Slicing the Pie

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Abstract. Given a circular unit disc and n lines such that any two of them intersect inside the disc, we show that there exists a cell of area at least $\pi/4n$.

1. Introduction

The result stated in the abstract was motivated by the following conjecture:

Conjecture 1 [2]. *Let n great circles in general position (i.e., no three having a point in common) divide the sphere into $m = n(n - 1) + 2$ spherical polygons of areas $b_1, b_2, \dots, b_m$. Then $\max b_i / \min b_i \rightarrow \infty$ as $n \rightarrow \infty$.*

Many interesting questions similar with the one above may be raised; we formulate some of them as open problems in a later section. Despite the fact that line arrangements in the plane and great circle arrangements on the sphere received a great deal of attention in the literature, to the best of our knowledge very little is known about the so-called “small face–big face problems.” Loosely speaking, given an arrangement of curves on a surface, we may ask for the existence of a certain cell, the measure of which is either much smaller or much larger than the average. The reader may consult [1] and [5] to get a flavor of these problems.

In this paper we present one such result. We hope that this will stimulate further research in this interesting and difficult area.

In Section 2 we introduce the notions needed in what follows. In particular, in Section 2.2 we follow the presentation from [3]. In Section 3 we define the associated polygon of an arrangement of lines and prove an interesting property for this polygon. In Section 4 we present the proof of our main result and in Section 5 we make a few remarks and present a list of open problems.

2. Basic Definitions

2.1. Arrangements of Lines

An *arrangement of lines* $\mathcal{A}$ is a finite family of $n = n(\mathcal{A})$ lines $L_1, L_2, \dots, L_n$ in the Euclidean plane $\mathbb{E}^2$. If no three lines are concurrent we say that the arrangement is *simple*. Moreover, the arrangement $\mathcal{A}$ is said to be in *general position* if it is simple and no two lines of $\mathcal{A}$ are parallel. In what follows we mostly deal with line arrangements of this latter type.

With an arrangement $\mathcal{A}$, we associate the two-dimensional *cell complex* into which the lines decompose $\mathbb{E}^2$. It is easy to prove that an arrangement in general position determines $\binom{n}{2}$ vertices, n^2 edges, and $1 + \binom{n+1}{2}$ faces (cells), $2n$ of which are unbounded.

For our purposes, it is convenient to consider the cells as open sets. We define the *interior* (*exterior*) of an arrangement of lines as being the union of all bounded (resp. unbounded) cells determined by the arrangement.

2.2. The Winding Number

Let P be an oriented polygon and let A be any point of $\mathbb{E}^2 \setminus P$. Let $R = [A, B)$ be a ray which contains no vertex of P . For each edge E_j of P we define an index $\psi(R, E_j)$ as follows:

$$\psi(R, E_j) = \begin{cases} 0, & \text{if } R \text{ does not intersect } E_j; \\ 1, & \text{if } E_j \text{ crosses } R \text{ in a counterclockwise direction as viewed from } A; \\ -1, & \text{if } E_j \text{ crosses } R \text{ in a clockwise direction as viewed from } A. \end{cases}$$

Then the *winding number* $w(R, P)$ of R with respect to P is defined as

$$w(R, P) = \sum_j \psi(R, E_j),$$

where summation is considered over all the edges of P (see Fig. 1).

Thus, $w(R, P)$ is the signed difference between the number of edges that cross R in a counterclockwise direction and the number that cross R in a clockwise direction. It is easy to see that $w(R, P)$ has the same value of all rays R with endpoint A that do not contain a vertex of P .

Therefore the value of $w(R, P)$ can more appropriately be denoted by $w(A, P)$. As a function of A , $w(A, P)$ is continuous and takes only integer values; hence it is constant as A ranges over any cell C of P . Denote its value for $A \in C$ by $w(C, P)$. The value of $w(C, P)$ is often called the *density* of the cell C with respect to P . It is clear that if A lies in the exterior of P , then $w(A, P) = 0$.

