Gallai–Sylvester Theorem for Pairwise Intersecting Unit Circles

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Abstract. For every finite family of (at least five) pairwise intersecting unit circles in the plane, there is an intersection point that lies on exactly two circles. This proves a conjecture of A. Bezdek.

1. Introduction

The celebrated Gallai–Sylvester theorem ([S], [G], and see also [BM], [M], and [PA] for a survey of the history and generalizations of this theorem), in its dual form, asserts that for every finite collection of lines in the real projective plane that do not form a pencil (i.e., do not share a common point) there is a point that lies on exactly two lines. A possible analogue for circles could be: in any finite collection of (at least two) unit circles that are pairwise intersecting (i.e., every two circles meet) there is an intersection point that belongs to exactly two circles.

Unfortunately, this statement is false. A simple counterexample consists of three distinct unit circles through a common point P , each two meeting in two points, and a fourth unit circle that passes through the remaining three intersection points. We call this family *exceptional* (see Fig. 1). Bezdek [B1] conjectured that this is the only counterexample to the statement above. (In [B1] Bezdek actually refers to the dual statement, namely, for any set of n points in the plane with diameter ≤ 2 there exists a unit circle which passes through precisely two points. See [BB] for some closely related results about circles determined by a small number of points.) He confirmed this under the additional condition that the distance between the centers of any two unit circles is less than $\sqrt{3}$. Bezdek used this condition to bound the number of lunes and lenses (definitions will follow) induced by the family of unit circles. This bound (which is actually four) and Euler formula give the desired result.

In this paper we prove Bezdek’s conjecture, namely, we prove the following theorem.

Fig. 1. An exceptional configuration.

Theorem 1.1. *Let $\mathcal{C}$ be a finite family of (at least two) pairwise intersecting unit circles in the plane. Then there is a point through which exactly two circles pass, unless $\mathcal{C}$ is exceptional.*

Remark. Under the same assumptions of Theorem 1.1, one can show rather easily that there exists an intersection point through which few circles pass (consult [BFT] for a short argument proving the existence of an intersection point incident to at most five unit circles). Even under weaker assumptions than in Theorem 1.1 one can show the existence of an intersection point incident to at most three circles [ALPS], [NPPS], as described in more detail in the concluding remarks section. Proving Theorem 1.1 requires a more delicate argumentation. One of the reasons for this is, apparently, the existence of a small counterexample to this theorem, namely the exceptional configuration.

There are several ways in which one can try to generalize Theorem 1.1. Consider first the case where $\mathcal{C}$ is a family of circles of arbitrary radii which are pairwise intersecting. There is an obvious counterexample to this generalization, simply take $\mathcal{C}$ to be a pencil of circles, that is, circles which pass through two given points. Then there are only two intersection points and each is incident to all circles in $\mathcal{C}$. We can impose natural restrictions on the family $\mathcal{C}$, such as that $\mathcal{C}$ is not a pencil, or that at most two circles pass through any two given points (as in the case of unit circles).

In each of those two cases we have a small counterexample to Theorem 1.1, other than the one given in Fig. 1. This example is given by Fig. 2.

There are six circles in the family $\mathcal{C}$ and a dotted one. The family of the six full circles does not have an intersection point of precisely two circles, and satisfies the condition that at most two circles pass through any two given points. If we add the dotted circle we overrule this condition but still $\mathcal{C}$ is not a pencil, and again there is no intersection point through which exactly two circles pass.

Finally, note that if the family $\mathcal{C}$ consists of unit circles which are not necessarily pairwise intersecting, then the analogue of Theorem 1.1 to this case is not valid. This is clearly true if $\mathcal{C}$ is a family of disjoint unit circles. If we insist that $\mathcal{C}$ be a connected

Fig. 2. An exceptional configuration.

family, then one can build a counterexample by extending the example given in Fig. 1. Namely, assume we already have a configuration $\mathcal{C}$ of $n \geq 1$ unit circles with no simple intersection point. Let Q be a point, on a circle $C \in \mathcal{C}$, which is also an extreme point of the convex hull of $\bigcup \mathcal{C}$. Choose three points P_1, P_2, P_3 on C , very close to each other and very close to Q . There are exactly three unit circles different from C which pass through exactly two of the points P_1, P_2, P_3 . These three unit circles meet at a common point P which is “far” from the original configuration. Adjoining these three unit circles to $\mathcal{C}$ will now produce a new configuration of $n + 3$ unit circles with no simple intersection point.

2. Definitions and Notation

Throughout this paper $\mathcal{C}$ denotes a finite family of at least three (clearly, Theorem 1.1 is valid when $\mathcal{C}$ consists of just two circles) unit circles in the Euclidean plane. We assume that every two (distinct) circles in $\mathcal{C}$ meet. Every two circles in $\mathcal{C}$ either *cross* (i.e., have two points in common) or *touch* (i.e., have a unique point in common). We denote by $\mathcal{A}(\mathcal{C})$ the planar map induced by $\mathcal{C}$. Its vertices $\mathcal{V}(\mathcal{C})$ are the intersection points of circles in $\mathcal{C}$, its edges are the closures of the components of $\bigcup \mathcal{C} \setminus \mathcal{V}(\mathcal{C})$, and its faces are the closures of the components of $\mathbb{R}^2 \setminus \bigcup \mathcal{C}$. We denote the (planar) graph of $\mathcal{A}(\mathcal{C})$ by $G_{\mathcal{C}}$. $G_{\mathcal{C}}$ may have multiple edges, but no loops (unless $\mathcal{C}$ consists of just two touching circles, a case which we excluded).

We usually denote a unit circle by C , possibly with some modifier (subscript or superscript). The closed disk bounded by that circle is denoted by D , and its center by O , with the same modifier.

For a circle C and points A, B on C that are not antipodal, we denote by $\text{arc}_C(AB)$ the closed shortest arc on C between A and B .

For two distinct points A and B in the plane we denote by $\overline{AB}$ the line through A and B . We denote by $\overrightarrow{AB}$ the closed ray which starts at A and includes B . $[AB]$ denotes the closed interval between A and B .

For every non-collinear point P, Q, R , we denote by $\Delta(PQR)$ the closed triangle with vertices P, Q , and R . $\angle PQR$ denotes the closed convex region bounded by the rays $\overrightarrow{QP}$ and $\overrightarrow{QR}$. $\angle PQR$ denotes the angular measure of $\angle PQR$. Therefore, $0 < \angle PQR < \pi$.

Definition 2.1. The *degree* $d(P)$ of a point $P \in \mathcal{V}(\mathcal{C})$ is the number of circles in $\mathcal{C}$ that pass through P . We call P a *simple* intersection point when $d(P) = 2$.

