Almost Disjoint Triangles in 3-Space*

Gyula Károlyi¹ and József Solymosi²

¹Institute for Operations Research, Department of Mathematics, ETH Zurich,
CH-8092 Zurich, Switzerland
karolyi@cs.elte.hu

²Department of Mathematics, University of California, San Diego,
9500 Gilman Drive, La Jolla, CA 92093-0112, USA
solymosi@math.ucsd.edu

Abstract. Two triangles are called *almost disjoint* if they are either disjoint or their intersection consists of one common vertex. Let $f(n)$ denote the maximum number of pairwise almost disjoint triangles that can be found on some vertex set of n points in 3-space. Here we prove that $f(n) = \Omega(n^{3/2})$.

1. Introduction

Two triangles are called *almost disjoint* if they are either disjoint or their intersection consists of one common vertex. Let $f(n)$ denote the maximum number of pairwise almost disjoint triangles that can be found on some vertex set of n points in 3-space. Note that in a set of almost disjoint triangles, any vertex can be incident to at most $(n-1)/2$ triangles, thus $f(n) = O(n^2)$. The study of this function was suggested by Kalai [3]. For other extremal problems concerning families of triangles in 3-space we refer to [1] and [5].

The problem is closely related to the existence of closed polyhedral 2-manifolds in 3-space with unusually large genus. A result of McMullen et al. [6] implies $f(n) = \Omega(n \log n)$. The main goal of this paper is to prove the following lower bound on $f(n)$.

Theorem 1. $f(n) = \Omega(n^{3/2})$.

* This research was done while the first author was on leave from Eötvös University, Budapest, and the research was partially supported by Hungarian Scientific Research Grants OTKA F030822 and AKP 2000–78 2.1. The work of the second author was done while on leave from SZTAKI, and was supported by the Berlin–Zürich European Graduate Program “Combinatorics, Geometry, and Computation”.

We present two constructions to prove this theorem, each based on the following idea. For every $m \geq 3$, first we construct a set $\mathcal{R} = \{R_1, R_2, \dots, R_m\}$ of m points and a set $\mathcal{Q} = \{Q_{ij} \mid 1 \leq i < j \leq m\}$ of $\binom{m}{2}$ points such that the triangles $R_i R_j Q_{ij}$ are almost disjoint. Then we “blow up” the construction by considering translates $\mathcal{R}^\kappa = \{R_1^\kappa, R_2^\kappa, \dots, R_m^\kappa\}$ ($1 \leq \kappa \leq m$) of $\mathcal{R}$ such that all the triangles $R_i^\kappa R_j^\kappa Q_{ij}$ are almost disjoint. Then we have

$$f\left(m^2 + \binom{m}{2}\right) \geq m \binom{m}{2}$$

for every $m \geq 3$, and the theorem follows.

The second construction, found by the first author, exploits both the combinatorial and the geometric properties of cyclic polytopes. Although it is quite involved, we think that an argument using cyclic polytopes is quite natural here, and may serve as a basis for an improvement on Theorem 1 if one can reduce the cardinality of the set $\mathcal{Q}$. While refereeing the original version of this paper, the second author found, based on some of the same principal ideas, a strikingly simple construction that does not depend on cyclic polytopes. Since we do not know if Theorem 1 is best possible, we found it reasonable to publish both proofs in the same paper. We also believe that both constructions can be useful in the study of further extremal problems concerning geometric hypergraphs.

We organize this paper as follows. The first construction, which is very easy to describe, is presented in Section 2. In Section 3 we give a concise introduction to cyclic polytopes and formulate the results that are essential to the second construction. In the last section we sketch an alternative proof of Theorem 1. Unfortunately, it is not clear how to transfer these constructions to closed polyhedral manifolds. Thus we address the following problem.

Problem 2. Is there, for n arbitrarily large, a closed polyhedral manifold in 3-space with n vertices whose genus is $\Omega(n^{3/2})$?