Given an oriented polygon $P = V_1 V_2 \dots V_n$ where each V_i has cartesian coordinates (x_i, y_i) for $i = 1 \dots n$, we define its *generalized area* $G(P)$ as

$$G(P) = \frac{1}{2} \cdot \left| \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i) \right|. \quad (1)$$

Fig. 1. For several rays issuing from A and B , the values of ψ are indicated near each intersection with the polygon P . In this example we have $w(A, P) = -1$, $w(B, P) = 2$.

For simple polygons $G(P)$ is just the usual area. Here and in the following, unless otherwise stated, all subscripts will be taken modulo n . It is well known that for every oriented polygon P ,

$$G(P) = \left| \sum_C w(C, P) \cdot \text{Area}(C) \right|, \quad (2)$$

the summation being over all cells C determined by P . This observation will be useful later.

3. The Polygon Associated with an Arrangement of Lines

Let $L_1, L_2, \dots, L_n$ be the lines of an arrangement $\mathcal{A}$ sorted in decreasing order of their slopes, where m_i is the slope of L_i . Hence,

$$m_1 > m_2 > \dots > m_{n-1} > m_n. \quad (3)$$

Two lines will be called *adjacent* if they have consecutive subscripts. In particular, L_1 and L_n are considered to be adjacent.

The vertices of the *associated polygon* $P(\mathcal{A})$ are the points determined by pairs of adjacent lines, $V_i = L_{i-1} \cap L_i$; the oriented edges are denoted by $E_i = [V_i V_{i+1}]$ (see Fig. 2) and the cartesian coordinates of V_i are denoted by x_i and y_i .

Our first result concerns a nice property of the associated polygon.

Fig. 2. The associated polygon $P(\mathcal{A})$ has a positive winding number with respect to each point contained in the interior of $\mathcal{A}$.

Theorem 1. *The winding number of every interior point of an arrangement of lines $\mathcal{A}$ with respect to the associated polygon $P(\mathcal{A})$ is at least 1.*

Proof. Let L_i and L_{i+1} be two adjacent lines of the arrangement and let DW_i be the open double-wedge determined by these two lines (see Fig. 2). Using induction, it is easy to prove that for any arrangement with at least three lines we have

$$\bigcup_{i=1}^n DW_i \supseteq \mathbb{E}^2 \setminus \bigcup_{i=1}^n L_i. \quad (4)$$

Let O be an interior point of $\mathcal{A}$. We have to show that $w(O, P(\mathcal{A})) \geq 1$. From (4) it follows that O is contained in some double-wedge DW_i . After an appropriate rigid motion we can assume that O is at the origin, is contained in the upper angle of DW_n , and that DW_n has a vertical symmetry axis. The possibilities are described in Fig. 3; in all four cases, the fact that O belongs to the upper part of the double-wedge DW_n implies that

$$y_1 - m_1 x_1 < 0 \quad \text{and} \quad y_n - m_n x_n < 0. \quad (5)$$

We compute $w(O, P(\mathcal{A}))$ by analyzing the intersections between the edges of $\mathcal{A}$ and the vertical halfline Oy . One may also assume that no vertex of $P(\mathcal{A})$ lies on the vertical halfline Oy since otherwise one may slightly perturb point O in such a way that it remains within the original cell while Oy contains no vertex, which means that none of the x_i 's is zero. Moreover, since O does not belong to any of the lines L_i , it follows that all quantities $y_i - m_i x_i$ are nonzero.

Fig. 3. The first and the last edges of the associated polygon.

For now, we first express the index $\psi(Oy, E_i)$ in terms of the coordinates of the endpoints of E_i , V_i and V_{i+1} :

$$\psi(Oy, E_i) = \begin{cases} 1, & \text{if } x_i > 0 > x_{i+1} \text{ and } y_i - m_i x_i > 0; \\ -1, & \text{if } x_i < 0 < x_{i+1} \text{ and } y_i - m_i x_i > 0; \\ 0, & \text{otherwise,} \end{cases}$$

or in a more compact form

$$\psi(Oy, E_i) = \frac{1}{4} [\operatorname{sgn}(y_i - m_i x_i) + 1] \cdot [\operatorname{sgn}(x_i) - \operatorname{sgn}(x_{i+1})].$$