For $k \geq 2$ denote by t_k the number of intersection points of degree k . The number V of vertices is $\sum_{k \geq 2} t_k$. Denote by f_k the number of faces of $G_{\mathcal{C}}$ which have k edges (we also regard the unbounded face). The number F of faces is $\sum_{k \geq 2} f_k$. Denote by E the number of edges of $G_{\mathcal{C}}$. We have

$$2E = \sum_{k \geq 2} k f_k \quad \text{as well as} \quad E = \sum_{k \geq 2} k t_k.$$

Therefore

$$2E = \sum_{k \geq 2} k f_k = 3F + \sum_{k \geq 2} (k - 3) f_k.$$

Clearly, $G_{\mathcal{C}}$ is connected and has no loops. Hence, by Euler's formula, $V - E + F = 2$. Therefore,

$$\begin{aligned} -6 &= -3V + 3E - 3F = -3V + E + (2E - 3F) \\ &= -3 \sum_{k \geq 2} t_k + \sum_{k \geq 2} k t_k + \sum_{k \geq 2} (k - 3) f_k \\ &= \sum_{k \geq 2} (k - 3) t_k + \sum_{k \geq 2} (k - 3) f_k. \end{aligned}$$

Rearranging, we get

$$t_2 = 6 + \sum_{k \geq 3} (k - 3) t_k + \sum_{k \geq 3} (k - 3) f_k - f_2. \quad (1)$$

According to this notation Theorem 1.1 asserts that $t_2 > 0$ (unless $\mathcal{C}$ is exceptional). As is easily seen from (1), this will follow if we show that f_2 is “small” relative to the other terms on the right-hand side of the equation. To this end we introduce the notion of lunes and lenses.

Definition 2.2. Let C_1, C_2 be two circles in $\mathcal{C}$. In case C_1 and C_2 cross, we call $L = D_1 \cap D_2$ a *lens* if it is a face with two edges in the planar graph $G_{\mathcal{C}}$. We then say that both C_1 and C_2 *support* L (see Fig. 3). The two intersection points of C_1 and C_2 are called the *vertices* of L . If C_1 and C_2 touch at P , we call P a *lens* if $d(P) \leq 3$. Also in this case we say that C_1 and C_2 *support* P . A lens that is not a single point is called a *proper* lens.

We call $D_1 \setminus \text{int } D_2$ a *lune* if it is a face with two edges of the planar graph $G_{\mathcal{C}}$. We then call $C_1 \setminus \text{int } D_2$ the *longest arc* of the lune and say that C_1 *supports* the longest arc of that lune.

Observation 2.3. *The length of the longest arc of a lune is greater than π .*

Clearly, every two-edge face is either a lune or a proper lens. In the next section we bound the number of lunes and lenses. We show (Lemma 3.4) that the number of lenses

Fig. 3. A lens and a lune.

in any configuration of n pairwise intersecting unit circles is at most n . In Lemma 3.2 we show that if C supports a lens, then it must contain an edge of the unbounded face of $\mathcal{A}(\mathcal{C})$. As a corollary of these two results we conclude (at the beginning of the proof of Theorem 1.1) that the number of lenses is less than or equal to the number of edges of the unbounded face of $\mathcal{A}(\mathcal{C})$. This is a crucial step towards showing that the right-hand side of (1) is positive. The lunes seem to make much less trouble. In Corollary 3.10 we bound the number of lunes by three. Section 4 completes the proof.

3. Some Lemmata

Lemma 3.1.

- (i) Let $C_1, C_2, C_3 \in \mathcal{C}$ such that $D_1 \cap D_2 \cap D_3$ is at most one point. Then $\Delta(O_1 O_2 O_3)$ is acute unless two of the circles touch and $D_1 \cap D_2 \cap D_3 \neq \emptyset$.
- (ii) Assume $C, C_1, C_2 \in \mathcal{C}$. If $D \cap D_1$ and $D \cap D_2$ are distinct lenses, then all angles of $\Delta(OO_1 O_2)$ are $\leq \pi/2$.

Proof. (i) We show that the angle at the vertex O_3 of $\Delta(O_1 O_2 O_3)$ is acute unless C_1 and C_2 touch at a point P , in which case $\angle O_1 O_3 O_2 < \pi/2$ iff $P \notin D_3$. The same argument will apply of course to the angles at O_1 and O_2 .

If C_1 and C_2 touch at a point P , then $\angle O_1 O_3 O_2 < \pi/2$ iff the distance from P to O_3 is > 1 or in other words iff $P \notin D_3$.

If C_1 and C_2 cross, denote by P the midpoint of $[O_1 O_2]$. $P \in \text{int } D_1 \cap \text{int } D_2$, and $\angle O_1 O_3 O_2 \geq \pi/2$ would imply that $P \in \text{int } D_3$ as well. This is impossible since $\text{int}(D_1 \cap D_2 \cap D_3) = \emptyset$.

Part (ii) follows immediately from (i), if no two of the circles C, C_1, C_2 touch or if $D_1 \cap D_2 \cap D_3 = \emptyset$. Assume therefore that $D_1 \cap D_2 \cap D_3 \neq \emptyset$, and that two of the circles touch at P . If the third circle passes through P , then $\Delta(OO_1 O_2)$ is a right triangle. If the third circle does not pass through P , then it must contain P in its interior. Consequently, this circle cannot create a lens with any of the two touching circles, a contradiction. $\square$

Fig. 4. Lemma 3.2.

Lemma 3.2. Assume $D_1 \cap D_2$ is a lens with vertices A, B (possibly $A = B$), and denote by Q the intersection point of $\overrightarrow{O_2 O_1}$ with C_1 . Then

1. C_1 contains an edge of the unbounded face of $\mathcal{A}(\mathcal{C})$.
2. If $C \in \mathcal{C}$ and $\text{int } D$ is disjoint from $D_1 \cap D_2$, then the arc $C_1 \cap D$ is included either in $\text{arc}_{C_1}(QA) \setminus \{Q\}$ or in $\text{arc}_{C_1}(QB) \setminus \{Q\}$. (If $A = B$ then by $\text{arc}_{C_1}(QA)$ and $\text{arc}_{C_1}(QB)$, we mean the two semicircles of C_1 from both sides of $[A, Q]$.)

Proof. 1. Let $\vec{r} \subset \overrightarrow{O_1 O_2}$ be the ray which starts at Q and does not include O_1 (see Fig. 4). It is enough to show that $\vec{r}$ does not meet any circle of $\mathcal{C}$ but C_1 .

Assume $C \in \mathcal{C} \setminus \{C_1\}$ meets $\vec{r}$ at a point P . $D_1 \cap D_2$ is a lens, and therefore either $D \supset D_1 \cap D_2$ or $(D_1 \cap D_2) \cap \text{int } D = \emptyset$.

