad (10)$$

In the case $j = 0$, $g_0^0 = 1$. Because Lemma 2 does not apply, we consider the cases $j = 0$ and $i - j = 0$ separately when we look at the right-hand side of (7).

We will frequently encounter polytopes of the form $(PQ)/F$ and $(BQ)/F$, where F is a face of Q . The proof of the following lemma is straightforward.

Lemma 3. *If F is a face of Q , then $(PQ)/F = P(Q/F)$ and $(BQ)/F = B(Q/F)$.*

Because of these equalities, we often omit the parentheses and write PQ/F and BQ/F .

Proof of Theorem 5. Our proof is by induction on i . As we saw at the beginning of Section 2, $g_0^0 * g_0^0 = 2g_0^1$, so the theorem holds for the base case $i = 0$.

Suppose the conjecture is true for all $(2k + 1)$ -polytopes, $k < i$. Consider the basis of $(2i + 1)$ -polytopes. Each word begins with either P or BP ; we consider these two cases separately.

Case 1: PQ . Suppose the polytope is of the form PQ , where Q is a $(2i)$ -polytope. On the left-hand side of (7) we have $(i + 2)g_i^{2i+1}(PQ) = (i + 2)g_i^{2i}(Q)$ by Lemma 1.

Now consider the right-hand side of (7) in the case $j \neq 0, i$,

$$g_j^{2j} * g_{i-j}^{2(i-j)}(PQ) = \sum_{\substack{F \subset PQ, \\ \dim(F)=2j}} g_j^{2j}(F)g_{i-j}^{2(i-j)}(P(Q/F)) = 0 \quad (11)$$

by Lemmas 3 and 2.

Suppose $j = i$. The facets of PQ are Q itself and pyramids over facets of Q , so

$$g_i^{2i} * g_0^0(PQ) = g_i^{2i}(Q) + \sum_{\substack{F \subset Q, \\ \dim(F)=2i-1}} g_i^{2i}(F)g_0^0(PQ/PF) \quad (12)$$

$$= g_i^{2i}(Q), \quad (13)$$

again by Lemma 2. So the $j = i$ term contributes $(i + 1)g_i^{2i}(Q)$ to the right-hand side of (7).

Now consider the case $j = 0$. The vertices of PQ are the vertex added in taking the pyramid over Q and the vertices of Q itself. Thus

$$g_0^0 * g_i^{2i}(PQ) = g_i^{2i}(Q) + \sum_{\substack{V \text{ a vert} \\ \text{of } Q}} g_i^{2i}(PQ/V) \quad (14)$$

$$= g_i^{2i}(Q) \quad (15)$$

again by Lemma 2.

So the right-hand side of (7) is $(i + 2)g_i^{2i}(Q)$, as desired, and the theorem holds for polytopes of the form PQ .

Case 2: BPQ . Now consider polytopes of the form BPQ , where Q is a $(2i - 1)$ -polytope. To show that the conjecture holds for them, we will need a pair of lemmas. Given that $h(BX, x) = (1 + x)h(X)$ [14, Remark 3], their proofs are straightforward.

Lemma 4. For $k \geq 1$,

$$g_i^k(BX) = h_i^{k-1}(X) - h_{i-2}^{k-1}(X) = g_i^{k-1}(X) + g_{i-1}^{k-1}(X). \quad (16)$$

Lemma 5. For $i \geq 1$,

$$g_i^{2i}(BX) = g_{i-1}^{2i-1}(X) \quad (17)$$

and in particular $g_i^{2i}(BPQ) = g_{i-1}^{2i-1}(PQ) = g_{i-1}^{2i-2}(Q)$.

Proof of Case 2. Consider the left-hand side of (7). By Lemmas 4, 2, and 1,

$$\begin{aligned} (i+2)g_i^{2i+1}(BPQ) &= (i+2)(g_i^{2i}(PQ) + g_{i-1}^{2i}(PQ)) \\ &= (i+2)g_{i-1}^{2i}(PQ) \\ &= (i+2)g_{i-1}^{2i-1}(Q) \\ &= (i+2)g_{i-1}^{2(i-1)+1}(Q), \end{aligned}$$

so we are set up for induction.

Consider the right-hand side of (7) for $j \neq 0, i$. Omitting the factor of $(j+1)$ we have

$$g_j^{2j} * g_{i-j}^{2(i-j)}(BPQ) = \sum_{\substack{F \subset PQ, \\ \dim(F)=2j}} g_j^{2j}(F) g_{i-j}^{2(i-j)}(BPQ/F) \quad (18)$$

$$= \sum_{\substack{F \subset Q, \\ \dim(F)=2j}} g_j^{2j}(F) g_{i-j}^{2(i-j)}(BPQ/F) \quad (19)$$

$$= \sum_{\substack{F \subset Q, \\ \dim(F)=2j}} g_j^{2j}(F) g_{i-j-1}^{2(i-j-1)}(Q/F) \quad (20)$$

$$= g_j^{2j} * g_{i-j-1}^{2(i-j-1)}(Q). \quad (21)$$

We get (18) by noting that each $2j$ -face of BPQ is either a $2j$ -face of PQ or a pyramid over a $(2j-1)$ -face of PQ , and the second set of faces contributes 0 to the sum. Similarly, expression (19) comes from noting that each face of PQ is either a $2j$ -face of Q or a pyramid over a $(2j-1)$ -face of Q , and the second set of faces contributes 0 to the sum. Expression (20) follows from Lemmas 5 and 1. So each $j \neq 0, i$ contributes $(j+1)(g_j^{2j} * g_{i-j-1}^{2(i-j-1)}(Q))$ to the right-hand side.

When $j = i$,

$$(i+1)(g_i^{2i} * g_0^0)(BPQ) = (i+1) \left(\sum_{\substack{F \subset BPQ, \\ \dim(F)=2i}} g_i^{2i}(F) g_0^0(BPQ/F) \right) = 0 \quad (22)$$

by Lemma 2 since every facet of BPQ is a pyramid over a facet of PQ .