It then follows that

$$4 w(O, P(\mathcal{A})) = \sum_{i=1}^n \operatorname{sgn}(y_i - m_i x_i) \cdot [\operatorname{sgn}(x_i) - \operatorname{sgn}(x_{i+1})] + \sum_{i=1}^n [\operatorname{sgn}(x_i) - \operatorname{sgn}(x_{i+1})].$$

The second sum is obviously zero and after regrouping the terms in the first sum we obtain

$$4w(O, P(\mathcal{A})) = \sum_{i=1}^n \{\operatorname{sgn}[x_{i+1}(y_{i+1} - m_{i+1}x_{i+1})] - \operatorname{sgn}[x_{i+1}(y_i - m_i x_i)]\}. \quad (6)$$

Now, since $y_{i+1} = y_i + m_i(x_{i+1} - x_i)$ the above relation becomes

$$4w(O, P(\mathcal{A})) = \sum_{i=1}^n \{\operatorname{sgn}[x_{i+1}(y_i - m_i x_i) + x_{i+1}^2(m_i - m_{i+1})] - \operatorname{sgn}[x_{i+1}(y_i - m_i x_i)]\}$$

which, given the fact that $m_i > m_{i+1}$ for all $i \in \{1, 2, \dots, n-1\}$, implies that the first $n-1$ terms of the above sum are all nonnegative. The last term is $\operatorname{sgn}[x_1(y_1 - m_1 x_1)] - \operatorname{sgn}[x_1(y_n - m_n x_n)]$ which is exactly zero by (5).

Therefore, the sum in (6) consists only of nonnegative terms and the only way $w(O, P(\mathcal{A}))$ can be zero is when all its terms are vanishing. This happens when, for every $i \in \{1, 2, \dots, n\}$, $(y_{i+1} - m_{i+1}x_{i+1})$ and $(y_i - m_i x_i)$ have the same sign. From this and (5) it follows that for all $i \in \{1, 2, \dots, n\}$ we have $y_i - m_i x_i < 0$ which is equivalent to the fact that none of the lines L_i intersects the halfline Oy ; consequently, in this case O is an exterior point of the arrangement $\mathcal{A}$ contradicting our initial assumption. This ends the proof. $\square$

4. The Main Result

Theorem 2. *Given a circular disc of unit radius and an arrangement $\mathcal{A}$ of n lines such that all intersection points lie inside the disc, there exists a cell of area at least $\pi/4n$.*

Proof. Obviously we can assume that no three lines are concurrent; otherwise we can slightly perturb the lines so that the areas of the cells are within ε of their initial value (for ε arbitrarily small) and the lines are in general position.

Here, the cells are the $1 + n(n+1)/2$ bounded connected components of $\mathbf{E}^2$ resulting when the lines and the circle are removed.

The main idea is to prove that the sum of the areas of the cells which have a circular arc as part of their boundary is “big,” more precisely at least half of the area of the entire circular disc. Since there are $2n$ such cells the claim in the theorem follows immediately. Equivalently, we will show that

$$\sum_C \operatorname{Area}(C) \leq \frac{\pi}{2}, \quad (7)$$

where the sum is over all bounded polygonal cells C determined by the arrangement $\mathcal{A}$.

Denote the lengths of the arcs determined by the lines $L_1, L_2, \dots, L_n$ on the circle by $2a_1, \dots, 2a_{2n}$ as in Fig. 4. Clearly, $0 < a_j < \pi$ for all $j = 1, \dots, n$ and $\sum_{j=1}^{2n} a_j = \pi$. Using (1) it follows after some tedious but straightforward computations that the generalized area of the associated polygon $P(\mathcal{A}) = V_1 V_2 \dots V_n$ can be written as

$$G(P(\mathcal{A})) = \left| \sum_{i=1}^n B_i \right|, \quad (8)$$

Fig. 4. The lines $L_1, L_2, \dots, L_n$ pairwise intersecting inside the unit circle.

where for each integer $1 \leq i \leq n$ we define

$$B_i = \frac{2 \sin a_i \sin a_{n+i} + (\sin^2 a_i + \sin^2 a_{n+i}) \cdot \cos(a_i + 2a_{i+1} + \dots + 2a_{n+i-1} + a_{n+i})}{2 \sin(a_i + a_{n+i})}.$$

We postpone the proof of equality (8) until Section 6.