If $D \supset D_1 \cap D_2$, then in particular D includes the point $R \in C_1$ that lies opposite to Q (see Fig. 4). It follows (since $P, R \in C \neq C_1$) that the diameter of D is then greater than two, which is impossible.

If, on the other hand, $(D_1 \cap D_2) \cap \text{int } D = \emptyset$, then, within the family $\mathcal{C}' = \{C_1, C_2, C\}$, $D \cap D_2$ is also a lens. By Lemma 3.1, $\angle O_2 O_1 O \leq \pi/2$ (see Fig. 4). This implies that the distance between P and O is greater than 1, i.e., $P \notin C$.

2. Let a denote the arc $C_1 \cap D$ of the circle C_1 . Since we assume $\text{int } D \cap (D_1 \cap D_2) = \emptyset$, $\text{arc}_{C_1}(AB) \cap \text{int } D = \emptyset$. Moreover, by the first part of Lemma 3.2 $Q \notin D$. It follows that a is included either in $\text{arc}_{C_1}(QA)$ or in $\text{arc}_{C_1}(QB)$. $\square$

Our next goal is to prove Lemma 3.4 which says that in an arrangement of n pairwise intersecting unit circles there are at most n lenses. We shall need the following auxiliary lemma.

Lemma 3.3. Assume $C_1, C_2, C_3, C_4 \in \mathcal{C}$ are four distinct circles. If the four intersections $D_1 \cap D_3$, $D_1 \cap D_4$, $D_2 \cap D_3$, and $D_2 \cap D_4$ are lenses, then C_1 and C_2 touch at a point P , and C_3 and C_4 touch at P as well.

Proof. We restrict our attention to the subfamily $\mathcal{C}' = \{C_1, C_2, C_3, C_4\}$. Note that if the intersection $D_i \cap D_j$ is not a lens in $\mathcal{C}'$, then it is not a lens in $\mathcal{C}$ as well. By Lemma 3.1, all

12 angles $\angle O_i O_j O_k$ ($i, j, k \in \{1, 2, 3, 4\}$) are less than or equal to $\pi/2$. However, this is possible only if O_1, O_2, O_3, O_4 are the vertices of a rectangle R . Let P the center of R . The four intervals $[P, O_i], i = 1, 2, 3, 4$, all have the same length d . If $d < 1$, then $P \in \text{int}(D_1 \cap D_2 \cap D_3 \cap D_4)$. This is impossible since already $\text{int}(D_1 \cap D_2 \cap D_3) = \emptyset$, as $D_1 \cap D_3$ and $D_2 \cap D_3$ are lenses. If $d > 1$, then the circles centered at opposite vertices of R are disjoint. There remains the case $d = 1$. In this case all four circles pass through P , and the two circles centered at opposite vertices of R touch at P . P is not a lens, because it lies on four circles. It follows that the vertex of R opposite to O_1 is O_2 , and O_4 lies opposite to O_3 . $\square$

Lemma 3.4. *If $\#(\mathcal{C}) = n$, then there are at most n lenses in $\mathcal{A}(\mathcal{C})$.*

Proof. By induction on n . The statement is clearly true for $n = 2, 3$. If there is a circle $C \in \mathcal{C}$ which supports at most one lens, then take $\mathcal{C}' = \mathcal{C} \setminus \{C\}$. By the induction hypothesis, $\mathcal{C}'$ has at most $n - 1$ lenses. Adding C to $\mathcal{C}'$, will produce at most one more lens (and could possibly kill a few lenses of $\mathcal{A}(\mathcal{C}')$).

If every circle in $\mathcal{C}$ supports exactly two lenses, then the number of lenses is n , since every lens is supported by exactly two circles.

The proof of Lemma 3.4 will follow now from the following lemma which asserts that if a circle $C \in \mathcal{C}$ supports three lenses, then there is another circle in $\mathcal{C}$ which supports just one lens.

Lemma 3.5. *Let $C, C_1, C_2, C_3 \in \mathcal{C}$ be four distinct circles. Assume that $D \cap D_i$ ($i = 1, 2, 3$) are lenses. Then for some $j \in \{1, 2, 3\}$, C_j supports no lens other than $D \cap D_j$.*

Proof. By Lemma 3.1 $\angle O_i O O_j \leq \pi/2$ for $1 \leq i < j \leq 3$. Assume, without loss of generality, that $O_2 \in \text{int} \angle O_1 O O_3$. Rotate (and if necessary flip) the plane so that the line $\overline{OO_2}$ is vertical, with O_2 above O , O_1 is to the left of $\overline{OO_2}$, and O_3 is to the right of $\overline{OO_2}$.

Let $Q = \overline{OO_2} \cap C_2$ and $P = \overline{O_2 O} \cap C$. Denote the intersection points of C and C_2 by A, B . If $D \cap D_2$ is not a proper lens, then $A = B \in \overline{OO_2}$. Otherwise, assume that A is to the left of $\overline{OO_2}$. In what follows, when $A = B$ $\text{arc}_{C_2}(AQ)$ denotes the semicircle of C_2 which is to the left of $\overline{OO_2}$, and $\text{arc}_{C_2}(BQ)$ denotes the semicircle of C_2 which is to the right of $\overline{OO_2}$.

We show that C_2 supports no lens other than $D \cap D_2$. Assume to the contrary, that $C_0 \in \mathcal{C}$, $C_0 \neq C$, and $D_0 \cap D_2$ is a lens.

By Lemma 3.1, $\Delta(OO_0O_2)$ is a proper triangle, hence $O_0 \notin \overline{OO_2}$. Assume, without loss of generality, that O_0 is to the right of $\overline{OO_2}$. We restrict our attention to the subfamily $\mathcal{C}' = \{C, C_0, C_1, C_2\}$ and intend to show that $D_0 \cap D_1$ is a lens in $\mathcal{C}'$, contradicting Lemma 3.3. (Indeed, by Lemma 3.3, this is possible only if C and C_0 touch at a point V , and C_1, C_2 touch at V as well. This is impossible, since O_1 is to the left of $\overline{OO_2}$ and O_0 is to the right of $\overline{OO_2}$.)

