Covering an Annulus by Strips*

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Abstract. Take a unit square and turn it into an annulus by cutting a parallel square hole in it. We prove that if the square hole has edge length $1 - 1/\sqrt{2} \approx 0.29$ and a finite number of strips cover the annulus, then after appropriate rearrangement the same strips can cover the unit square as well.

1. Introduction and Results

The well-known result of Tarski on covering a circular disk with strips (a *strip* is a closed plane region between two parallel lines) states that if the disk is covered by strips, then the sum of their widths is at least the diameter of the disk. If equality occurs, then the strips must be parallel. Tarski [3] conjectured and Bang [1] proved the following generalization (here we state the planar version only): should a convex region K (a *region* is a compact planar set with non-empty interior) be covered by a finite collection of strips, then the sum of the widths of the strips must be at least the minimal width of K (the *minimal width* of K is the smallest possible distance between two parallel supporting lines of K). Later several further generalizations have been obtained. In 1999 we proposed to investigate a variation not yet considered.

We state our question first for the case of a circular disk, partly because of the above history, but mainly because this is the question's most striking form:

Problem. Take a circular disk and turn it into an annulus by cutting a sufficiently small circular hole at its center. Prove that if the strips can cover the annulus, then after an

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appropriate rearrangement they can cover the whole disk as well. If this turns out to be true, then the natural question of the maximum size of the hole arises. It might be interesting to know if one can get similar result without requiring that the hole is cut at the center.

Let O denote the origin of the plane E^2 . For a given convex set K and $\varepsilon > 0$, let εK denote a *homothetic copy* of K consisting of all points X such that $\overrightarrow{OX} = \varepsilon \overrightarrow{OY}$, where $Y \in K$. Now we state the general conjecture.

Conjecture. For each convex region K there is an $\varepsilon > 0$ such that if εK lies in the interior of K and the annulus $K \setminus \varepsilon K$ is covered by finitely many strips, then the sum of the widths of the strips is at least the minimal width of K .

In this paper we prove the following:

Theorem 1. *Let C be the unit square and let ε be equal to $1 - 1/\sqrt{2} \approx 0.29$. If εC lies in the interior of the square C and the annulus $C \setminus \varepsilon C$ is covered by finitely many strips, then the sum of the widths of the strips is at least 1.*

Theorem 2. *The Conjecture is true for each polygon whose incircle is tangent to two of its parallel sides. In particular, it is true for regular polygons with an even number of sides.*

Remark 1. Assume that in the Problem above the circle has radius 1, the circular hole at the center has radius ε , and the annulus is covered by n strips $S_1, S_2, \dots, S_n$ such that S_i is of width w_i . A simple fact (discovered already by Archimedes) is that if two parallel planes intersect a sphere of radius 1, then the surface area between the planes is $2\pi d$, where d is the distance between the planes. Project orthogonally each strip onto the surface of that hemisphere whose great circle is the given unit circle (this is exactly how Tarski's problem was solved). The images of the strips cover the image of the annulus. Thus, we get that $\sum_{i=1}^n \pi w_i \geq 2\pi(\sqrt{1 - \varepsilon^2})$ which is less than the bound 2π needed for our conjecture.

Remark 2. We conjecture that Theorem 1 remains true if one replaces $\varepsilon = 1 - 1/\sqrt{2} \approx 0.29$ by $\varepsilon = \frac{1}{2}$.

Remark 3. Concerning the higher-dimensional version of our conjecture, we also expect an affirmative answer. As W. Kuperberg pointed out, for certain convex bodies in dimension three the solution is simpler than that of their planar relatives. For example, if a spherical annulus is covered by slabs, then the surface of the outside sphere must be covered by the slabs. In view of the simple argument mentioned in Remark 1, we already know that this implies that the sum of the widths of the slabs is at least the diameter of the sphere. By the same reason, if the diameter of the insphere of the convex body is equal to the minimal width of the convex body (cubes and regular octahedrons have this property) and the hole inside the convex body avoids the surface of the insphere we also get an affirmative answer.

2. Proofs

Proof of Theorem 1. Let us recall a difficult, but elementary fact, which appears in the paper of Eggleston [2] and whose proof is related to an earlier argument of Bang [1]:

Let $S_1, S_2, \dots, S_n$ be a finite number of strips such that S_i is of width w_i . Let K be a centrally symmetric convex polygon of $2n$ sides such that each pair of opposite sides is perpendicular to one of the strips S_i and have length w_i . It is easily constructed. Assume the origin O is the center of K . Denote by u_i that vector which is perpendicular to S_i and has magnitude $\frac{1}{2}w_i$. It turns out that one of the 2^n points, say P , of the form $\varepsilon_1 u_1 + \varepsilon_2 u_2 + \dots + \varepsilon_n u_n$ (where each ε_i can be ± 1) does not belong to the interior of any strip S_i ($i = 1, \dots, n$). We will use the following immediate consequence: if no two of the strips are parallel and no three of the boundary lines pass through the same point, then the strips cannot cover a neighborhood of the point P and thus they cannot cover a neighborhood of K .