When $j = 0$,

$$g_0^0 * g_i^{2i}(BPQ) = 2g_i^{2i}(PQ) + \sum_{\substack{V \text{ a vert} \\ \text{of } PQ}} g_i^{2i}(BPQ/V) \quad (23)$$

$$= g_i^{2i}(BQ) + \sum_{\substack{V \text{ a vert} \\ \text{of } Q}} g_i^{2i}(BPQ/V) \quad (24)$$

$$= g_{i-1}^{2i-1}(Q) + \sum_{\substack{V \text{ a vert} \\ \text{of } Q}} g_{i-1}^{2i-2}(Q/V) \quad (25)$$

$$= g_{i-1}^{2i-1}(Q) + g_0^0 * g_{i-1}^{2(i-1)}(Q). \quad (26)$$

We get (23) by considering first the two vertices added in bipyramiding over PQ and then the vertices of PQ itself. By Lemma 2 the first term is 0. We get the first term of (24) from the vertex V of PQ added in pyramiding over Q ; note $PQ/V = Q$ in this case. The second term comes from the remaining vertices of PQ , which are just vertices of Q . Expression (25) follows from Lemmas 5 and 1.

Putting this all together, we see that the right-hand side of (7) is

$$\begin{aligned} g_{i-1}^{2i-1}(Q) + \sum_{j=0}^{i-1} (j+1)g_j^{2j} * g_{i-j-1}^{2(i-j-1)}(Q) &= g_{i-1}^{2(i-1)+1}(Q) + ((i-1)+2)(g_{i-1}^{2i-1}(Q)) \\ &= (i+2)(g_{i-1}^{2i-1}(Q)) \end{aligned} \quad (27)$$

by induction. We also have that the left-hand side of (7) is $(i+2)g_{i-1}^{2(i-1)+1}(Q)$. So the theorem holds for polytopes of the form BPQ .

The theorem holds for all $(2i+1)$ -polytopes in the basis, and therefore it is true for all $(2i+1)$ -polytopes. $\square$

We note that this result makes sense in light of a theorem of Braden and MacPherson [8]. They introduced the *relative g -polynomial* $g(Q, F)$ as a generalization of the toric g -vector. Here we follow their notation and let $g(Q) = g(Q, x)$.

Definition 2. The family of polynomials $g(Q, F)$ associated to a polytope Q and a face F of Q is defined inductively by the following relation: for all Q, F we have

$$\sum_{F \subseteq E \subseteq Q} g(E, F)g(Q/E) = g(Q). \quad (28)$$

Note the induction begins with $Q = F$, giving $g(F, F) = g(F)$. In general the left-hand side gives $g(Q, F) \cdot 1$ plus terms involving $g(E, F)$ with $\dim(E) < \dim(Q)$.

Theorem 6 [8].

$$g(Q, F) = \sum_{F \subseteq F' \subseteq Q} (-1)^{\dim Q - \dim F'} g(F') \overline{g}(Q/F'). \quad (29)$$

Consider this equality in the case $\dim(Q) = 2i + 1$ and $F = \emptyset$. When $F' = \emptyset$, we have $\bar{g}(Q)$ on the right-hand side, and when $F' = Q$, we have $g(Q)$, and both of these polynomials have degree at most i . Similarly, if $\dim(F') = 2j$, the F' term has degree at most i . However, if $\dim(F') = 2j + 1$, $j \neq -1, i$, the F' term has degree at most $i - 1$. So $g(Q, \emptyset)$ has degree at most i . Recall that $\bar{g}_k^{2k} = g_k^{2k}$. Then we have that the coefficient of x^i in $g(Q, \emptyset)$ is

$$\begin{aligned} g_i^{2i+1}(Q) + \bar{g}_i^{2i+1}(Q) + \sum_{j=0}^i \sum_{\substack{\emptyset \subset F' \subset Q \\ \dim F' = 2j}} (-1)^{2i+1-2j} g_j^{2j}(F') g_{i-j}^{2i-2j}(Q/F') \\ = g_i^{2i+1}(Q) + \bar{g}_i^{2i+1}(Q) - \sum_{j=0}^i g_j^{2j} * g_{i-j}^{2i-2j}(Q). \end{aligned}$$

However, we also have

Theorem 7 [19]. *For any polytope $Q \neq \emptyset$,*

$$\sum_{\emptyset \subseteq F' \subseteq Q} (-1)^{\dim F'} g(F') \bar{g}(Q/F') = 0. \quad (30)$$

This gives us $g(Q, \emptyset) = 0$. Thus (30) becomes

$$g_i^{2i+1}(Q) + \bar{g}_i^{2i+1}(Q) = \sum_{j=0}^i g_j^{2j} * g_{i-j}^{2i-2j}(Q), \quad (31)$$

which is just the sum of (7) and (8) divided by $(i + 2)$.

3. Some cd-Index Inequalities That Are Not Implied by Toric g -Vector Inequalities

We have seen that convolutions of $g_i^{d_i}$ s and $\bar{g}_i^{d_i}$ s are non-negative for rational polytopes. Kalai [14] conjectured that all non-negative forms arise in this way, but Meisinger [15] gave a counterexample. He showed that $f_2^6 - 35 \geq 0$, which holds for all 6-polytopes, is not implied by the non-negativity of the g convolutions. He further conjectured

Conjecture 1 [15]. *For $d \geq 6$ the linear inequalities $f_j - \binom{d+1}{j+1} \geq 0$ ($2 \leq j \leq d - 3$) are not a non-negative combination of convolutions of the linear forms g_i^k and $\bar{g}_i^k$ ($0 \leq k \leq d, 0 \leq i \leq \lfloor k/2 \rfloor$).*

We will prove this conjecture and show that the non-negativity of these forms is actually implied by inequalities derived from the **cd**-index. This means that these **cd**-index inequalities are not implied by the non-negativity of the g convolutions.