Next come four applications of Lemma 3.2. With C_0, C_2, C standing for C, C_1, C_2 , respectively, and since O_0 is to the right of $\overline{OO_2}$, we have that

$$C_2 \cap D_0 \subseteq \text{arc}_{C_2}(QB) \setminus \{Q\}. \quad (2)$$

Similarly,

$$\begin{array}{ll} \text{with } C_0, C, C_2 \text{ we get} & C \cap D_0 \subseteq \text{arc}_C(PB) \setminus \{P\}, \\ \text{with } C_1, C_2, C \text{ we get} & C_2 \cap D_1 \subseteq \text{arc}_{C_2}(QA) \setminus \{Q\}, \\ \text{with } C_1, C, C_2 \text{ we get} & C \cap D_1 \subseteq \text{arc}_C(PA) \setminus \{P\}. \end{array}$$

Therefore,

$$C \cap D_0 \cap D_1, C_2 \cap D_0 \cap D_1 \subseteq \begin{cases} \emptyset & \text{if } A \neq B, \\ \{A\} & \text{if } A = B. \end{cases} \quad (3)$$

If $A = B$, then $\{A\}$ is a lens and so $A \notin C_0 \cap C_1$. Thus if $A \in D_0 \cap D_1$, then A is an interior point of D_0 or D_1 , but this is not possible because $D_2 \cap D_0$ and $D \cap D_1$ are both lenses. Therefore $C \cap D_0 \cap D_1$ and $C_2 \cap D_0 \cap D_1$ are always empty which proves that $D_0 \cap D_1$ is a lens in $\mathcal{C}'$. $\square$

Our next aim is to study the lunes in $\mathcal{A}(\mathcal{C})$, and more specifically the relationship between lunes and lenses in $\mathcal{A}(\mathcal{C})$.

We denote the length of a circular arc S by $|S|$.

Lemma 3.6. *Let C, C' be distinct circles in $\mathcal{C}$, with $C \cap C' = \{A, B\}$ (possibly $A = B$, if C and C' touch). Assume $|\text{arc}_C(AB)| = |\text{arc}_{C'}(AB)| = \alpha$. Denote by L, L' the rays tangent to $\text{arc}_C(AB)$ and $\text{arc}_{C'}(AB)$ at A . Then*

1. *The angle between L and L' is α .*
2. *The intersection of C (or C') with the convex angular region bounded by L and L' has length 2α .*

Proof. See Fig. 5. Note that the ray $\overrightarrow{AB}$ makes an angle $\frac{1}{2}\alpha$ with each of the tangents L, L' . $\square$

Lemma 3.7. *Suppose $C \in \mathcal{C}$ supports the longest arc of a lune. Then every lens that is included in D must be supported by C . Moreover, if C_1 and C_2 are two circles which touch at a point $P \in D$, then either $C = C_1$ or $C = C_2$.*

Fig. 5. Lemma 3.6.

Proof. It is enough to prove that if $C_1, C_2 \in \mathcal{C} \setminus \{C\}$, then $D_1 \cap D_2 \not\subseteq D$.

Suppose $D \setminus \text{int } D'$ is a lune. Then the arc $C \cap D'$ is included in an open semicircle S , and for any circle $C_i \in \mathcal{C} \setminus \{C\}$ we have $C \cap D_i \subset C \cap D' \subset S$. Assume first that $C \cap D_1$ and $C \cap D_2$ do not have interior points in common.

Let $C \cap C_1 = \{A', B'\}$ (possibly $A' = B'$) and $C \cap C_2 = \{A'', B''\}$ (possibly $A'' = B''$), and assume that the points A', B', A'', B'' appear in this order on S (possibly $B' = A''$). Let $|C \cap D_1| = \alpha \geq 0$, $|\text{arc}_C(B'A'')| = \beta \geq 0$, $|C \cap D_2| = \gamma \geq 0$. It follows from Observation 2.3 that $\alpha + \beta + \gamma < \pi$.

Draw the ray L_1 tangent to $C_1 \cap D$ at B' , and the ray L_2 tangent to $C_2 \cap D$ at A'' . Since $2\alpha + \beta + 2\gamma < 2\pi$, by Lemma 3.6, these rays do not cross inside D , and they separate $D \cap D_1$ from $D \cap D_2$. If C_1 and C_2 touch at B' , then L_1 and L_2 coincide. In this case $\beta = 0$ and $\alpha + \gamma = \pi$ contradicting the assumption that $\alpha + \beta + \gamma < \pi$.

Therefore, C_1 and C_2 cross, and since $D \cap D_1$ is separated by a line from $D \cap D_2$, we conclude that $D_1 \cap D_2 \not\subseteq D$.

If the arcs $C \cap D_1$ and $C \cap D_2$ do have interior points in common, then clearly $D_1 \cap D_2 \not\subseteq D$. $\square$

From this point on we assume that every intersection point of circles in $\mathcal{C}$ has degree ≥ 3 .

Lemma 3.8. *Let $C_1, C_2 \in \mathcal{C}$ be two different circles, then either $D_1 \cap D_2$ includes a lens or there is a circle which touches C_1 or C_2 at a point on the boundary of $D_1 \cap D_2$.*

Proof. Clearly, if C_1 and C_2 touch, then there is nothing to prove. Assume then that C_1 and C_2 cross and $C_1 \cap C_2 = \{A, B\}$. We assume that no circle in $\mathcal{C}$ touches C_1 or C_2 at a point on the boundary of $D_1 \cap D_2$.

Let G'_C be the planar subgraph of G_C which is induced by the vertices of G_C which are in $D_1 \cap D_2$. We regard the complement of $D_1 \cap D_2$ as the unbounded face of G'_C .

For every $k \geq 2$ let u_k denote the number of vertices inside $\text{int } (D_1 \cap D_2)$ through which exactly k circles pass. Let v_k denote the number of vertices, different from A and B , on the boundary of $D_1 \cap D_2$ through which exactly k circles pass. Denote by w_k the number of faces in G'_C which have exactly k edges.

Let V, E, F denote the number of vertices, edges, and faces of G'_C , respectively. Then $V = 2 + \sum_{k \geq 2} u_k + v_k$ and $F = \sum_{k \geq 2} w_k$. Counting the number of edges in two different ways we get

$$2E = \sum_{k \geq 2} k w_k = 3F + \sum_{k \geq 2} (k - 3) w_k. \quad (4)$$

Since we assume that $D_1 \cap D_2$ includes no lens, all faces of G'_C have at least three edges. Note that the unbounded face has exactly $2 + \sum_{k \geq 2} v_k$ edges. Therefore,

$$\sum_{k \geq 2} (k - 3) w_k \geq \sum_{k \geq 2} v_k - 1. \quad (5)$$

Observe that the degree in G'_C of a vertex counted by u_k is $2k$ and the degree of a vertex counted by v_k is $k + 1$. Let $\deg(A)$ and $\deg(B)$ denote the degrees in G'_C of A and

