The Number of Directions Determined by Points in the Three-Dimensional Euclidean Space*

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Abstract. Let X be a set of n points in the three-dimensional Euclidean space such that no three points in X are on the same line and there is no plane containing all points in X . An old conjecture states that pairs of points in X determine at least $2n - 3$ directions. We prove the weaker result that X determines at least $1.75n - 2$ directions.

1. Introduction

Let X be a point set in the three-dimensional Euclidean space $\mathbb{R}^3$. We say that X is in *general position* if no three points in X are on the same line and there is no plane containing all points in X . The directions determined by X are the directions of the lines going through pairs of points in X . We say that a direction δ determined by X has *multiplicity* k if there are exactly k pairs of points in X such that the line through these pairs of points is in direction δ . This note is about the following conjecture.

Conjecture 1. *If $X \subseteq \mathbb{R}^3$ is in general position and $|X| = n$, then X determines at least $2n - 3$ directions.*

The first version of the conjecture in print (in the form that X is non-planar, but more than two points of X may be on the same line) seems to appear in [Sc]. This problem is similar to the problem of Erdős that asks for the minimal number of lines determined

* The research by the second author was partially supported by the NSF.

by n non-collinear points (see [St]). As far as we know, Erdős's problem is not solved in dimensions greater than two.

Finding the minimal number of directions determined by n points on the plane, when no three of them are collinear, is easy.

Proposition 2. *If X is a set of n points on the plane such that no three points are on the same line, then X determines at least n directions.*

Proof. Let $p_1, p_2, p_3 \in X$ be three consecutive points among the points of X which are on the boundary of the convex hull of X . The lines through p_2 determine $n - 1$ directions, and the direction of the line through p_1 and p_3 is different from these. $\square$

The extremal configurations for Proposition 2 were determined by Jamison [Ja3]. Proposition 2 has the following non-trivial strengthening.

Theorem 3 [Un]. *If X is a set of n points on the plane such that not all of them are on the same line, then X determines at least $2\lfloor n/2 \rfloor$ directions.*

We cannot prove Conjecture 1, but we shall prove the following two theorems.

Theorem 4. *If $X \subseteq \mathbb{R}^3$ is a set of n points in general position and there is a direction determined by X of multiplicity k , then X determines at least $2n - 2 - k$ directions.*

Theorem 5. *If $X \subseteq \mathbb{R}^3$ is a set of n points in general position, then it determines a direction with multiplicity at most $n/4$.*

Corollary 6. *If $X \subseteq \mathbb{R}^3$ is a set of n points in general position, then it determines at least $1.75n - 2$ directions.*

We finish this Introduction with three more conjectures. The last two of these are similar to Conjecture 6.3 of [Ja2].

Conjecture 7. *If $X \subseteq \mathbb{R}^3$ is in general position, then it determines a direction of multiplicity at most 2.*

Conjecture 8. *If $X \subseteq \mathbb{R}^3$ is a set of n points which are not all in the same plane, then X determines at least $2n - 5$ directions.*

Conjecture 9. *If $X \subseteq \mathbb{R}^d$, $d \geq 2$, is a set of n points which are not all in the same hyperplane and no three points of X are on the same line, then X determines at least $(d - 1)n - d(d - 2)$ directions.*

2. Examples

Conjecture 1, if true, is sharp. We know of one infinite series and four sporadic extremal configurations. Of course, any affine image of these configurations is also a suitable example.

- (1) Let n be even and $X_{1,n} = \{p_1, \dots, p_n\}$ such that
 - (i) $p_1, \dots, p_{n-2}$ are the vertices of a regular $(n-2)$ -gon Y ;
 - (ii) p_{n-1}, p_n are on the line orthogonal to the plane of Y which goes through the center o of Y ; and
 - (iii) o is the middle point of the segment between p_{n-1} and p_n .
- (2) $X_2 = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1), (-1, -1, -1)\}$ (i.e., the vertex set of a cube).
- (3) $X_3 = X_2 \cup \{(2, 0, 0), (-2, 0, 0)\}$.
- (4) $X_4 = X_3 \cup \{(0, 2, 0), (0, -2, 0)\}$ (X_4 also can be obtained as an affine image of the mid-points of the edges of a cube).
- (5) $X_5 = X_4 \cup \{(0, 0, 2), (0, 0, -2)\}$ (X_5 also can be obtained as an affine image of the symmetric difference of the vertex sets of the unit cube and its translate by the vector $(1, 1, 1)$).

Theorem 3 is sharp. Four infinite families and more than 100 sporadic extremal configurations are listed in [Ja1] and [JH].

Conjecture 8, if true, is also sharp. Examples (1)–(5) above are centrally symmetric, and adding the center we obtain an extremal configuration for Conjecture 8. Also, any centrally symmetric configuration from the lists in [Ja1] and [JH] can be augmented to a double cone X as in example (1), which determines $2|X| - 5$ directions. The same augmentation procedure can be continued to obtain higher-dimensional configurations which determine only a few directions (see Theorem 6.2 of [Ja2]).

3. Proof of the Theorems

Proof of Theorem 4. Let $X \subseteq \mathbb{R}^3$ be a set of n points in general position, and let δ be a direction determined by X of multiplicity k . Let Y be the projection of X onto a plane orthogonal to δ . Then Y consists of $n - k$ points, and k of these are “double” (i.e., the projection of two points of X). Let p be one of these double points, and let $L_1, \dots, L_m$ be the lines determined by Y that go through p . Let k_i denote the number of points of X whose projection is on L_i , but the projection is different from p . Since the lines L_i contain the projections of all points of X , we have

$$\sum_{i=1}^m k_i = n - 2. \quad (1)$$

Clearly, if for two pairs of points in X the directions determined by their projections in Y are different, then the directions determined by them in X are also different.

Since X is not planar, the points in Y are not all on the same line. Therefore, by Theorem 3, the points in Y determine at least $n - k - 1$ directions. At least $n - k - 1 - m$

of these are not the directions of the lines $L_1, \dots, L_m$. Our next observation is that by Proposition 2, the $k_i + 2$ points of X whose projections are on line L_i determine at least $k_i + 1$ directions different from δ . Finally, counting also the direction δ and using (1), we obtain that X determines at least

$$(n - k - 1 - m) + \sum_{i=1}^m (k_i + 1) + 1 = 2n - 2 - k$$

directions. □

Proof of Theorem 5. Let $X \subseteq \mathbb{R}^3$ be a set of n points in general position, and let $\Delta = \{\delta_1, \dots, \delta_m\}$ be the set of directions determined by X , considered as a subset of the projective plane. Since X is not planar, not all points of Δ are on the same line. Therefore, by the Gallai–Sylvester theorem (see [St]) there are $\delta_i, \delta_j \in \Delta$ such that no other points of Δ are on the line determined by δ_i and δ_j . Let k_i and k_j denote the multiplicity of the direction δ_i and δ_j in X , respectively. Let X_i be the subset of $2k_i$ points in X which is the union of the k_i pairs determining the direction δ_i . We define the subset X_j of $2k_j$ points analogously, using the pairs of points determining the direction δ_j . The sets X_i and X_j are disjoint, since if $p, q \in X$ determined the direction δ_i and $p, r \in X$ determined the direction δ_j then q and r would determine a direction which is on the line through δ_i and δ_j . Hence $2k_i + 2k_j \leq n$ and so at least one of k_i, k_j must be no greater than $n/4$. □

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Received February 9, 2001, and in revised form July 11, 2001. Online publication October 29, 2002.