The **cd**-index is another way to encode the information in the flag f -vector. It was introduced by Fine and developed by Bayer and Klapper [4]. To define it, we first need the *flag h -vector*, given by $h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T$ for $S \subseteq \{0, 1, \dots, d - 1\}$. Now let

$\mathbf{a}$ and $\mathbf{b}$ be non-commuting variables and let $u_S = u_0 u_1 \cdots u_{d-1}$, where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Then we can define the **ab-index** Ψ of a polytope Q by

$$\Psi(Q) = \sum_S h_S u_S. \quad (32)$$

As a consequence of the generalized Dehn–Sommerville equations, Ψ can always be written in terms of $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. When it is written this way, it is known as the **cd-index**. For example, when $d = 1$, $\Psi(Q) = \mathbf{c}$; when $d = 2$, $\Psi(Q) = \mathbf{c}^2 + (f_0(Q) - 2)\mathbf{d}$; and when $d = 3$, $\Psi(Q) = \mathbf{c}^3 + (f_0(Q) - 2)\mathbf{dc} + (f_1(Q) - f_0(Q))\mathbf{cd}$. If we assign a weight of 1 to each $\mathbf{c}$ and 2 to each $\mathbf{d}$, each word in the **cd-index** of a d -polytope has weight d .

Denote the coefficient of the **cd** word $\mathbf{w}$ by $[\mathbf{w}]$. Meisinger [15, Appendix D] generated a list of all $[\mathbf{w}]$ for $3 \leq d \leq 8$. The **cd-index** behaves nicely under duality; $[\mathbf{w}](Q^*) = [\bar{\mathbf{w}}](Q)$, where $\bar{\mathbf{w}}$ is $\mathbf{w}$ written backwards [4]. We can see this in dimension 3: $[\mathbf{dc}] = f_0 - 2$ and $[\mathbf{cd}] = f_1 - f_0 = f_2 - 2$, and $f_0(Q^*) = f_2(Q)$.

How are $[\mathbf{wc}]$ and $[\mathbf{wd}]$ related to $[\mathbf{w}]$? To answer this question, we need to introduce yet another vector, the *flag k -vector* [6]. For S sparse, let $k_S = \sum_{T \subseteq S} (-2)^{|S|-|T|} f_T$. Billera, Ehrenborg, and Readdy [5] showed

Theorem 8 [6]. *Let $\mathbf{w} = \mathbf{c}^{n_1} \mathbf{dc}^{n_2} \mathbf{dc}^{n_3} \cdots \mathbf{c}^{n_p} \mathbf{dc}^{n_{p+1}}$ and define $m_0, \dots, m_p$ by $m_0 = 0$ and $m_i = m_{i-1} + n_i + 2$. Then*

$$[\mathbf{w}] = \sum_{i_1, \dots, i_p} (-1)^{(m_1-i_1)+(m_2-i_2)+\cdots+(m_p-i_p)} k_{i_1, i_2, \dots, i_p}, \quad (33)$$

where the sum is over all p -tuples $(i_1, i_2, \dots, i_p)$ such that $m_{j-1} \leq i_j \leq m_j - 2$.

For further discussion of the flag k -vector, see [9].

Lemma 6. *If $[\mathbf{w}] = \sum_{S \in Sp(d)} \alpha_S f_S^d$, then $[\mathbf{wc}] = \sum_{S \in Sp(d)} \alpha_S f_S^{d+1}$.*

This is clear because n_{p+1} is not involved in the expression for $[\mathbf{w}]$ in terms of the flag k -vector. So, for example, we have $[\mathbf{cdc}^k] = f_1^{k+3} - f_0^{k+3}$ for all $k \geq 0$.

Lemma 7. *If $[\mathbf{w}] = \sum_{S \in Sp(d)} \alpha_S f_S^d$, then $[\mathbf{dw}] = \sum_{S \in Sp(d)} \alpha_S (f_{\{0\} \cup (S+2)}^{d+2} - 2f_{(S+2)}^{d+2})$.*

Proof. In the flag k -vector expression for $[\mathbf{dw}]$, $n_1 = 0$, so $i_1 = 0$. Then each

$$(-1)^{(m_1-i_1)+(m_2-i_2)+\cdots+(m_p-i_p)} k_{i_1, i_2, \dots, i_p}$$

in $[\mathbf{w}]$ corresponds to

$$(-1)^{(2-0)+(m_1+2-(i_1+2))+(m_2+2-(i_2+2))+\cdots+(m_p+2-(i_p+2))} k_{0, i_1+2, i_2+2, \dots, i_p+2}$$

in $[\mathbf{dw}]$. Further, since every subset of $\{0, i_1 + 2, i_2 + 2, \dots, i_p + 2\}$ is either a subset of $(S + 2) = \{i_1 + 2, i_2 + 2, \dots, i_p + 2\}$ or $\{0\}$ union such a subset, each f_T in $k_{i_1, i_2, \dots, i_p}$ corresponds to $f_{\{0\} \cup (T+2)} - 2f_{(T+2)}$ in $k_{0, i_1+2, i_2+2, \dots, i_p+2}$. Therefore, $\alpha_S f_S^d$ in $[\mathbf{w}]$ contributes $\alpha_S (f_{\{0\} \cup (S+2)}^{d+2} - 2f_{(S+2)}^{d+2})$ to $[\mathbf{dw}]$. $\square$

For example, $[\mathbf{cd}] = f_1^3 - f_0^3$ and $[\mathbf{dcd}] = f_{03}^5 - 2f_3^5 - f_{02}^5 + 2f_2^5$.
By duality, we have

Corollary 1. *Given $[\mathbf{w}]$ written in the sparse basis, we can find $[\mathbf{wd}]$. Each $\alpha_T f_T^d$ in $[\mathbf{w}]$ contributes $\alpha_T(f_{T \cup \{d+1\}}^{d+2} - 2f_T^{d+2})$ to $[\mathbf{wd}]$.*

Notice that $f_{T \cup \{d+1\}}^{d+2}$ is not part of the sparse basis, so to write $[\mathbf{wd}]$ in terms of that basis we need to apply the generalized Dehn–Sommerville equations.

Stanley [20] showed that if $\mathbf{w}$ has weight d , then $[\mathbf{w}]$ is non-negative for all d -polytopes. Furthermore,

Theorem 9 [5]. *For Q a d -polytope,*

$$[\mathbf{w}](Q) - [\mathbf{w}](\Delta^d) \geq 0,$$

where Δ^d is the d -simplex.

So the $\mathbf{cd}$ -index gives us more linear inequalities for flag f -vectors.