B , respectively. Counting the sum of the degrees of all vertices in two different ways we obtain

$$2E = \deg(A) + \deg(B) + \sum_{k \geq 2} 2ku_k + (k+1)v_k. \quad (6)$$

By the Euler formula $V - E + F = 2$, so that plugging in (4)–(6) we obtain

$$\begin{aligned} -12 &= -6V + 6E - 6F = -6V + 2E + 2(2E - 3F) \\ &= -6 \left(2 + \sum_{k \geq 2} u_k + v_k \right) + \deg(A) + \deg(B) \\ &\quad + \sum_{k \geq 2} (2ku_k + (k+1)v_k) + 2 \sum_{k \geq 2} (k-3)w_k \\ &= -12 + \deg(A) + \deg(B) + \sum_{k \geq 2} ((2k-6)u_k + (k-5)v_k) \\ &\quad + 2 \sum_{k \geq 2} (k-3)w_k \\ &\geq -12 + \deg(A) + \deg(B) + \sum_{k \geq 2} ((2k-6)u_k + (k-5)v_k) \\ &\quad + 2 \left(\sum_{k \geq 2} v_k - 1 \right) \\ &= -14 + \deg(A) + \deg(B) + \sum_{k \geq 2} ((2k-6)u_k + (k-3)v_k). \end{aligned}$$

Rearranging, we get

$$\begin{aligned} 2u_2 + v_2 &\geq -2 + \deg(A) + \deg(B) + \sum_{k \geq 3} ((2k-6)u_k + (k-3)v_k) \\ &\geq -2 + \deg(A) + \deg(B) \geq 2. \end{aligned} \quad (7)$$

The last inequality is true because $\deg(A), \deg(B) \geq 2$. Every vertex counted by u_2 is an intersection point of precisely two circles in $\mathcal{C}$. Moreover, since we assume that no circle in $\mathcal{C}$ touches C_1 or C_2 at a point on the boundary of $D_1 \cap D_2$, every vertex counted by v_2 is an intersection point of precisely two circles in $\mathcal{C}$. We assume that through every intersection point in $\mathcal{A}(\mathcal{C})$ at least three circles pass and hence $u_2 = v_2 = 0$. In view of (7), this is a contradiction. $\square$

Corollary 3.9. *Assume that $C_1, C_2 \in \mathcal{C}$ and each of C_1, C_2 supports the longest arc of a lune, then either $D_1 \cap D_2$ is a proper lens or C_1 and C_2 touch.*

Proof. By Lemma 3.8, either $D_1 \cap D_2$ includes a lens l or it includes a tangency point P of two circles $C_3, C_4 \in \mathcal{C}$. In the first case, using Lemma 3.7, l is supported by C_1 as well as by C_2 and hence $l = D_1 \cap D_2$. In the second case since $P \in D_1$, by the second part of Lemma 3.7, C_1 is one of C_3, C_4 , and similarly C_2 is one of C_3, C_4 . Hence, C_1 and C_2 touch at P . $\square$

Corollary 3.10. *There are at most three lunes in $\mathcal{A}(\mathcal{C})$.*

Proof. Assume $C_1, C_2, C_3, C_4 \in \mathcal{C}$ and each of them supports the longest arc of some lune. By Corollary 3.9, for every $1 \leq i < j \leq 4$, either $D_i \cap D_j$ is a proper lens or C_i and C_j touch.

Note that if C_i and C_j touch at a point P and C_k also passes through P , then, by the second part of Lemma 3.7, C_k coincides with C_i or C_j . This means that all six intersections $D_i \cap D_j$ are lenses which contradicts Lemma 3.4. $\square$

4. Proof of the Main Theorem

Proof of Theorem 1.1. Assume to the contrary that every intersection point $P \in \mathcal{A}(\mathcal{C})$ has $d(P) \geq 3$.

Denote by m the number of lenses and by l the number of lunes. The faces of $G_{\mathcal{C}}$ counted by f_2 are exactly the *proper* lenses and the lunes. Therefore $f_2 \leq m + l$. Let u denote the number of edges of the unbounded face of $\mathcal{A}(\mathcal{C})$. We claim that $m \leq u$. Indeed, let $C_1, \dots, C_s$ be all the circles in $\mathcal{C}$ that support a lens. By Lemma 3.2, the unbounded face has at least $u \geq s$ edges. By Lemma 3.4, there are at most $m \leq s$ lenses.

Substituting this in (1) we get

$$t_2 = 6 + \sum_{k \geq 3} (k-3)t_k + \sum_{k \geq 3} (k-3)f_k - f_2 \geq 6 + (m-3) - (m+l) = 3-l,$$

where the inequality is true because $f_2 \leq m+l$, $\sum_{k \geq 3} (k-3)t_k \geq 0$, and $\sum_{k \geq 3} (k-3)f_k \geq u-3 \geq m-3$, due to the unbounded face.

Since we assume $t_2 = 0$ we must have $l = 3$ (because of Corollary 3.10). Note also that all inequalities must be equalities. Hence we deduce the following equalities: $f_2 = m + l$, implying that all lenses are proper lenses. $\sum_{k \geq 3} (k-3)t_k = 0$, implying that every intersection point has degree 3. $\sum_{k \geq 3} (k-3)f_k = m-3$, implying that the unbounded face has exactly m edges. Hence (following our argument showing that $u \geq m$) every circle which includes an edge of the unbounded face must support a lens. Moreover, every bounded face which is neither a lens nor a lune must be a triangle (that is, a face with just three edges). An important corollary is that no two circles in $\mathcal{C}$ touch. For if two circles in $\mathcal{C}$ touch at a point P , then $d(P) = 3$ implies that P is a lens which is not a proper lens and we obtain a contradiction. From now on we assume that every two circles in $\mathcal{C}$ cross.

Let C_1, C_2, C_3 be the three circles which support the longest arcs of the three lunes in $\mathcal{A}(\mathcal{C})$. By Corollary 3.9, for every $1 \leq i < j \leq 3$ $D_i \cap D_j$ is a proper lens. Denote $l_1 = D_2 \cap D_3, l_2 = D_1 \cap D_3, l_3 = D_1 \cap D_2$.

Claim 4.1. *Let $C \in \mathcal{C}$. Then D includes at least one of l_1, l_2, l_3 .*

Proof. If C is one of C_1, C_2, C_3 , then clearly D includes two of l_1, l_2, l_3 . Let $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$ and assume that D does not include any of l_1, l_2, l_3 . Hence, D is disjoint from the interior of each of l_1, l_2, l_3 .

$D \cap \text{int } l_2 = \emptyset$ implies that $\text{int } D \cap \text{int } D_1 \cap \text{int } D_3 = \emptyset$. It follows that $D_3 \cap \text{int } (D \cap D_1) = \emptyset$. Similarly, $D \cap \text{int } l_3 = \emptyset$ implies $D_2 \cap \text{int } (D \cap D_1) = \emptyset$. In other words, if we restrict our attention to the subfamily $\mathcal{C}' = \{C, C_1, C_2, C_3\}$ then $D \cap D_1$ is a lens.