We will need the following:

Claim. With $a_j, j = 1, \dots, 2n$, and $B_i, i = 1, \dots, n$, defined as above we have

$$-\sin\left(\frac{a_i + a_{n+i}}{2}\right) \leq B_i \leq \frac{\sin(a_i + a_{n+i})}{2}. \quad (9)$$

Proof. For proving the left inequality we notice first that

$$B_i \geq \frac{2 \sin a_i \sin a_{n+i} - (\sin^2 a_i + \sin^2 a_{n+i})}{2 \sin(a_i + a_{n+i})} = -\frac{(\sin a_i - \sin a_{n+i})^2}{2 \sin(a_i + a_{n+i})}.$$

It would be thus sufficient to show that

$$(\sin a_i - \sin a_{n+i})^2 \leq 2 \sin(a_i + a_{n+i}) \sin\left(\frac{a_i + a_{n+i}}{2}\right)$$

which is equivalent to

$$\sin^2\left(\frac{a_i - a_{n+i}}{2}\right) \cos\left(\frac{a_i + a_{n+i}}{2}\right) \leq \sin^2\left(\frac{a_i + a_{n+i}}{2}\right).$$

However, since $a_i, a_{n+i} \in (0, \pi)$ we have the stronger inequality

$$0 \leq \sin a_i \sin a_{n+i} = \sin^2\left(\frac{a_i + a_{n+i}}{2}\right) - \sin^2\left(\frac{a_i - a_{n+i}}{2}\right).$$

The right inequality will follow from the stronger

$$B_i \leq \frac{\sin(a_i + a_{n+i}) \cos(a_i - a_{n+i})}{2}.$$

After eliminating the denominator this can be written in the equivalent form

$$\begin{aligned} & \sin^2(a_i + a_{n+i}) \cos(a_i - a_{n+i}) - 2 \sin a_i \sin a_{n+i} \\ & \geq (\sin^2 a_i + \sin^2 a_{n+i}) \cos(a_i + 2a_{i+1} + \cdots + 2a_{n+i-1} + a_{n+i}). \end{aligned} \quad (10)$$

The left-hand side of the above inequality can be written as

$$\cos(a_i - a_{n+i}) - \cos^2(a_i + a_{n+i}) \cos(a_i - a_{n+i}) - \cos(a_i - a_{n+i}) + \cos(a_i + a_{n+i})$$

which after simplifying the similar terms becomes

$$\cos(a_i + a_{n+i}) \cdot [1 - \cos(a_i + a_{n+i}) \cos(a_i - a_{n+i})] = \cos(a_i + a_{n+i}) \cdot (\sin^2 a_i + \sin^2 a_{n+i}).$$

Therefore, (10) is equivalent to proving that

$$(\sin^2 a_i + \sin^2 a_{n+i})(\cos(a_i + a_{n+i}) - \cos(a_i + 2a_{i+1} + \cdots + 2a_{n+i-1} + a_{n+i})) \geq 0$$

which reduces to showing that

$$\begin{aligned} & (\sin^2 a_i + \sin^2 a_{n+i}) \sin(a_{i+1} + \cdots + a_{n+i-1}) \\ & \cdot \sin(a_i + a_{i+1} + \cdots + a_{n+i-1} + a_{n+i}) \geq 0. \end{aligned}$$

The last inequality holds by our hypothesis assumptions. This ends the proof of the claim. $\square$

Returning now to the proof of the theorem, we first notice that the above claim immediately implies that

$$-\frac{\pi}{2} < \sum_{i=1}^n B_i < \frac{\pi}{2}. \quad (11)$$

Indeed, by using Jensen's inequality for the sine function on $(0, \pi)$ and the claim we obtain

$$\sum_{i=1}^n B_i \leq \frac{1}{2} \sum_{i=1}^n \sin(a_i + a_{n+i}) \leq \frac{n}{2} \cdot \sin \frac{\pi}{n} < \frac{\pi}{2}.$$

Similarly,

$$\sum_{i=1}^n B_i \geq - \sum_{i=1}^n \sin \left(\frac{a_i + a_{n+i}}{2} \right) \geq -n \cdot \sin \frac{\pi}{2n} > -\frac{\pi}{2}.$$

From (8) it follows that $G(P(\mathcal{A})) < \pi/2$. Now using (2) and Theorem 1 we obtain

$$\sum_C \text{Area}(C) \leq \sum_C w(C, P) \cdot \text{Area}(C) < \frac{\pi}{2},$$

where the summation is over all bounded cells C determined by $P(\mathcal{A})$. This proves (7) and with it the entire theorem.