2. The First Construction

Let $\mathcal{R} = \{R_1, R_2, \dots, R_m\}$ be a set of $m \geq 3$ points in general position in 3-space, that is, no three collinear and no four coplanar. Let $\mathbf{v}$ be a vector that is not parallel to any plane $R_i R_j R_k$. If $\varepsilon > 0$ is small enough, then with the notation $R_i' = R_i + (m-1)\varepsilon\mathbf{v}$, the parallelograms $R_i R_j R_j' R_i'$ and $R_k R_l R_l' R_k'$ are disjoint unless the segments $R_i R_j$ and $R_k R_l$ have a common endpoint R_s , in which case their common points form the segment $R_s R_s'$.

Fix such an ε . Choose, for every $1 \leq i < j \leq m$, a point Q_{ij} very close to the centre of the parallelogram $R_i R_j R_j' R_i'$ such that the pyramids $R_i R_j R_j' R_i' Q_{ij}$ and $R_k R_l R_l' R_k' Q_{kl}$ are disjoint unless the segments $R_i R_j$ and $R_k R_l$ have a common endpoint R_s , in which case their common points again form the segment $R_s R_s'$.

Consider, for $1 \leq \kappa \leq m$, the points $R_i^\kappa = R_i + (\kappa-1)\varepsilon\mathbf{v}$. One can readily check that two distinct triangles $R_i^\kappa R_j^\kappa Q_{ij}$ and $R_k^\lambda R_l^\lambda Q_{kl}$ are disjoint unless either $\{i, j\} = \{k, l\}$, in which case their only common point is Q_{ij} , or $\{i, j\} \cap \{k, l\} = \{s\}$ and $\kappa = \lambda$, in which case their only common point is R_s^κ . Thus, any two of the $m \binom{m}{2}$ triangles $R_i^\kappa R_j^\kappa Q_{ij}$ ($1 \leq \kappa \leq m, 1 \leq i < j \leq m$) are indeed almost disjoint.

3. Cyclic Polytopes

Let $0 \leq t_0 < t_1 < \dots < t_{m+2}$, and consider the points $R_i = (t_i, t_i^2, t_i^3) \in \mathbb{R}^3$ on the moment curve. If $m \geq 1$, then $R_0, R_1, \dots, R_{m+2}$, and also every subsequence of length ≥ 4 form the vertex array of a cyclic polytope. It is known that no four points on the moment curve are coplanar, in particular, cyclic polytopes are simplicial polytopes. If $T_1, T_2, \dots, T_\ell$ is the vertex array of the cyclic polytope $\mathcal{C} = \text{conv}\{T_1, T_2, \dots, T_\ell\}$, then Gale's evenness condition implies that for $1 \leq i < j < k \leq \ell$, $T_i T_j T_k$ is a face of $\mathcal{C}$ if and only if either $i = 1$ and $k = j + 1$ or $k = \ell$ and $j = i + 1$. For Gale's evenness condition, and for the introduction of cyclic polytopes in general we refer to the monographs [2] and [7].

For every $i < j < k < m + 2$, let H_{ijk} denote the plane incident to points R_i, R_j, R_k , and let H_{ijk}^+ denote that closed half-space, supported by the plane H_{ijk} , which contains the point R_{m+2} . Similarly we denote by H_{ijk}^- the closed half-space, supported by H_{ijk} , that does not contain R_{m+2} . If $a, b \notin \{i, j, k\}$, then $R_a, R_b \notin H_{ijk}$, and it follows from Gale's evenness condition that H_{ijk} separates R_a from R_b if and only if an odd number of the indices i, j, k separate a and b in the usual ordering of natural numbers. We also note that none of the planes H_{ijk} is parallel to the (x, y) -plane. We say that H_{ijk} *weakly separates* sets $\mathcal{A}$ and $\mathcal{B}$ if one of the sets is contained in H_{ijk}^+ and the other is in H_{ijk}^- . For every two point $P \neq Q$, let $r(PQ)$ be the closed ray which starts at P and passes through Q , and we define ray $r^*(PQ)$ by removing the segment PQ , open at the end Q , from the ray $r(PQ)$. Thus, $r^*(PQ) = r(QR)$ for any point R incident to line PQ such that Q lies between P and R .