In the same way we obtain that also $D \cap D_2$ and $D \cap D_3$ are lenses (within $\mathcal{A}(\mathcal{C}')$). This contradicts Lemma 3.4. $\square$

$D_1 \cap D_2 \cap D_3$ has an empty interior so it is either empty or a single point.

Case 1: $D_1 \cap D_2 \cap D_3$ is a point P . Let $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$. By Claim 4.1, D must include at least one of l_1, l_2, l_3 . Hence $P \in D$. We claim that $P \in \text{int } D$. Assume to the contrary that $P \in C$. Then D includes either one or two lenses of l_1, l_2, l_3 . If D includes just one lens, say l_1 , then it must be disjoint from the interiors of l_2 and l_3 . This is possible only if C touches C_1 at P , which is impossible since every two circles in $\mathcal{C}$ cross. If D includes two lenses, say l_2 and l_3 , and is disjoint from the interior of the third, l_1 , then it is easily seen by inspection that C must coincide with C_1 , contradicting our assumption on C .

Hence $P \in \text{int } D$. It follows now that D includes all lenses l_1, l_2, l_3 . Denote the remaining intersection points of the circles C_1, C_2 , and C_3 by P_1, P_2, P_3 (see Fig. 1). We regard the affine plane as the two-dimensional vector space $\mathbb{R}^2$ with P as its origin. Then $P_1 = O_2 + O_3$, $P_2 = O_1 + O_3$, and $P_3 = O_1 + O_2$.

Therefore the distance of each of P_1, P_2 , and P_3 to the point $O_1 + O_2 + O_3$ is 1, which shows that the circle through P_1, P_2 , and P_3 is a unit circle. Moreover,

$$\Delta(P_1 P_2 P_3) = (O_1 + O_2 + O_3) - \Delta(O_1 O_2 O_3).$$

Hence, $\Delta(P_1 P_2 P_3)$ is congruent to $\Delta(O_1 O_2 O_3)$, which is an acute triangle by Lemma 3.1. It follows that the circumcircle of $\Delta(P_1 P_2 P_3)$ is a unit circle which we denote by S .

Let $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$. D includes l_1, l_2 , and l_3 , and so in particular $D \supset \Delta(P_1 P_2 P_3)$. It follows that $C = S$.

In other words $\#(\mathcal{C}) \leq 4$. If $\mathcal{C} = \{S, C_1, C_2, C_3\}$ then $\mathcal{C}$ is exceptional. Otherwise, $\mathcal{C} = \{C_1, C_2, C_3\}$ and each of P_1, P_2 , and P_3 has degree 2.

Case 2: $D_1 \cap D_2 \cap D_3 = \emptyset$. Denote by B_1, B_2, B_3 and A_1, A_2, A_3 the intersection points of C_1, C_2 , and C_3 as indicated in Fig. 6.

Lemma 4.2. *Let C, C_1, C_2 be three pairwise crossing unit circles. If $D \supset D_1 \cap D_2$, then $\angle O_1 O O_2 > \pi/2$.*

Proof. Denote by A, B the intersection points of C_1 and C_2 . Let S_A, S_B be the unit disks centered at A, B , respectively. Note that $O \in S_A \cap S_B$. Observe that the boundaries of S_A and S_B intersect at O_1 and O_2 . $S_A \cap S_B$ is included in the disk whose diameter is $[O_1 O_2]$. Hence $\angle O_1 O O_2 > \pi/2$. $\square$

Claim 4.3. *Assume $C \in \mathcal{C}$ and D includes l_1 but not l_2 or l_3 . Then C intersects C_1 in two points on $\text{arc}_{C_1}(B_2 B_3)$.*

Remark. By symmetry, Claim 4.3 is true if we change the role of l_1 by l_2 or l_3 , and, correspondently, C_1 by C_2 or C_3 .

Fig. 6. Case 2.

Proof. We show that O is inside $\angle O_2 O_1 O_3$. D is disjoint from $\text{int } l_2 = \text{int } (D_1 \cap D_3)$ and from $\text{int } l_3 = \text{int } (D_1 \cap D_2)$. Therefore, by Lemma 3.1, the triangles $\Delta(O_1 O_2 O)$ and $\Delta(O_1 O_3 O)$ are acute triangles. Moreover, by Lemma 3.1, all the angles of $\Delta(O_1 O_2 O_3)$ are acute. If $O \notin \angle O_2 O_1 O_3$, then either $O \in \angle O_1 O_3 O_2$ or $O \in \angle O_1 O_2 O_3$. We assume, without loss of generality, that $O \in \angle O_1 O_3 O_2$. Since $O \notin \angle O_2 O_1 O_3$, it follows that $O_3 \in \angle O_1 O O_2$. By Lemma 4.2, $\angle O_2 O O_3 > \pi/2$. We obtain a contradiction since now $\angle O_2 O O_3 \leq \angle O_2 O O_1 < \pi/2$.

We conclude that $\overrightarrow{O_1 O}$ must be between $\overrightarrow{O_1 O_2}$ and $\overrightarrow{O_1 O_3}$. Since D is disjoint from $\text{int } l_2$ and $\text{int } l_3$, $D \cap C_1$ must be disjoint from the relative interiors of both arcs $D_2 \cap C_1$ and $D_3 \cap C_1$. The center of the arc $C_1 \cap D$ is between the centers of $C_1 \cap D_2$ and $C_1 \cap D_3$. It follows now that $D \cap C_1$ must be included in $\text{arc}_{C_1}(B_2 B_3)$. $\square$

Claim 4.4. Assume that $C \in \mathcal{C} \setminus \{C_1, C_2, C_3\}$. If C supports a lens, then D includes exactly one of the lenses l_1, l_2 , or l_3 .

Proof. By Claim 4.1, D includes at least one of l_1, l_2 , or l_3 . Assume, without loss of generality, that $D \supset l_1 \cup l_2$ and that $D \cap D'$ is a lens. D includes also $\text{arc}_{C_3}(B_1 B_2)$. D' must include only l_3 (and not any of l_1, l_2) for otherwise $D \cap D'$ includes either l_1 or l_2 and therefore is not a lens.

By Claim 4.3, $\text{int } D'$ meets the relative interior of $\text{arc}_{C_3}(B_1 B_2)$, implying that $\text{int } (D \cap D')$ also meets it. This is contradicting the assumption that $D \cap D'$ is a lens. $\square$

Claim 4.5. Each of A_1, A_2 , and A_3 is included in the interior of some unit disk bounded by a circle from $\mathcal{C}$.