From Lemma 6, Corollary 1, and the Euler–Poincaré relation, we can see that

$$[\mathbf{c}^j \mathbf{dc}^{d-j-2}] = f_j - f_{j-1} + \cdots + (-1)^j f_0 - 1 + (-1)^{j+1} \quad (34)$$

so

$$f_j = [\mathbf{c}^j \mathbf{dc}^{d-j-2}] + [\mathbf{c}^{j-1} \mathbf{dc}^{d-j-1}] + 2[\mathbf{c}^d]. \quad (35)$$

Therefore we have

Theorem 10. *For Q a d -polytope, Billera and Ehrenborg’s $\mathbf{cd}$ -index inequalities imply the following previously known (see Section 10.2 of [12]) inequalities:*

$$f_j(Q) - f_j(\Delta^d) = f_j(Q) - \binom{d+1}{j+1} \geq 0. \quad (36)$$

If $j \in \{0, 1, d-2, d-1\}$, then this inequality can be expressed as a non-negative linear combination of g -convolutions: $f_0^d - (d+1) = g_1^d$, $f_1^d - \binom{d+1}{2} = \frac{1}{2}(g_0^0 * g_1^{d-1} + dg_1^d)$, $f_{d-2}^d - \binom{d+1}{d-1} = \frac{1}{2}(\overline{g}_1^{d-1} * g_0^0 + d\overline{g}_1^d)$, and $f_{d-1}^d - (d+1) = \overline{g}_1^d$ [15]. Additionally, $f_2^5 - 15 = \frac{3}{2}(g_1^5 + \overline{g}_1^5) + \frac{1}{2}(g_0^0 * g_1^4 + \overline{g}_1^4 * g_0^0)$.

We prove the following theorem and in the process prove Meisinger’s conjecture.

Theorem 11. *For Q a d -polytope, $d \geq 6$, and $2 \leq j \leq d-4$, the inequality $[\mathbf{c}^j \mathbf{dc}^{d-j-2}](Q) - [\mathbf{c}^j \mathbf{dc}^{d-j-2}](\Delta^d) \geq 0$ is not implied by the non-negativity of the g convolutions.*

In his proof of the case $d = 6$, $j = 2$ of his conjecture, Meisinger introduced the point

$$\begin{aligned} X_6 &= (f_0, f_1, f_2, f_3, f_4, f_{02}, f_{03}, f_{04}, f_{13}, f_{14}, f_{24}, f_{024}) \\ &= (7, 21, 0, 0, 21, 105, 350, 315, 620, 840, 630, 2520), \end{aligned}$$

which satisfies all the inequalities generated by the g convolutions but obviously violates the inequality $f_2 - 35 \geq 0$. It also violates $[\mathbf{ccdc}](Q) - [\mathbf{ccdc}](\Delta^d) = f_2 - f_1 + f_0 - 21 \geq 0$. Therefore, neither $f_2 - 35$ nor $f_2 - f_1 + f_0 - 21$ can be a non-negative combination of g convolutions.

If we know the flag f -vector of a polytope Q , we can find the flag f -vector of PQ , the pyramid over Q . Using the conventions $f_{-1} = 1 = f_d$ and $f_{S \cup \{i\} \cup \{i\}} = f_{S \cup \{i\}}$ we have

Lemma 8. *If PQ is the pyramid over the polytope Q , then*

$$f_{i_1, i_2, \dots, i_k}(PQ) = f_{i_1, i_2, \dots, i_k}(Q) + \sum_{j=1}^k f_{i_1, \dots, i_j, i_{j+1}-1, \dots, i_k-1}(Q) \quad (37)$$

and in particular

$$f_j(PQ) = f_j(Q) + f_{j-1}(Q). \quad (38)$$

Proof. Every face of PQ is either a face of Q or the pyramid over a face of Q . If a face $F_l \subset PQ$ is the pyramid over $F'_l \subset Q$ and $F_l \subset F_m \subset PQ$, then $F_m = PF'_m$ for some $F'_m \subset Q$. So every chain $F_{i_1} \subset F_{i_2} \subset \dots \subset F_{i_k}$ in PQ corresponds to a chain $F_{i_1} \subset \dots \subset F_{i_j} \subset F'_{i_{j+1}} \subset \dots \subset F'_{i_k}$ in Q for some j , $0 \leq j \leq k-1$. $\square$

We use Lemma 8 to define $\text{Pyr}(X)$, the flag f -vector of the “pyramid” over any vector X of length $a_d - 1$.

Definition 3. Let X be a vector in the affine span of the flag f -vectors of d -polytopes. Define $\text{Pyr}(X)$ by

$$f_{i_1, i_2, \dots, i_k}(\text{Pyr}(X)) = f_{i_1, i_2, \dots, i_k}(X) + \sum_{j=1}^k f_{i_1, \dots, i_j, i_{j+1}-1, \dots, i_k-1}(X). \quad (39)$$

Lemma 9. *Let X be a vector in the span of the flag f -vectors of $(d-1)$ -polytopes. Then*

$$g_{i_1}^{d_1} * g_{i_2}^{d_2} * \dots * g_{i_k}^{d_k}(\text{Pyr}(X)) = \sum_{j=0}^{k-1} g_{i_1}^{d_1} * \dots * g_{i_j}^{d_j} * g_{i_{j+1}}^{d_{j+1}-1} * g_{i_{j+2}}^{d_{j+2}} * \dots * g_{i_{k-1}}^{d_{k-1}} * g_{i_k}^{d_k}(X), \quad (40)$$

where $k-1 + \sum_{j=1}^k d_j = d$.

Proof. Let Q be a $(d-1)$ -polytope. Consider

$$\begin{aligned} & g_{i_1}^{d_1} * g_{i_2}^{d_2} * \dots * g_{i_{k-1}}^{d_{k-1}} * g_{i_k}^{d_k}(PQ) \\ &= \sum_{F_1 \subset F_2 \subset \dots \subset F_{k-1} \subset PQ} g_{i_1}^{d_1}(F_1) g_{i_2}^{d_2}(F_2/F_1) \dots g_{i_{k-1}}^{d_{k-1}}(F_{k-1}/F_{k-2}) g_{i_k}^{d_k}(PQ/F_{k-1}), \end{aligned} \quad (41)$$

where $\dim(F_l) = l-1 + \sum_{j=1}^l d_j$.