Lemma 3. *Suppose $t_{i+1} - t_i \geq 2(t_{i+2} - t_{i+1})$ for $i = 0, 1, \dots, m$, where $m \geq 3$. Then, for every $0 \leq r < s < t < u < v < m + 2$, the lines $R_{m+2}R_r$ and $R_{m+2}R_t$ are not parallel to the plane H_{suv} . Moreover, $r(R_{m+2}R_t) \subseteq H_{suv}^+$ and $r^*(R_{m+2}R_r) \subseteq H_{suv}^-$.*

Intuitively, this must be true if the multiplicative constant 2 is replaced by a very large number. From the analysis of the proof that follows one can readily check that it can be replaced by the somewhat smaller unique positive root of the polynomial $x^4 - 2x^3 + x - 1$.

Proof. Since H_{suv} separates R_{m+2} and R_r , the statements involving point R_r are obvious. The statements involving R_t follow from the more general fact that, given any subsequence x, a, y, z, b of $0, 1, 2, \dots, m + 2$, the plane H_{xyz} does not separate points R_a and R_b (Gale's evenness condition) and the distance δ_a of point R_a from the plane H_{xyz} is larger than the distance δ_b of R_b from H_{xyz} .

We denote by $V(y_1, \dots, y_k)$ the Vandermonde matrix

$$V(y_1, \dots, y_k) = \begin{pmatrix} 1 & y_1 & \dots & y_1^{k-1} \\ 1 & y_2 & \dots & y_2^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_k & \dots & y_k^{k-1} \end{pmatrix}.$$

To see that R_b is indeed closer to H_{xyz} than R_a we only have to show that the fraction δ_a/δ_b is greater than 1, and we do this as follows:

$$\frac{\delta_a}{\delta_b} = \frac{|\text{Det } V(t_a, t_x, t_y, t_z)|}{|\text{Det } V(t_b, t_x, t_y, t_z)|} = \frac{t_a - t_x}{t_b - t_x} \cdot \frac{t_y - t_a}{t_b - t_y} \cdot \frac{t_z - t_a}{t_b - t_z} > \frac{1}{2} \cdot 1 \cdot 3 > 1,$$

since

$$t_b - t_y = \sum_{i=y}^{b-1} (t_{i+1} - t_i) \leq \sum_{j=1}^{b-y} \frac{1}{2^j} (t_y - t_{y-1}) < t_y - t_{y-1} \leq t_y - t_a,$$

and in a similar way

$$t_b - t_x = (t_b - t_a) + (t_a - t_x) < (t_a - t_{a-1}) + (t_a - t_x) \leq 2(t_a - t_x),$$

$$t_b - t_z < t_z - t_{z-1} \leq \frac{1}{3}((t_z - t_{z-1}) + (t_{z-1} - t_{z-2})) \leq \frac{1}{3}(t_z - t_a). \quad \square$$

Now we are able to state the main result of this section.

Lemma 4. *Let $m \geq 3$, and consider points $R_i = (t_i, t_i^2, t_i^3)$ for $i = 0, 1, \dots, m+2$, where $t_i = 2 - 2^{-i}$. There is a finite system $\mathcal{H}$ of hyperplanes not parallel to the (x, y) -plane such that for every 4-tuple of indices (i, j, k, l) with $1 \leq i < j \leq m$, $1 \leq k < l \leq m$, $i < k$, there exists a hyperplane $H(ijkl) \in \mathcal{H}$ that is not parallel to lines $R_{m+2}R_i$ and $R_{m+2}R_k$, and which weakly separates $r^*(R_{m+2}R_i) \cup \{R_j\}$ from $r^*(R_{m+2}R_k) \cup \{R_l\}$.*

Proof. Note first that the numbers t_i satisfy the condition of Lemma 3. We will show that in fact the statement of the lemma is valid for the system $\mathcal{H} = \{H_{xyz} \mid 0 \leq x < y < z \leq m+1\}$. For convenience, when writing $H = H_{xyz}$ we denote the half-spaces H_{xyz}^+ and H_{xyz}^- by H^+ and H^- , respectively. We distinguish between four cases:

- (i) $j = k$. In this case let $H(ijkl) = H_{(i-1)kl}$ and apply Lemma 3 with $s = i - 1$, $t = i$, $u = j = k$, $v = l$ to conclude that $H = H(ijkl)$ is not parallel to line $R_{m+2}R_i$ and $r^*(R_{m+2}R_i) \subset r(R_{m+2}R_i) \subset H^+$. Moreover, $R_j = R_k \in H = H^+ \cap H^-$. Since $R_{m+2} \in H^+ \setminus H$, this also implies that line $R_{m+2}R_k$ is not parallel to H and $r^*(R_{m+2}R_k) \subset H^-$. We also see that $R_l \in H \subset H^-$. Putting all this information together we obtain that $H \in \mathcal{H}$ is not parallel to lines $R_{m+2}R_i$ and $R_{m+2}R_k$, $r^*(R_{m+2}R_i) \cup \{R_j\} \subset H^+$ and $r^*(R_{m+2}R_k) \cup \{R_l\} \subset H^-$.
- (ii) $j \leq k - 1$. In this case $k > i + 1$. Let $H(ijkl) = H_{(k-1)l(l+1)}$ and apply Lemma 3 with $r = i$, $s = k - 1$, $t = k$, $u = l$, $v = l + 1$. Then we immediately obtain that $H = H(ijkl)$ is not parallel to lines $R_{m+2}R_i$ and $R_{m+2}R_k$, $r^*(R_{m+2}R_i) \subset H^-$ and $r^*(R_{m+2}R_k) \subset r(R_{m+2}R_k) \subset H^+$. Moreover, if $j = k - 1$, then $R_j \in H$, and if $j < k - 1$, then H separates R_j and R_{m+2} by Gale's evenness condition; in any case $R_j \in H^-$. On the other hand, $R_l \in H \subset H^+$, and the result follows.
- (iii) $j > k > i + 1$. In this case let $H(ijkl) = H_{(i+1)j(j+1)}$ and apply Lemma 3 with $r = i$, $s = i + 1$, $t = k$, $u = j$, $v = j + 1$. As in the previous case, we obtain that $H = H(ijkl)$ is not parallel to lines $R_{m+2}R_i$ and $R_{m+2}R_k$, $r^*(R_{m+2}R_i) \subset H^-$ and $r^*(R_{m+2}R_k) \subset r(R_{m+2}R_k) \subset H^+$. Moreover, $R_j \in H \subset H^-$, and $R_l \in H^+$.

because either $l \in \{j, j+1\}$, in which case $R_l \in H$, or $k < l < m+2$, and l is separated from $m+2$ by either no or by exactly two indices in the set $\{i+1, j, j+1\}$, in which case Gale's evenness condition implies $R_l \in H^+$.

- (iv) $j > k = i+1$. In this case let $H(ijkl) = H_{ij(j+1)}$ and apply Lemma 3 with $s = i, t = k, u = j, v = j+1$ to conclude that $H = H(ijkl)$ is not parallel to line $R_{m+2}R_k$ and $r^*(R_{m+2}R_k) \subset r(R_{m+2}R_k) \subset H^+$. We can check that $R_l \in H^+$ in exactly the same way as in the previous case. On the other hand, $R_i, R_j \in H \subset H^-$, and since $R_{m+2} \in H^+ \setminus H$, we can argue that H is not parallel to line $R_{m+2}R_i$, and $r^*(R_{m+2}R_i) \subset H^-$. Thus the proof of the lemma is complete. $\square$

For technical reasons we also need the following simple lemma.