Proof. We prove the claim for A_1 . Assume that the claim is not true for A_1 . Let $C'_2 \in \mathcal{C}$ be the circle so that $D_2 \setminus \text{int } D'_2$ is a lune. Since $A_1 \in D'_2$, $A_1 \notin \text{int } D'_2$ implies that C'_2 passes through A_1 . Similarly, let $C'_3 \in \mathcal{C}$ be the circle so that $D_3 \setminus \text{int } D'_3$ is a lune, then C'_3 passes through A_1 . However, $d(A_1) = 3$ and C_2, C_3 pass through A_1 . It follows that $C'_2 = C'_3$. Denote $C' = C'_2 = C'_3$. D' includes all lenses l_1, l_2, l_3 . Denote the intersection points of C' with C_2 and C_3 other than A_1 by Y, Z , respectively (see Fig. 6).

The face whose two edges are $\text{arc}_{C_3}(A_1 B_1)$ and $\text{arc}_{C'}(A_1 Y)$ must have exactly three edges so there must be a circle (other than C_2) in $\mathcal{C}$ which passes through Y and B_1 . Similarly, there must be a circle (other than C_3) in $\mathcal{C}$ which passes through Z and B_1 . Since $d(B_1) = 3$ and C_2 and C_3 pass through B_1 , it follows that the same circle $C'' \in \mathcal{C}$ passes through Y, B_1 , and Z .

Observe that $Y \neq A_3$ for otherwise there would be four circles through this point, namely C_1, C_2, C' , and C'' . It is easily seen now that the face whose edge is $\text{arc}_{C''}(Y B_1)$ which is adjacent to the face $A_1 Y B_1$ has more than three edges (as no other circles pass through Y nor B_1), a contradiction. $\square$

Let Δ_0 denote the closed region which is bounded by the arcs $\text{arc}_{C_1}(B_2 B_3)$, $\text{arc}_{C_2}(B_3 B_1)$, and $\text{arc}_{C_3}(B_1 B_2)$.

Lemma 4.6. *Let $C \in \mathcal{C}$. Then $D \cap \Delta_0 \cap \overline{B_1 A_1} \neq \emptyset$. (Similarly for $\overline{B_2 A_2}$ and $\overline{B_3 A_3}$.)*

Proof. If $D \supset l_1$, then clearly $B_1 \in D \cap \Delta_0 \cap \overline{B_1 A_1}$. If $D \supset (l_2 \cup l_3)$, then D includes $\text{arc}_{C_1}(B_2 B_3)$. Therefore, $D \cap \Delta_0 \cap \overline{B_1 A_1}$ includes the intersection point of $\text{arc}_{C_1}(B_2 B_3)$ with $\overline{B_1 A_1}$ (indeed, this intersection point exists because $\overline{B_1 A_1}$ separates l_2 and l_3).

The only case which is left to check is where D includes only one of l_2 or l_3 , and not l_1 . Assume, without loss of generality, that D includes l_3 and not l_1 or l_2 . By Claim 4.3, D includes a subarc of $\text{arc}_{C_3}(B_1 B_2)$. Since D is convex, D includes an interval connecting B_3 to a point on $\text{arc}_{C_3}(B_1 B_2)$. This interval intersects $\overline{B_1 A_1}$ inside Δ_0 . $\square$

Claim 4.7. *For each of the lenses l_1, l_2 , and l_3 there is a circle $C \in \mathcal{C}$ such that D includes that lens and not the two others.*

Proof. We prove the claim for l_1 . Let Q be the intersection point of a circle from $\mathcal{C}$ and the ray $\overrightarrow{B_1 A_1}$, which has maximum distance from B_1 . One of the circles C through Q includes an edge of the unbounded face. By Claim 4.5, $Q \neq A_1$. Consequently, C is not one of C_1, C_2, C_3 . We show that D includes l_1 and not any of l_2 or l_3 .

By Lemma 4.6, $D \cap \Delta_0 \cap \overline{B_1 A_1} \neq \emptyset$. It follows, from the convexity of D , that $D \supset [B_1 A_1]$. Hence $D \supset l_1$.

Observe that if D includes l_2 or l_3 , then, by Claim 4.4, C does not support a lens. This is a contradiction because C includes an edge of the unbounded face. $\square$

We will need the following geometric observation in what follows.

Lemma 4.8. *Let C, C_1, C_2 be three distinct unit circles. Assume that C_1 and C_2 cross, and $D \supset D_1 \cap D_2$. Denote the intersection points of C_1 and C_2 by M, N . Denote the*

Fig. 7. Lemma 4.8.

intersection points of C and C_1 by X_1, Y_1 and those of C and C_2 by X_2, Y_2 (as indicated in Fig. 7). Let $\theta = |\text{arc}_C(X_1X_2)|$ and let $\alpha = |\text{arc}_{C_1}(MN)|$. Then

$$|\text{arc}_{C_1}(Y_1M)| + |\text{arc}_{C_2}(Y_2M)| = \pi + \alpha - \theta.$$

Proof. Rotate (and if necessary flip) the plane so that $\overline{O_1O_2}$ is horizontal, O_1 is to the right of O_2 , and X_1, X_2 are below $\overline{O_1O_2}$. We show how to deal with the case where O is on or above $\overline{O_1O_2}$. The other case can be treated similarly. Note that each of the quadrilaterals $OX_1O_1Y_1$ and $OX_2O_2Y_2$ is a rhombus.

$$\begin{aligned} |\text{arc}_{C_2}(MY_2)| + |\text{arc}_{C_1}(MY_1)| &= \alpha + \angle O_1O_2Y_2 + \angle O_2O_1Y_1 \\ &= \alpha + \angle O_1O_2O + \angle OO_2Y_2 + \angle O_2O_1O + \angle OO_1Y_1 \\ &= \alpha + \angle O_1O_2O + \angle O_2OX_2 + \angle O_2O_1O + \angle O_1OX_1 \\ &= \alpha + \angle O_1O_2O + \angle O_2O_1O + \angle O_2OO_1 - \theta \\ &= \pi + \alpha - \theta. \end{aligned} \quad \square$$

Denote by $\alpha_1, \alpha_2, \alpha_3$ the lengths of the arcs $C_1 \cap D_2$, $C_2 \cap D_3$, and $C_3 \cap D_1$, respectively. Denote by $\theta_1, \theta_2, \theta_3$ the lengths of $\text{arc}_{C_1}(B_2B_3)$, $\text{arc}_{C_2}(B_3B_1)$, and $\text{arc}_{C_3}(B_1B_2)$, respectively. By reducing to the extreme case where C_1, C_2 , and C_3 are pairwise touching we observe that for every $1 \leq i \leq 3$, $\theta_i < \pi/3$. For $i = 1, 2, 3$ let $C'_i \in \mathcal{C}$ be a circle which is guaranteed by Claim 4.7, for l_i . Thus, D'_i includes l_i alone from l_1, l_2 , and l_3 .