From the proof of Lemma 8, we know that for each chain $F_1 \subset F_2 \subset \cdots \subset F_{k-1} \subset PQ$ there is a $0 \leq j \leq k-1$ such that

$$\begin{aligned} (F_1 \subset \cdots \subset F_j \subset F_{j+1} \cdots \subset F_{k-1} \subset PQ) \\ = (F_1 \subset \cdots \subset F_j \subset PF'_{j+1} \cdots \subset PF'_{k-1} \subset PQ), \end{aligned} \quad (42)$$

where $F_1 \subset \cdots \subset F_j \subseteq F'_{j+1} \subset \cdots \subset F'_{k-1} \subset Q$.

It is easy to see that $(PF_l)/(PF_{l-1}) = F_l/F_{l-1}$. Therefore by Lemmas 3 and 1 we have

$$\begin{aligned} g_{i_1}^{d_1}(F_1) \cdots g_{i_j}^{d_j}(F_j/F_{j-1}) g_{i_{j+1}}^{d_{j+1}}(F_{j+1}/F_j) \cdots g_{i_{k-1}}^{d_{k-1}}(F_{k-1}/F_{k-2}) g_{i_k}^{d_k}(PQ/F_{k-1}) \\ = g_{i_1}^{d_1}(F_1) \cdots g_{i_j}^{d_j}(F_j/F_{j-1}) g_{i_{j+1}}^{d_{j+1}}(PF'_{j+1}/F_j) \cdots \\ g_{i_{k-1}}^{d_{k-1}}(PF'_{k-1}/PF'_{k-2}) g_{i_k}^{d_k}(PQ/PF'_{k-1}) \\ = g_{i_1}^{d_1}(F_1) \cdots g_{i_j}^{d_j}(F_j/F_{j-1}) g_{i_{j+1}}^{d_{j+1}}(P(F'_{j+1}/F_j)) \cdots \\ g_{i_{k-1}}^{d_{k-1}}(F'_{k-1}/F'_{k-2}) g_{i_k}^{d_k}(Q/F'_{k-1}) \\ = g_{i_1}^{d_1}(F_1) \cdots g_{i_j}^{d_j}(F_j/F_{j-1}) g_{i_{j+1}}^{d_{j+1}-1}(F'_{j+1}/F_j) \cdots \\ g_{i_{k-1}}^{d_{k-1}}(F'_{k-1}/F'_{k-2}) g_{i_k}^{d_k}(Q/F'_{k-1}). \end{aligned} \quad (43)$$

Two special cases need comment. First, if $d_{j+1} = 2i_{j+1} \neq 0$, then $g_{i_{j+1}}^{d_{j+1}-1}(F'_{j+1}/F_j) = 0$. Second, if $F'_{j+1} = F_j$, we have $d_{j+1} = 0 = i_{j+1}$. Then in (43) we have g_0^{-1} which we define as $f_\emptyset^{-1}$; we have $f_\emptyset^{-1} * M = M$ for M a linear form.

Equation (43) only involves faces of Q . If we sum this over all possible chains

$$F_1 \subset \cdots \subset F_j \subseteq F'_{j+1} \subset \cdots \subset F'_{k-1} \subset Q,$$

we get $g_{i_1}^{d_1} * \cdots * g_{i_j}^{d_j} * g_{i_{j+1}}^{d_{j+1}-1} * \cdots * g_{i_{k-1}}^{d_{k-1}} * g_{i_k}^{d_k}(Q)$. Finally, if we sum over all j , we get

$$g_{i_1}^{d_1} * g_{i_2}^{d_2} * \cdots * g_{i_{k-1}}^{d_{k-1}} * g_{i_k}^{d_k}(PQ) = \sum_{j=0}^{k-1} g_{i_1}^{d_1} * \cdots * g_{i_j}^{d_j} * g_{i_{j+1}}^{d_{j+1}-1} * \cdots * g_{i_{k-1}}^{d_{k-1}} * g_{i_k}^{d_k}(Q). \quad (44)$$

We have shown this equality for Q a $(d-1)$ -polytope. However, since it is true for all $(d-1)$ -polytopes, it is true for all points in the span of their flag f -vectors. $\square$

Proof of Theorem 11. Starting with X_6 , recursively define $X_d = \text{Pyr}(X_{d-1})$. We will show that X_d satisfies all the inequalities generated by the g convolutions but violates $[\mathbf{c}^j \mathbf{d}^{d-j-2}](Q) - [\mathbf{c}^j \mathbf{d}^{d-j-2}](\Delta^d) \geq 0$ for $2 \leq j \leq d-4$, and therefore these $\mathbf{cd}$ -index inequalities are not implied by the g inequalities.

Our proof is by induction on d . We have already shown that the statement holds for dimension 6, our base case. Suppose it is true for dimension $d-1$. We know Lemma 9

holds for X_{d-1} . Since

$$g_{i_1}^{d_1} * \cdots * g_{i_j}^{d_j} * g_{i_{j+1}}^{d_{j+1}-1} * \cdots * g_{i_{k-1}}^{d_{k-1}} * g_{i_k}^{d_k}(X_{d-1}) \geq 0$$

for $0 \leq j \leq k-1$, we have $g_{i_1}^{d_1} * g_{i_2}^{d_2} * \cdots * g_{i_{k-1}}^{d_{k-1}} * g_{i_k}^{d_k}(X_d) \geq 0$.

Note that Lemma 9 also holds if some of the g_i are replaced by $\bar{g}_i$, since we know $\bar{g}_i^n(PQ) = g_i^n((PQ)^*) = g_i^n(P(Q^*)) = g_i^{n-1}(Q^*) = \bar{g}_i^{n-1}(Q)$.

So for each dimension $d \geq 6$, we have constructed a point X_d that satisfies all the g inequalities. Now we will show it violates $f_j - \binom{d+1}{j+1} \geq 0$ for $2 \leq j \leq d-3$ and $[\mathbf{c}^j \mathbf{d} \mathbf{c}^{d-j-2}](Q) - [\mathbf{c}^j \mathbf{d} \mathbf{c}^{d-j-2}](\Delta^d) \geq 0$ for $2 \leq j \leq d-4$.