Lemma 5. *Suppose that $A = (a, a^2, a^3)$, $B = (b, b^2, b^3)$ and $C = (c, c^2, c^3)$ are points on the moment curve such that $0 < a < b < c$. Denote by A' the point where the line BC intersects the plane parallel to the (x, y) -plane and incident to A . Then the first two coordinates of the vector $\overrightarrow{AA'}$ are positive.*

Proof. One can readily check that $c^2 + cb + b^2 > b^2 + ba + a^2$ and $(b+a)(c^2 + cb + b^2) > (c+b)(b^2 + ba + a^2)$. Consequently,

$$b - a > (c - b) \frac{b^3 - a^3}{c^3 - b^3} \quad \text{and} \quad b^2 - a^2 > (c^2 - b^2) \frac{b^3 - a^3}{c^3 - b^3},$$

and the result follows. $\square$

4. The Second Construction

Let points $R_0, R_1, \dots, R_{m+2}$ be as in Lemma 4. The geometric picture we have in mind is that if we pick the points Q_{ij} for $1 \leq i < j \leq m$ along the rays $r_i = r^*(R_{m+2}R_i)$, then two triangles $R_iR_jQ_{ij}$ and $R_kR_lQ_{kl}$ cannot cross each other, assuming that the distances between consecutive points R_i and R_{i+1} decrease considerably. This is indeed justified by Lemma 4. To be more precise, we can prove the following:

Corollary 6. *Let $\varphi > 0$ be a sufficiently small angle and denote by $\mathcal{C}_i$ the half-cone whose axis is r_i and whose apex angle is 2φ . Suppose that points Q_{ij} are interior points of $\mathcal{C}_i$ for $1 \leq i < j \leq m$. Then for every 4-tuple of indices (i, j, k, l) with $1 \leq i < j \leq m$, $1 \leq k < l \leq m$, $i < k$, triangles $R_iR_jQ_{ij}$ and $R_kR_lQ_{kl}$ are almost disjoint.*

Proof. It follows from Lemma 4 that $H(ijkl)$ weakly separates the two triangles. Moreover, points Q_{ij} and Q_{kl} are not incident to $H(ijkl)$. Therefore the only common points of the two triangles belong to the intersection of segments R_iR_j and R_kR_l . Since they are also weakly separated by $H(ijkl)$ and no four points of the moment curve are coplanar, we only have two possibilities. First, if points R_i, R_j, R_k, R_l are all distinct, then were the intersection of the two segments non-empty, it would consist of one of these points,

which is not possible since no three points of the moment curve are collinear; thus, in this case the two triangles are disjoint. In the other case, R_j equals one of the endpoints of $R_k R_l$, when the intersection of the two segments, and hence also of the two triangles, is $\{R_j\}$, proving that the two triangles are indeed almost disjoint. $\square$

Note that this corollary leaves the case $i = k$ uncovered. Thus, we have to investigate how to position points Q_{ij} in $\mathcal{C}_i$ (where $1 \leq i \leq m-1$ is fixed) such that each pair of the triangles $\{R_i R_j Q_{ij} \mid i < j \leq m\}$ is also almost disjoint. To do this, consider for every $i < j \leq m$, the half-plane G_{ij} bounded by line $\ell_i = R_{m+2} R_i$ and which contains ray $r(R_i R_j)$. Since no three points on the moment curve are collinear, these half-planes are uniquely defined, and they are also pairwise distinct, since no four points on the moment curve are coplanar. Moreover, if g_{ij} denotes the ray where G_{ij} intersects the horizontal plane H_0 through R_{m+2} , then it follows from Lemma 5 that g_{ij} lies inside the positive quadrant of H_0 . Assume that $Q_{ij} \in G_{ij} \setminus \ell_i$, then triangles $R_i R_j Q_{ij}$ and $R_i R_k Q_{ik}$ lie in the different half-planes G_{ij} and G_{ik} , respectively. Therefore any point common to both triangles must lie on the intersection of these two planes, that is, on ℓ_i . However, ℓ_i contains only one vertex of each triangle, namely R_i , and it follows that the triangles are almost disjoint.