D'_1 includes l_1 and is disjoint from the interiors of both l_2 and l_3 . Therefore C'_1 separates the interior of l_1 from the interior of l_3 .

Let $F_{12} = C'_1 \cap \text{arc}_{C_2}(B_3B_1)$, and similarly define $F_{13}, F_{21}, F_{23}, F_{31}$, and F_{32} (see Fig. 8).

Also let $\theta'_1 = |\text{arc}_{C'_1}(F_{12}F_{13})|$, $\theta'_2 = |\text{arc}_{C'_2}(F_{21}F_{23})|$, and $\theta'_3 = |\text{arc}_{C'_3}(F_{31}F_{32})|$.

Claim 4.9. $\theta'_1, \theta'_2, \theta'_3 \leq \max(\theta_1, \theta_2, \theta_3) < \pi/3$.

Proof. We prove the claim for $\theta'_1 = |\text{arc}_{C'_1}(F_{12}F_{13})|$. The length of $\text{arc}_{C'_1}(F_{12}F_{13})$ depends monotonously on the length of the interval $[F_{12}F_{13}]$. Observe that $[F_{12}F_{13}]$ is included in $\Delta(B_1B_2B_3)$ and therefore the length of $[F_{12}F_{13}]$ is less than or equal to the

Fig. 8. Claim 4.9.

maximum of the lengths of the intervals $[B_3 B_1]$, $[B_1 B_2]$, and $[B_2 B_3]$. Consequently, the length of $\text{arc}_{C'_1}(F_{12} F_{13})$ is at most $\max(\theta_1, \theta_2, \theta_3)$. $\square$

Let F'_{12} denote the second intersection point of C'_1 and C_2 (other than F_{12}). Similarly define F'_{13} , F'_{21} , F'_{23} , F'_{31} , and F'_{32} .

$D'_1 \supset D_2 \cap D_3$. Therefore, using Lemma 4.8 with C'_1 , C_2 , and C_3 , We obtain

$$|\text{arc}_{C_2}(B_1 F'_{12})| + |\text{arc}_{C_3}(B_1 F'_{13})| = \pi + \alpha_1 - \theta'_1 > \frac{2\pi}{3} + \alpha_1. \quad (8)$$

Similarly, using Lemma 4.8 twice more, once with C'_2 , C_1 , C_3 and once with C'_3 , C_1 , C_2 , we obtain

$$|\text{arc}_{C_1}(B_2 F'_{21})| + |\text{arc}_{C_3}(B_2 F'_{23})| = \pi + \alpha_2 - \theta'_2 > \frac{2\pi}{3} + \alpha_2, \quad (9)$$

$$|\text{arc}_{C_1}(B_3 F'_{31})| + |\text{arc}_{C_2}(B_3 F'_{32})| = \pi + \alpha_3 - \theta'_3 > \frac{2\pi}{3} + \alpha_3. \quad (10)$$

Since C_1 supports the longest arc of a lune, $\text{arc}_{C_1}(B_2 F'_{21})$, $\text{arc}_{C_1}(B_3 F'_{31})$, and $\text{arc}_{C_1}(B_2 B_3)$ are included in an open semicircle on C_1 . Therefore,

$$|\text{arc}_{C_1}(B_2 F'_{21})| + |\text{arc}_{C_1}(B_3 F'_{31})| + |\text{arc}_{C_1}(B_2 B_3)| < \pi. \quad (11)$$

A similar argument for C_2 and C_3 gives

$$|\text{arc}_{C_2}(B_3 F'_{32})| + |\text{arc}_{C_2}(B_1 F'_{12})| + |\text{arc}_{C_2}(B_3 B_1)| < \pi, \quad (12)$$

$$|\text{arc}_{C_3}(B_1 F'_{13})| + |\text{arc}_{C_3}(B_2 F'_{23})| + |\text{arc}_{C_3}(B_1 B_2)| < \pi. \quad (13)$$

Summing up (11)–(13), we get

$$\begin{aligned}
 3\pi &> (|\operatorname{arc}_{C_2}(B_1F'_{12})| + |\operatorname{arc}_{C_3}(B_1F'_{13})|) \\
 &\quad + (|\operatorname{arc}_{C_1}(B_2F'_{21})| + |\operatorname{arc}_{C_3}(B_2F'_{23})|) \\
 &\quad + (|\operatorname{arc}_{C_1}(B_3F'_{31})| + |\operatorname{arc}_{C_2}(B_3F'_{32})|) \\
 &\quad + (\theta_1 + \theta_2 + \theta_3) \\
 &> 2\pi + \alpha_1 + \alpha_2 + \alpha_3 + \theta_1 + \theta_2 + \theta_3 = 3\pi,
 \end{aligned} \tag{14}$$

by (8)–(10) and since $\alpha_1 + \alpha_2 + \alpha_3 + \theta_1 + \theta_2 + \theta_3$ is the sum of the internal angles of $\Delta(O_1O_2O_3)$. We thus reached the desired contradiction. $\square$

5. Concluding Remarks

It is interesting to note that Theorem 1.1 remains true if $\mathcal{C}$ is a family of (at least five) pairwise intersecting translates of the boundary of a centrally symmetric convex body. The proof goes through the same lines as the proof presented in this paper. Clearly, one cannot use the notion of an angle as it is used above any more. However, the main lemmata and the structure of the proof remain identical. When an angle whose apex is a center of some disk is measured, one should usually take the length of the corresponding arc on the boundary of that disk (assuming that all bodies are normalized so that the length of their boundaries is 2π).

We can state this result in the following form.

Theorem 5.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$, and let A be a finite set of points with diameter ≤ 2 . Then there exist a point p in the plane such that $\|p - a\| = 1$ is true for precisely two elements $a \in A$.*

As mentioned in the Introduction, Theorem 1.1 does not generalize to the case of circles with different radii. In [ALPS], however, we did manage to prove the following weaker form of a generalization of Theorem 1.1.

Theorem 5.2. *Let $\mathcal{C}$ be a family of n pairwise intersecting circles in the plane. If n is sufficiently large and $\mathcal{C}$ is not a pencil, then there exists an intersection point incident to at most three circles.*

In [NPPS] we generalized Theorem 5.2 to the case where $\mathcal{C}$ is a family of pseudocircles.

Finally, we already noted in the Introduction that the analogue of Theorem 1.1 to the case where $\mathcal{C}$ is a family of unit circles, which are not necessarily pairwise intersecting, is not valid. However, all known counterexamples to this case are quite sparse. In this context we conjecture the following.

Conjecture 5.3. *Let $\mathcal{C}$ be a family of n unit circles in a bounded region. If n is large enough, then there exists an intersection point of degree 2.*

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