In the case $d = 6$ we have $f_j(X_d) = f_j(\Delta^d)$ for $j \in \{-1, 0, 1, d-2, d-1, d\}$ and $f_j(X_d) < f_j(\Delta^d)$ for $2 \leq j \leq d-3$. Since both X_d and Δ^d satisfy (38), induction on d gives us these statements for all $d \geq 6$, so we have proved Meisinger's conjecture.

By (34) and (38), we have

$$\begin{aligned} [\mathbf{c}^j \mathbf{d} \mathbf{c}^{d-j-2}](X_d) &= f_j(X_d) - f_{j-1}(X_d) + \cdots + (-1)^j f_0(X_d) - 1 + (-1)^{j+1} \\ &= (f_j(X_{d-1}) + f_{j-1}(X_{d-1})) - (f_{j-1}(X_{d-1}) + f_{j-2}(X_{d-1})) \\ &\quad + \cdots + (-1)^j (f_0(X_{d-1}) + 1) - 1 + (-1)^{j+1} \\ &= f_j(X_{d-1}) - 1. \end{aligned} \tag{45}$$

Similarly,

$$[\mathbf{c}^j \mathbf{d} \mathbf{c}^{d-j-2}](\Delta^d) = f_j(\Delta^{d-1}) - 1. \tag{46}$$

However, we have just shown that $f_j(X_{d-1}) < f_j(\Delta^{d-1})$ for $2 \leq j \leq (d-1)-3$. So X_d satisfies all the g inequalities but violates $[\mathbf{c}^j \mathbf{d} \mathbf{c}^{d-j-2}](Q) - [\mathbf{c}^j \mathbf{d} \mathbf{c}^{d-j-2}](\Delta^d) \geq 0$ for $2 \leq j \leq d-4$, and therefore these inequalities are not implied by the non-negativity of the g convolutions. $\square$

Meisinger [15, Theorem 4.6] asserts that for $3 \leq d \leq 10$ every coefficient in the $\mathbf{cd}$ -index is a non-negative combination of g convolutions. Our Theorem 11 does not directly contradict this, since it involves $[\mathbf{w}](Q) - [\mathbf{w}](\Delta^d) \geq 0$ rather than $[\mathbf{w}](Q) \geq 0$. However, computations with Maple and PORTA indicate that $[\mathbf{ccdc} \mathbf{c}]$, $[\mathbf{ccdc} \mathbf{ccc}]$, $[\mathbf{ccdc} \mathbf{cccd}]$, $[\mathbf{ccdc} \mathbf{cccc}]$, $[\mathbf{cccd} \mathbf{cccc}]$, and their duals are not non-negative combinations of g convolutions.

4. Some Toric g -Vector Inequalities That Are Not Implied by $\mathbf{cd}$ -Index Inequalities

We now know that for $d \geq 6$ not all of the $\mathbf{cd}$ -index inequalities are implied by the g inequalities. What about implications in the other direction? For large enough d , are all of the g inequalities implied by the $\mathbf{cd}$ inequalities? The short answer is “no.” The longer answer is

Theorem 12. *For $d \geq 3$, the inequality $g_1^2 * g_0^{d-3} = f_{02}^d - 3f_2^d \geq 0$ is not implied by the $\mathbf{cd}$ -index inequalities.*

Proof. Our proof follows the same outline as the proof of Theorem 11. We work by induction on d .

Our base case is dimension $d = 3$. The **cd** inequalities are $[\mathbf{ccc}](Q) - [\mathbf{ccc}](\Delta^d) = 1 - 1 \geq 0$, $[\mathbf{cd}](Q) - [\mathbf{cd}](\Delta^d) = f_1(Q) - f_0(Q) - 2 \geq 0$, and $[\mathbf{dc}](Q) - [\mathbf{dc}](\Delta^d) = f_0(Q) - 4 \geq 0$. Let $Y_3 = (f_0^3, f_1^3) = (5, 10)$. Notice that Y_3 satisfies the **cd**-index inequalities. Using the generalized Dehn–Sommerville equations, $f_{02}^3 - 3f_2^3 = 2f_1^3 - 3(f_1^3 - f_0^3 + 2)$, so $g_1^2 * g_0^{d-3}(Y_3) \not\geq 0$ and this g inequality is not implied by the **cd**-index inequalities.

Now recursively define $Y_d = \text{Pyr}(Y_{d-1})$. Suppose that Y_{d-1} satisfies the $(d-1)$ -dimensional **cd**-index inequalities but does not satisfy $g_1^2 * g_0^{(d-1)-3} \geq 0$. We will show that Y_d satisfies all the d -dimensional **cd**-index inequalities but does not satisfy $g_1^2 * g_0^{d-3} \geq 0$.

Ehrenborg and Readdy [11] studied the effect of pyramiding on the **cd**-index of a polytope and found

Theorem 13 [11]. *Let Q be a $(d-1)$ -polytope. Then*

$$\Psi(PQ) = \frac{1}{2}(\Psi(Q)\mathbf{c} + \mathbf{c}\Psi(Q) + D(\Psi(Q))), \quad (47)$$

where D is a derivation given by $D(\mathbf{c}) = 2\mathbf{d}$, $D(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$, and $D(\mathbf{wx}) = D(\mathbf{w})\mathbf{x} + \mathbf{w}D(\mathbf{x})$ for $\mathbf{w}$ and $\mathbf{x}$ any words in $\mathbf{c}$ and $\mathbf{d}$.

As in our proof of Lemma 9, since this is true for all $(d-1)$ -polytopes, it is true for all points in the span of their flag f -vectors. Hence it is true for $\text{Pyr}(Y_{d-1}) = Y_d$. By the induction hypothesis, we know that Y_{d-1} satisfies all of the **cd**-index inequalities, and hence the polynomial $\Psi(Y_{d-1}) - \Psi(\Delta^{d-1})$ has non-negative coefficients. Theorem 13 shows that the coefficients of $\Psi(Y_d) - \Psi(\Delta^d) = \Psi(\text{Pyr}(Y_{d-1})) - \Psi(P\Delta^{d-1})$ are non-negative linear combinations of the coefficients of $\Psi(Y_{d-1}) - \Psi(\Delta^{d-1})$, and hence are also non-negative. So Y_d satisfies all of the **cd**-index inequalities.