Now we can prove Theorem 1. Let C be a unit square whose interior contains the homothetic square $C_0 = (1 - 1/\sqrt{2})C$ and assume, without loss of generality, that C has a pair of horizontal sides. Let $S_1, S_2, \dots, S_n$ be a given collection of n strips such that S_i is of width w_i . We suppose to the contrary that $w_1 + w_2 + \dots + w_n < 1$ and show that

(1) *The union of strips S_i does not contain the annulus $C \setminus C_0$.*

This will complete the proof. Without loss of generality, assume that the strips are in general position, i.e., no two of the strips are parallel and no three of the boundary lines of the strips pass through the same point. Otherwise we could expand the covered parts by moving the strips slightly so that the sum of the widths is still less than 1, while the strips are in general position. Consider K again. Let R be the rectangle enclosed by the vertical and horizontal supporting lines of K . Translate R together with the inscribed polygon K until the left upper vertices of the rectangles C and R coincide. Since the perimeter of K is less than 2, we have that both sides of R have length less than 1. Thus, R lies in C . It is also easy to see that one of the sides of R is shorter than $1/\sqrt{2}$. To see this, choose a vertex of K on each side of R and connect them counterclockwise. This quadrilateral also has a perimeter less than 2. Since among right triangles with a common hypotenuse the isosceles one has the largest perimeter, we get that the perimeter of R is less than $2\sqrt{2}$. Thus one of the sides of R is shorter than $1/\sqrt{2}$. Without loss of generality assume that the vertical sides of R have length $x < 1/\sqrt{2}$.

The crucial part of the proof is to introduce an additional strip S_{n+1} : choose S_{n+1} so that (i) it is horizontal (if K has a pair of horizontal sides, then to avoid degeneracy we choose a direction sufficiently close to horizontal), (ii) it has a width w such that $1 - x > w > 1 - 1/\sqrt{2}$, and (iii) it covers the inner square C_0 of the annulus, while the strips $S_1, S_2, \dots, S_{n+1}$ remain in general position.

Denote by K' the convex polygon associated with the family of the given $n + 1$ strips. In practice a pair of vertical sides of length $< 1/\sqrt{2}$ was added to the sides of K to get the polygon K' . Thus K' can be translated to lie in the interior of C too. There is a $\delta > 0$ such that the δ -neighborhood of K' belongs to C . According to the quoted fact of Eggleston, the strips $S_1, S_2, \dots, S_{n+1}$ do not cover the δ -neighborhood of K' . Let X be

one of the points not covered by the strips. Since S_{n+1} covers C_0 , we have that X belongs to the annulus $C \setminus C_0$. This proves (1). $\square$

Proof of Theorem 2. An argument similar to that used in the proof of Theorem 1 works. Let P be a polygon whose incircle C (if the incircle is not unique, then let C be one of them) is tangent to two parallel sides. Let the diameter of C be equal to 1. Denote the given strips by $S_1, S_2, \dots, S_n$. Denote the width of S_i by w_i ($i = 1, \dots, n$). Assume that the strips are in general position, meaning the same as that in the proof of Theorem 1. Let K be a centrally symmetric convex polygon of $2n$ sides, such that (i) each pair of opposite sides is perpendicular to one of the strips S_i and have length w_i and (ii) the center of K coincides with the center of the incircle C . We suppose to the contrary that $w_1 + w_2 + \dots + w_n < 1$ and show that

(2) *The union of strips S_i does not contain the annulus $P \setminus \varepsilon P$, where ε is a sufficiently small positive number depending on the polygon P only.*

First we explain how to choose ε : Since K is centrally symmetric and has a perimeter < 2 , it lies within the incircle C . Hence there exists an ε_1 such that K does not overlap the ε_1 neighborhoods of the vertices of P . If the diameter of a centrally symmetric region of perimeter less than 2 approaches 1, then its area approaches 0. Thus, for a sufficiently small $\varepsilon_2 < \varepsilon_1$, K cannot invade the ε_2 -neighborhoods of two nonparallel sides of P (otherwise K would contain a parallelogram whose area is too large). We will prove that if the hole εP is so small that the diameter of the circumradius of εP is less than ε_2 , then the union of strips does not contain the annulus $P \setminus \varepsilon P$.

Consider now the annulus $P \setminus \varepsilon P$ and introduce an additional strip S_{n+1} : If the distance between the boundaries of K and P is smaller than ε_2 , then let s be one of the at most two sides of K which are closer to P than ε_2 , otherwise let s be any side of P . Let S_{n+1} be a strip of width ε_2 which covers the inner polygon εP and whose boundary lines are perpendicular to the side s . To guarantee that $S_1, S_2, \dots, S_{n+1}$ are in general position we might need to make a small change in the position of the strip S_{n+1} . Consider the convex polygon K' associated with the family of these $n + 1$ strips. Translate K' to lie in the interior of P . This can be done due to the condition that K has at most one pair of parallel sides, whose distance from P is less than ε_2 . Thus, there is a $\delta > 0$ such that the δ -neighborhood of K' lies in P . According to the quoted fact of Eggleston the strips $S_1, S_2, \dots, S_{n+1}$ do not cover a δ -neighborhood of K' . Let X be one of the points not covered by the strips. Since S_{n+1} covers εP , we have that X belongs to the annulus $P \setminus \varepsilon P$. This proves (2). $\square$

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