5. Conclusions and Open Problems

The above result is optimal, up to a constant multiplicative factor; this can be easily seen by considering the long diagonals of a regular $2n$ -gon as the n lines of our arrangement. In this case all cells have area $\leq \pi/2n$.

As an immediate consequence, if the lines in Theorem 2 are in general position and $b_1, b_2, \dots, b_p$ denote the areas of the $p = 1 + n(n-1)/2$ cells, we have

$$\frac{\max b_i}{\min b_i} \geq \frac{\pi/4n}{\pi/[1 + n(n+1)/2]} \geq \frac{n+1}{8}.$$

An attempt to generalize our result is given by the following:

Conjecture 2. *An arrangement of n lines in general position in the plane determines $s = \binom{n-1}{2}$ bounded cells of areas $b_1, b_2, \dots, b_s$. Then $\max b_i / \min b_i \rightarrow \infty$ as $n \rightarrow \infty$.*

The major difference between this conjecture and Theorem 2 is the absence of good candidates for the role of “big cells.” New ideas seem to be needed. We can eventually try to generalize the above conjecture for the case of *pseudolines*, but this, if true, seems to be hopelessly difficult. Instead, maybe the following problem has a much better chance to be solved first.

Conjecture 3. *Let $f_1, f_2, \dots, f_{n-1}: [0, \pi] \rightarrow (-1, 1)$ be $n-1$ differentiable functions satisfying the following properties:*

- (a) *For every $1 \leq i \leq n-1$, the function f_i has at most one local extrema in $(0, \pi)$.*
- (b) *For every pair of indices $1 \leq i < j \leq n-1$, the equation $f_i(x) = f_j(x)$ has exactly one solution; the graphs properly cross at this point.*
- (c) *For every three distinct indices $1 \leq i < j < k \leq n-1$, the equation $f_i(x) = f_j(x) = f_k(x)$ has no solution.*

Then, if $b_1, b_2, \dots, b_p$ denote the areas of the $p = 1 + n(n-1)/2$ cells in which the rectangle $[0, \pi] \times [-1, 1]$ is divided by the graphs of the functions $f_1, f_2, \dots, f_{n-1}$, we have $\max b_i / \min b_i \rightarrow \infty$ as $n \rightarrow \infty$.

It is interesting to note that Conjecture 3 is stronger than the original conjecture of L. Fejes Tóth mentioned in the Introduction. The argument is sketched below.

Given n great circles in general position on a unit sphere S start by choosing a diameter that belongs to exactly one of these circles; with no loss of generality assume that this diameter is vertical. Then let C be the vertical cylinder of height 2 tangent to S . Then S and C touch in the horizontal great circle E on the sphere S , which is the equator of S ; we can assume that E is not one of the initial great circles. Let a and b be the corresponding north and south poles of S ; the straight vertical segment ab is the common axis of S and C . Let x be any point of S , other than the poles a and b . There is then a unique point y on the axis ab at the same height as x . When the horizontal segment yx is produced beyond x , its continuation will cut the cylinder C at some point z , at the same height as x and y . We call z the axial projection of x , see Fig. 5(a)). This is the famous Archimedean projection and it is a well-known fact that it preserves area (a nice elementary proof can be found in [4]).

By our choice, the image of the circle whose diameter was originally selected will consist of two vertical generators of the cylinder C while all the other circles will project into centrally symmetric curves.