Now consider the g inequality. Using Definition 3 and the generalized Dehn–Sommerville equation $f_{01}^{d-1} = 2f_1^{d-1}$, we have

$$\begin{aligned} g_1^2 * g_0^{d-3}(Y_d) &= f_{02}^d(\text{Pyr}(Y_{d-1})) - 3f_2^d(\text{Pyr}(Y_{d-1})) \\ &= f_{02}^{d-1}(Y_{d-1}) + f_{01}^{d-1}(Y_{d-1}) + f_1^{d-1}(Y_{d-1}) \\ &\quad - 3f_2^{d-1}(Y_{d-1}) - 3f_1^{d-1}(Y_{d-1}) \\ &= f_{02}(Y_{d-1}) - 3f_2(Y_{d-1}), \end{aligned}$$

which is negative by the induction hypothesis. $\square$

5. Convolutions of **cd**-Index Inequalities

It is natural to ask if convolutions of **cd** coefficients might generate more inequalities whose non-negativity is not implied by previously known inequalities. We will show that convolutions of **cd** coefficients can be written as non-negative linear combinations of

other $\mathbf{cd}$ coefficients. This means that convolutions of inequalities of the form $[\mathbf{w}](Q) \geq 0$ yield nothing new. Convolutions of the $[\mathbf{w}](Q) \geq [\mathbf{w}](\Delta^d)$ inequalities are still open to investigation.

Theorem 14. *For $\mathbf{w}$ and $\mathbf{x}$ words in $\mathbf{c}$ and $\mathbf{d}$, we have*

$$[\mathbf{wd}] * [\mathbf{dx}] = 2[\mathbf{wdcdx}], \quad (48)$$

$$[\mathbf{wc}] * [\mathbf{dx}] = [\mathbf{wddx}] + 2[\mathbf{wccdx}], \quad (49)$$

$$[\mathbf{wd}] * [\mathbf{cx}] = [\mathbf{wddx}] + 2[\mathbf{wdccx}], \quad (50)$$

$$[\mathbf{wc}] * [\mathbf{cx}] = [\mathbf{wcdx}] + [\mathbf{wdcx}] + 2[\mathbf{wcccx}]. \quad (51)$$

Proof. We begin with the base cases:

$$[\mathbf{d}] * [\mathbf{d}] = (f_0^2 - 2f_\emptyset^2) * (f_0^2 - 2f_\emptyset^2) = -2f_{02}^5 + 2f_{03}^5 - 4f_3^5 + 4f_2^5 = 2[\mathbf{dcd}],$$

$$[\mathbf{c}] * [\mathbf{d}] = (f_\emptyset^1) * (f_0^2 - 2f_\emptyset^2) = f_{02}^4 - 2f_1^4 = [\mathbf{dd}] + 2[\mathbf{ccd}],$$

$$[\mathbf{d}] * [\mathbf{c}] = (f_0^2 - 2f_\emptyset^2) * (f_\emptyset^1) = f_{02}^4 - 2f_2^4 = [\mathbf{dd}] + 2[\mathbf{dcc}],$$

$$[\mathbf{c}] * [\mathbf{c}] = (f_\emptyset^1) * (f_\emptyset^1) = f_1^3 = [\mathbf{cd}] + [\mathbf{dc}] + 2[\mathbf{ccc}].$$

Now we prove that $[\mathbf{u}] * [\mathbf{vc}] = [([\mathbf{u}] * [\mathbf{v}])\mathbf{c}]$ and $[\mathbf{du}] * [\mathbf{v}] = [\mathbf{d}([\mathbf{u}] * [\mathbf{v}])]$. Say $[\mathbf{u}] = \sum_{S \in Sp(m)} \alpha_S f_S^m$ and $[\mathbf{v}] = \sum_{T \in Sp(n)} \beta_T f_T^n$.

By Lemma 6, we know that $[\mathbf{vc}] = \sum_{T \in Sp(n)} \beta_T f_T^{n+1}$. So

$$[\mathbf{u}] * [\mathbf{v}] = \sum_{S \in Sp(m)} \sum_{T \in Sp(n)} \alpha_S \beta_T f_S^m * f_T^n = \sum_{S \in Sp(m)} \sum_{T \in Sp(n)} \alpha_S \beta_T f_{S \cup \{m\} \cup (T+m+1)}^{m+n+1} \quad (52)$$

and

$$[\mathbf{u}] * [\mathbf{vc}] = \sum_{S \in Sp(m)} \sum_{T \in Sp(n)} \alpha_S \beta_T f_{S \cup \{m\} \cup (T+m+1)}^{m+n+2}. \quad (53)$$

Since S and T are sparse, if $0 \notin T$, then $S \cup \{m\} \cup (T+m+1)$ is sparse. If $T = \{0\} \cup T'$, the generalized Dehn–Sommerville equations that express $f_{S \cup \{m, m+1\} \cup (T'+m+1)}^{m+n+1}$ and $f_{S \cup \{m, m+1\} \cup (T'+m+1)}^{m+n+2}$ in terms of the sparse basis are the same. So if $[\mathbf{u}] * [\mathbf{v}] = \sum_{U \in Sp(m+n+1)} \gamma_U f_U^{m+n+1}$, then $[\mathbf{u}] * [\mathbf{vc}] = \sum_{U \in Sp(m+n+1)} \gamma_U f_U^{m+n+2}$. However, by Lemma 6 we also have $[[\mathbf{u}] * [\mathbf{v}]]\mathbf{c} = \sum_{U \in Sp(m+n+1)} \gamma_U f_U^{m+n+2}$, so $[\mathbf{u}] * [\mathbf{vc}] = [[\mathbf{u}] * [\mathbf{v}]]\mathbf{c}$.