In order to be able to blow up the construction, we must have more freedom in selecting the position of Q_{ij} . To this end choose, for every $1 \leq i \leq m-1$, some small but positive angle γ_i and rotate G_{ij} through the angle γ_i in both directions about ℓ_i to obtain half-planes G_{ij}^l and G_{ij}^r that are also bounded by line ℓ_i and are such that their intersections with the plane H_0 still lie in the first quadrant. If γ_i is small enough, then any two of the wedges $\mathcal{W}_{ij}$ ($i < j \leq m$) bounded by the half-planes G_{ij}^l and G_{ij}^r have only ℓ_i in common. Consequently, if $Q_{ij} \in \mathcal{W}_{ij} \setminus \ell_i$, then triangle $R_i R_j Q_{ij}$ lies in $\mathcal{W}_{ij}$ and its only common point with ℓ_i is its vertex R_i . So in this case we can argue as before that these triangles are almost disjoint. Thus we can summarize the results we have so far as

Lemma 7. *Assume that for every $1 \leq i < j \leq m$, Q_{ij} is an interior point of $\mathcal{C}_i \cap \mathcal{W}_{ij}$. Then any two of the $\binom{m}{2}$ triangles $R_i R_j Q_{ij}$ are almost disjoint.*

Now we sketch how to blow up the construction. Here we omit the technical details which are rather tedious; the interested reader can find them in [4]. Choose a “long” horizontal vector $\mathbf{v}$ which is in general position with respect to the points R_i in an appropriate sense. Let $\mathcal{R} = \{R_1, R_2, \dots, R_m\}$ and define $R_i^\kappa = R_i + (\kappa - 1)\mathbf{v}$ and $\mathcal{R}^\kappa = \mathcal{R} + (\kappa - 1)\mathbf{v}$ for $1 \leq \kappa \leq m$. Consider also the regions

$$\mathcal{Q}_{ij} = \bigcap_{\kappa=1}^m ((\mathcal{C}_i \cap \mathcal{W}_{ij}) + (\kappa - 1)\mathbf{v}).$$

It can be shown that $\mathcal{Q}_{ij}$ is an infinite domain which contains points of arbitrarily small third coordinate, that is, points arbitrarily far from H_0 . It follows from Lemma 7 that if $Q_{ij} \in \mathcal{Q}_{ij}$ for every $1 \leq i < j \leq m$, then for every fixed $1 \leq \kappa \leq m$, the triangles $R_i^\kappa R_j^\kappa Q_{ij}$ are almost disjoint. If the vector $\mathbf{v}$ is long enough, then for every $1 \leq i < j \leq m$ and $1 \leq \kappa < \lambda \leq m$ there is a plane through Q_{ij} that separates $\mathcal{R}^\kappa$ from $\mathcal{R}^\lambda$, and therefore the triangles $R_i^\kappa R_j^\kappa Q_{ij}$ and $R_i^\lambda R_j^\lambda Q_{ij}$ are almost disjoint. Moreover (this is the most

technical part to show) one can position the points Q_{ij} in Q_{ij} such that if $\{i, j\} \neq \{k, l\}$ and $\kappa \neq \lambda$, then there is a plane through Q_{ij} and Q_{kl} that separates $\mathcal{R}^\kappa$ from $\mathcal{R}^\lambda$, and therefore the triangles $R_i^\kappa R_j^\kappa Q_{ij}$ and $R_k^\lambda R_l^\lambda Q_{kl}$ are also almost disjoint. Once again, the heavy details can be found in [4].

References

1. T.K. Dey and J. Pach, Extremal problems for geometric hypergraphs, *Discrete Comput. Geom.* **19** (1998), 473–484.
2. B. Grünbaum, *Convex Polytopes*, Interscience, London, 1967.
3. G. Kalai, Personal communication (1995).
4. Gy. Károlyi, Almost disjoint triangles in 3-space, Technical Report, Zurich, 2001. See <http://www.ifor.math.ethz.ch/pub/tr>.
5. Gy. Károlyi and E. Welzl, Crossing-free segments and triangles in point configurations, *Discrete Applied Math.* **115** (2001), 77–88.
6. P. McMullen, Ch. Schulz, and J.M. Wills, Polyhedral 2-manifolds in E^3 with unusually large genus, *Israel J. Math.* **46** (1983), 127–144.
7. G.M. Ziegler, *Lectures on Polytopes*, GTM 152, Springer-Verlag, New York, 1995.

Received November 20, 2001, and in revised form February 19, 2002. Online publication October 29, 2002.