If C is now slit open along one of the vertical generators mentioned above, it can be unrolled onto the plane as a rectangle of length 2π and height 2. Straightforward computations show that the images of the remaining $n-1$ great circles after unrolling can be expressed as graphs of functions of the following form, $f_{\alpha, \beta}: [0, 2\pi] \rightarrow (-1, 1)$, where $\alpha \in [0, 2\pi)$, $\beta > 0$, and

$$f_{\alpha, \beta}(x) = \frac{\cos(x - \alpha)}{\sqrt{\cos^2(x - \alpha) + \beta^2}}.$$

Fig. 5. (a) The Archimedean projection. (b) A family of seven great circles after projection and unrolling; the vertical circle projects onto the boundary and the dotted symmetry axis of the rectangle.

An example of such a family of functions is presented in Fig. 5(b)). Obviously, given the symmetry it is enough to consider just the interval $[0, \pi]$. It is easy to check that the functions restricted to this interval have the properties mentioned in the hypothesis of Conjecture 3.

6. Proof of (8)

For every $1 \leq i \leq n$, denote by A_i and A_{n+i} the points in which line L_i intersects the circle (see Fig. 6). Also, for every $1 \leq i \leq n$, we introduce the following notations:

$$\begin{aligned} b_i &= a_{i+1} + a_{i+2} + \cdots + a_{n+i-3} + a_{n+i-2}, \\ b_{n+i} &= a_{n+i+1} + \cdots + a_{2n} + a_1 + \cdots + a_{i-2}. \end{aligned} \quad (12)$$

In what follows all the subscripts of the quantities a_k and b_k are taken modulo $2n$. Recall that x_i and y_i represent the cartesian coordinates of the vertex V_i . We want to express $x_i y_{i+1} - x_{i+1} y_i$ in terms of a_{i-1} , a_i , b_i , a_{n+i-1} , a_{n+i} , and b_{n+i} .

Denote by λ_i , respectively μ_i , the ratios in which the point V_i , respectively V_{i+1} , divides the chord $A_i A_{n+i}$. That is,

$$\lambda_i = \frac{|A_i V_i|}{|A_i A_{n+i}|} \quad \text{and} \quad \mu_i = \frac{|A_i V_{i+1}|}{|A_i A_{n+i}|}. \quad (13)$$

Fig. 6. Three consecutive lines L_{i-1} , L_i , and L_{i+1} and the arcs determined by them on the circle.

It follows that

$$x_i = \lambda_i x_{A_i} + (1 - \lambda_i) x_{A_{n+i}}, \quad y_i = \lambda_i y_{A_i} + (1 - \lambda_i) y_{A_{n+i}}$$

and, similarly,

$$x_{i+1} = \mu_i x_{A_i} + (1 - \mu_i) x_{A_{n+i}}, \quad y_{i+1} = \mu_i y_{A_i} + (1 - \mu_i) y_{A_{n+i}}.$$

A short computation gives that

$$\begin{aligned} x_i y_{i+1} - x_{i+1} y_i &= (\lambda_i - \mu_i) (x_{A_i} y_{A_{n+i}} - x_{A_{n+i}} y_{A_i}) \\ &= -(\lambda_i - \mu_i) \sin(2a_i + 2b_i + 2a_{n+i-1}). \end{aligned} \quad (14)$$

Applying the sine theorem to the triangles $A_i V_i A_{i-1}$ and $A_i A_{n+i} A_{i-1}$ we obtain

$$\frac{|A_i V_i|}{|A_{i-1} A_i|} = \frac{\sin(a_i + b_i)}{\sin(a_{i-1} + a_{n+i-1})}, \quad \text{respectively} \quad \frac{|A_i A_{n+i}|}{|A_{i-1} A_i|} = \frac{\sin(a_i + b_i + a_{n+i-1})}{\sin a_{i-1}},$$

which implies that

$$\lambda_i = \frac{\sin(a_i + b_i)}{\sin(a_{i-1} + a_{n+i-1})} \cdot \frac{\sin a_{i-1}}{\sin(a_i + b_i + a_{n+i-1})}. \quad (15)$$