Now consider $[\mathbf{du}] * [\mathbf{v}]$. By Lemma 7, if $[\mathbf{u}] = \sum_{S \in Sp(m)} \alpha_S f_S^m$, then $[\mathbf{du}] = \sum_{S \in Sp(m)} \alpha_S (f_{\{0\} \cup (S+2)}^{m+2} - 2f_{(S+2)}^{m+2})$. So we have $[\mathbf{u}] * [\mathbf{v}] = \sum_{S \in Sp(m)} \sum_{T \in Sp(n)} \alpha_S \beta_T f_S^m * f_T^n$ and $[\mathbf{du}] * [\mathbf{v}] = \sum_{S \in Sp(m)} \sum_{T \in Sp(n)} \alpha_S \beta_T (f_{\{0\} \cup (S+2)}^{m+2} - 2f_{(S+2)}^{m+2}) * f_T^n$. As in the $[\mathbf{u}] * [\mathbf{vc}]$ case, we apply the generalized Dehn–Sommerville equations if $0 \in T$. So if $[\mathbf{u}] * [\mathbf{v}] = \sum_{U \in Sp(m+n+1)} \gamma_U f_U^{m+n+1}$, then we have $[\mathbf{du}] * [\mathbf{v}] = \sum_{U \in Sp(m+n+1)} \gamma_U (f_{\{0\} \cup (U+2)}^{m+n+3} - 2f_{(U+2)}^{m+n+3})$. However, by Lemma 7 we also have $[\mathbf{d}([\mathbf{u}] * [\mathbf{v}])] = \sum_{U \in Sp(m+n+1)} \gamma_U (f_{\{0\} \cup (U+2)}^{m+n+3} - 2f_{(U+2)}^{m+n+3})$, so $[\mathbf{du}] * [\mathbf{v}] = [\mathbf{d}([\mathbf{u}] * [\mathbf{v}])]$.

By duality, we also have $[\mathbf{cu}] * [\mathbf{v}] = [\mathbf{c}([\mathbf{u}] * [\mathbf{v}])]$ and $[\mathbf{u}] * [\mathbf{vd}] = [([\mathbf{u}] * [\mathbf{v}])\mathbf{d}]$. To show that $[\mathbf{wd}] * [\mathbf{dx}] = 2[\mathbf{wdcdx}]$, we start with our base case $[\mathbf{d}] * [\mathbf{d}] = 2[\mathbf{dcd}]$ and multiply by appropriate $\mathbf{c}$'s and $\mathbf{d}$'s on the right and left until we have $\mathbf{w}$ and $\mathbf{x}$. The same holds for $[\mathbf{wc}] * [\mathbf{dx}]$, $[\mathbf{wd}] * [\mathbf{cx}]$, and $[\mathbf{wc}] * [\mathbf{cx}]$, and thus we have proved the theorem. $\square$

This means that the coefficients of the $\mathbf{cd}$ -index form an algebra; in fact, they generate the algebra defined by Billera and Liu in [7].

Reading [16, Proposition 21] also proved (48) using a change of basis argument. Ehrenborg [10] suggested that all of Theorem 14 could be proved using coproducts. Let $\delta_{\mathbf{w}}(Q) = [\mathbf{w}](Q)$ if $\text{weight}(\mathbf{w}) = \dim(Q)$ and 0 otherwise. Then we can write $[\mathbf{wd}] * [\mathbf{dx}](Q) = \sum_{\emptyset \subset F \subset Q} \delta_{\mathbf{wd}}(F) \delta_{\mathbf{dx}}(Q/F)$ and similar statements for the remaining three cases. Ehrenborg and Readdy [11] showed that the $\mathbf{cd}$ -index is a Newtonian coalgebra map, and this plus some straightforward calculations will also prove Theorem 14.

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References

1. M. Bayer and L. J. Billera, Counting faces and chains in polytopes and posets. *Contemp. Math.* **34** (1984), 207–252.
2. M. Bayer and L. J. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets. *Invent. Math.* **79** (1985), 143–157.
3. M. Bayer and R. Ehrenborg, On the toric h -vectors of partially ordered sets. *Trans. Amer. Math. Soc.* **352** (2000), 4515–4531.
4. M. Bayer and A. Klapper, A new index for polytopes. *Discrete Comput. Geom.* **6** (1991), 33–47.
5. L. J. Billera and R. Ehrenborg, Monotonicity of the $\mathbf{cd}$ -index for polytopes. *Math. Z.* **233** (2000), 421–441.
6. L. J. Billera, R. Ehrenborg, and M. Readdy, The $\mathbf{cd}$ -index of zonotopes and arrangements, in *Mathematical Essays in Honor of Gian-Carlo Rota*, pp. 23–40. Birkhäuser, Boston, MA, 1998.
7. L. J. Billera and N. Liu, Noncommutative enumeration in graded posets. *J. Algebra Combin.* **12** (2000), 7–24.
8. T. Braden and R. MacPherson, Intersection homology of toric varieties and a conjecture of Kalai. *Comment. Math. Helv.* **74** (1999), 442–455.
9. R. Ehrenborg, A note on the flag k -vector, Preprint.
10. R. Ehrenborg, Personal communication.
11. R. Ehrenborg and M. Readdy, Coproducts and the $\mathbf{cd}$ -index. *J. Algebra Combin.* **48** (1998), 273–299.
12. B. Grünbaum, *Convex Polytopes*. Wiley, London, 1967.
13. G. Kalai, Rigidity and the lower bound theorem, I. *Invent. Math.* **88** (1987), 125–151.
14. G. Kalai, A new basis of polytopes. *J. Combin. Theory Ser. A* **49** (1988), 191–208.
15. G. Meisinger, Flag Numbers and Quotients of Convex Polytopes. Ph.D. Thesis, Universität Passau, Passau, October 1993.
16. N. Reading, Bases for the flag f -vectors of Eulerian posets, Preprint.
17. R. Stanley, *Enumerative Combinatorics*, Vol. 1. Cambridge University Press, Cambridge, 1986.

18. R. Stanley, Generalized h -vectors, intersection cohomology of toric varieties, and related results, in *Commutative Algebra and Combinatorics*, pp. 187–213. Advances in Studies in Pure Mathematics, Vol. 11. Kinokunia, Tokyo, and North-Holland, Amsterdam, 1987.
19. R. Stanley, Subdivisions and local h -vectors. *J. Amer. Math. Soc.* **5** (1992), 805–851.
20. R. Stanley, Flag f -vectors and the $\mathbf{cd}$ -index. *Math Z.* **216** (1994), 483–499.
21. G. Ziegler, *Lectures on Polytopes*. Graduate Texts in Mathematics, Vol. 152. Springer-Verlag, New York, 1995.

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