Similarly, the sine theorem applied to the triangles $A_i V_{i+1} A_{i+1}$ and $A_i A_{n+i} A_{i+1}$ gives

$$\frac{|A_i V_{i+1}|}{|A_{i+1} A_i|} = \frac{\sin(a_{i-1} + b_{n+i})}{\sin(a_i + a_{n+i})}, \quad \text{respectively} \quad \frac{|A_i A_{n+i}|}{|A_{i+1} A_i|} = \frac{\sin(a_i + b_i + a_{n+i-1})}{\sin a_i},$$

from which

$$\mu_i = \frac{\sin(a_{i-1} + b_{n+i})}{\sin(a_i + a_{n+i})} \cdot \frac{\sin a_i}{\sin(a_i + b_i + a_{n+i-1})}. \quad (16)$$

Using (15) and (16), equality (14) implies that

$$\begin{aligned} \frac{1}{2}(x_i y_{i+1} - x_{i+1} y_i) &= -(\lambda_i - \mu_i) \sin(a_i + b_i + a_{n+i-1}) \cos(a_i + b_i + a_{n+i-1}) \\ &= \left[\frac{\sin a_i \sin(a_{i-1} + b_{n+i})}{\sin(a_i + a_{n+i})} - \frac{\sin a_{i-1} \sin(a_i + b_i)}{\sin(a_{i-1} + a_{n+i-1})} \right] \\ &\quad \cdot \cos(a_i + b_i + a_{n+i-1}). \end{aligned} \quad (17)$$

Analogously, using the sine theorem in the triangles $A_{n+i} V_i A_{n+i-1}$ and $A_{n+i} A_i A_{n+i-1}$, respectively $A_{n+i} V_{i+1} A_{n+i+1}$ and $A_{n+i} A_i A_{n+i+1}$, we obtain

$$1 - \lambda_i = \frac{\sin(a_{n+i} + b_{n+i})}{\sin(a_{i-1} + a_{n+i-1})} \cdot \frac{\sin a_{n+i-1}}{\sin(a_i + b_i + a_{n+i-1})}, \quad (18)$$

respectively,

$$1 - \mu_i = \frac{\sin(a_{n+i-1} + b_i)}{\sin(a_i + a_{n+i})} \cdot \frac{\sin a_{n+i}}{\sin(a_i + b_i + a_{n+i-1})}. \quad (19)$$

Using (18) and (19), equality (14) becomes

$$\begin{aligned}
 \frac{1}{2}(x_i y_{i+1} - x_{i+1} y_i) &= [(1 - \lambda_i) - (1 - \mu_i)] \sin(a_i + b_i + a_{n+i-1}) \\
 &\quad \cdot \cos(a_i + b_i + a_{n+i-1}) \\
 &= \left[\frac{\sin a_{n+i-1} \sin(a_{n+i} + b_{n+i})}{\sin(a_{i-1} + a_{n+i-1})} - \frac{\sin a_{n+i} \sin(a_{n+i-1} + b_i)}{\sin(a_i + a_{n+i})} \right] \\
 &\quad \cdot \cos(a_i + b_i + a_{n+i-1}). \tag{20}
 \end{aligned}$$

Adding (17) and (20) we obtain

$$\begin{aligned}
 x_i y_{i+1} - x_{i+1} y_i &= \frac{[\sin a_i \sin(a_{i-1} + b_{n+i}) - \sin a_{n+i} \sin(a_{n+i-1} + b_i)] \cos(a_i + b_i + a_{n+i-1})}{\sin(a_i + a_{n+i})} \\
 &\quad + \frac{[\sin a_{n+i-1} \sin(a_{n+i} + b_{n+i}) - \sin a_{i-1} \sin(a_i + b_i)] \cos(a_i + b_i + a_{n+i-1})}{\sin(a_{i-1} + a_{n+i-1})}. \tag{21}
 \end{aligned}$$

After regrouping the terms with the same denominator, it follows that

$$\frac{1}{2} \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i) = \sum_{i=1}^n \frac{p_i + q_i}{2 \sin(a_i + a_{n+i})},$$

where, for every $1 \leq i \leq n$,

$$p_i = [\sin a_i \sin(a_{i-1} + b_{n+i}) - \sin a_{n+i} \sin(a_{n+i-1} + b_i)] \cos(a_i + b_i + a_{n+i-1}) \tag{22}$$

and

$$\begin{aligned}
 q_i &= [\sin a_{n+i} \sin(a_{n+i+1} + b_{n+i+1}) - \sin a_i \sin(a_{i+1} + b_{i+1})] \\
 &\quad \cdot \cos(a_{i+1} + b_{i+1} + a_{n+i}). \tag{23}
 \end{aligned}$$

Therefore, for proving (8) it would suffice to show that

$$\begin{aligned}
 p_i + q_i &= 2 \sin a_i \sin a_{n+i} + (\sin^2 a_i + \sin^2 a_{n+i}) \\
 &\quad \cdot \cos(a_i + 2a_{i+1} + \cdots + 2a_{n+i-1} + a_{n+i}). \tag{24}
 \end{aligned}$$

On one hand, expanding p_i from (22) we obtain

$$\begin{aligned}
 p_i &= -\frac{1}{2} \sin a_{n+i} [\sin(2a_{n+i-1} + 2b_i + a_i) - \sin a_i] \\
 &\quad + \frac{1}{2} \sin a_i [\sin(a_{i-1} + a_i + b_i + a_{n+i-1} + b_{n+i}) \\
 &\quad + \sin(a_{i-1} + b_{n+i} - a_i - b_i - a_{n+i-1})]
 \end{aligned}$$

which, given that $b_{n+i} = \pi - (a_{i-1} + a_i + b_i + a_{n+i-1} + a_{n+i})$, becomes

$$\begin{aligned}
 p_i &= \sin a_i \sin a_{n+i} + \frac{1}{2} \sin a_i \sin(2a_i + 2b_i + 2a_{n+i-1} + a_{n+i}) \\
 &\quad - \frac{1}{2} \sin a_{n+i} \sin(a_i + 2b_i + 2a_{n+i-1}). \tag{25}
 \end{aligned}$$

Similarly, expanding q_i from (23) and using that $b_{n+i+1} = \pi - (a_i + a_{i+1} + b_{i+1} + a_{n+i} + a_{n+i+1})$ we obtain

$$\begin{aligned} q_i &= \sin a_i \sin a_{n+i} + \frac{1}{2} \sin a_{n+i} \sin(a_i + 2a_{i+1} + 2b_{i+1} + 2a_{n+i}) \\ &\quad - \frac{1}{2} \sin a_i \sin(2a_{i+1} + 2b_{i+1} + a_{n+i}). \end{aligned} \quad (26)$$

Now combining (25) and (26) results in

$$\begin{aligned} p_i + q_i &= 2 \sin a_i \sin a_{n+i} \\ &\quad + \frac{1}{2} \sin a_i [\sin(2a_i + 2b_i + 2a_{n+i-1} + a_{n+i}) - \sin(2a_{i+1} + 2b_{i+1} + a_{n+i})] \\ &\quad + \frac{1}{2} \sin a_{n+i} [\sin(a_i + 2a_{i+1} + 2b_{i+1} + 2a_{n+i}) \\ &\quad \quad - \sin(a_i + 2b_i + 2a_{n+i-1})] \end{aligned} \quad (27)$$

which after transforming the sums into products inside the square brackets becomes

$$\begin{aligned} p_i + q_i &= 2 \sin a_i \sin a_{n+i} \\ &\quad + \sin a_i \sin(a_i + b_i + a_{n+i-1} - a_{i+1} - b_{i+1}) \\ &\quad \cdot \cos(a_i + a_{i+1} + b_i + b_{i+1} + a_{n+i-1} + a_{n+i}) \\ &\quad + \sin a_{n+i} \sin(a_{i+1} + b_{i+1} + a_{n+i} - b_i - a_{n+i-1}) \\ &\quad \cdot \cos(a_i + a_{i+1} + b_i + b_{i+1} + a_{n+i-1} + a_{n+i}). \end{aligned}$$

Now using (12) the above equality becomes

$$\begin{aligned} p_i + q_i &= 2 \sin a_i \sin a_{n+i} + (\sin^2 a_i + \sin^2 a_{n+i}) \\ &\quad \cdot \cos(a_i + 2a_{i+1} + \cdots + 2a_{n+i-1} + a_{n+i}) \end{aligned} \quad (28)$$

which is exactly (24), the equality to be shown.

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