Quotients of Some Finite Universal Locally Projective Polytopes

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Abstract. This paper classifies the quotients of a finite and locally projective polytope of type $\{4, 3, 5\}$. Seventy quotients are found, including three regular polytopes, and nine other section regular polytopes which are themselves locally projective. The classification is done with the assistance of GAP, a computer system for algebraic computation. The same techniques are also applied to two finite locally projective polytopes respectively of type $\{3, 5, 3\}$ and $\{5, 3, 5\}$. No nontrivial quotients of the latter are found.

1. Introduction

In [5] and [6] it was shown how abstract polytopes may be classified in terms of string C-groups W , and certain conjugacy classes of subgroups of those C-groups. In this paper the results of those two papers are applied to some particular string C-groups, and some properties of the polytopes discovered are given. Most of the work was done using GAP version 3 release 4.4 [10].

Understanding this paper will require some basic knowledge of abstract polytopes and C-groups, and of the results of [5] and [6]. To avoid unnecessary repetition, the reader is recommended to read at least the Introduction to those papers before continuing. An up-to-date general reference on abstract regular polytopes is [9]. The key result that will be used here is Theorem 5.5 of [6], which reads as follows.

Theorem 1.1. *There is a one-to-one correspondence between polytopes acted on by a string C-group W via the flag action, and conjugacy classes of subgroups N of W satisfying*

- (i) $w^{-1}Nw \cap s_i H_i$ is empty for all $w \in W$ and for all i ,
- (ii) $W_{>i} \cap w^{-1}Nw W_{<j} \subseteq w^{-1}Nw(W_{>i} \cap W_{<j})H_k$ for all $w \in W$ and for $i < k < j$, and
- (iii) $W_{>i} \cap NW_{<i+1} \subseteq N$ for all i .

Here, a *C-group* is a group W generated by a set S of involutions which satisfies $\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle$ for all $I, J \subseteq S$. If $S = \{s_0, \dots, s_{n-1}\}$, and if $(s_i s_j)^2 = 1$ whenever $i \neq j, j \pm 1$, then W is a *string C-group*. Then $H_i = \langle s_j : j \neq i \rangle$, $W_{>i} = \langle s_j : j > i \rangle$ and $W_{<i} = \langle s_j : j < i \rangle$. Finally, the *flag action* of W on the flags of a polytope $\mathcal{P}$ is defined for any flag $\Phi = \{\Phi_{-1}, \Phi_0, \dots, \Phi_{n-1}, \Phi_n\}$ via Φ^{s_i} is the unique flag $\Phi' = \{\Phi'_{-1}, \Phi'_0, \dots, \Phi'_{n-1}, \Phi'_n\}$ of $\mathcal{P}$ such that $\Phi_j = \Phi'_j$ if and only if $i \neq j$ —at least, if this can be extended to a well-defined group action.

Subgroups N satisfying the conditions of the theorem are called *semispars* subgroups of W .

2. The String C-Group

In this section the string C-group under examination is introduced.

Let $S = \{s_0, s_1, s_2, s_3\}$, and let $W = \langle S \rangle$ satisfy all the relations of the Coxeter group $[4, 3, 5]$, as well as the additional relator $(s_1 s_2 s_3)^5$. Let $v_1 = s_0 s_1 s_2 s_3 s_2 s_1$, $v_2 = s_1 v_1 s_1$ and $v_3 = s_2 v_2 s_2$. Let $H_3 = \langle s_0, s_1, s_2 \rangle$, and $V = \langle v_1, v_2, v_3 \rangle$. It may be shown that conjugation by s_0, s_1 or s_2 permutes the set $\{v_1, v_2, v_3, v_1^{-1}, v_2^{-1}, v_3^{-1}\}$ (see later).

Lemma 2.1. $W = VH_3$.

Proof. Let $w \in W$. Then w may be written $w = h_1 s_3 h_2 s_3 \cdots h_k s_3 h_{k+1}$, where each h_i is an element of H_3 . Since $s_3 = s_2 s_1 s_0 v_1 s_1 s_2$, we may in turn write $w = h'_1 v_1 h'_2 v_1 \cdots h'_k v_1 h'_{k+1}$ where again, each h'_i is an element of H_3 . Any conjugate of v_1 by an element of H_3 is either a generator of V or its inverse, so we may write $w = v_{i_1}^{\varepsilon_1} v_{i_2}^{\varepsilon_2} \cdots v_{i_k}^{\varepsilon_k} h'_1 h'_2 \cdots h'_k h'_{k+1}$, where $h'_1 \cdots h'_j v_1 = v_{i_j}^{\varepsilon_j} h'_1 \cdots h'_j$. Thus $w \in VH_3$ as required. $\square$

The reason for writing W in this form will become clear later. Note that it is not claimed here that W is a semidirect product of V and H_3 .

For $I \subseteq S$, let $W_I = \langle I \rangle$. As mentioned earlier, a string C-group is a group which is generated by involutions, whose Coxeter diagram is a string (that is, for which $(s_i s_j)^2 = 1$ whenever $i \neq j, j \pm 1$), and which satisfies the so-called “intersection property”, that $W_I \cap W_J = W_{I \cap J}$ for all $I, J \subseteq S$. String C-groups are important, because there is a one-to-one correspondence between string C-groups and regular polytopes, and because of their use in classifying (nonregular) polytopes.

Theorem 2.2. *The group W satisfies the intersection property, and is therefore a string C-group.*

Proof. Corollary 5.3 of [8] may be applied (by letting the $\mathcal{K}$ of that corollary be the hemi-icosahedron) to show that there exists a (universal) regular polytope whose facets are cubes and whose vertex figures are hemi-icosahedra. We may deduce that the group of this polytope must be a quotient $W' = W/N$ of W , since it must satisfy at least the

relations listed for W . We may also deduce that the natural homomorphism from W to W' is a monomorphism from the W_I to the corresponding $W'_{I'}$, for proper subsets I of S , and therefore W also satisfies the intersection property. $\square$

It follows that W' in fact equals W , since W' is the group of the universal polytope whose facets are cubes and whose vertex figures are hemi-icosahedra, and W is the group of a polytope with the same facets and vertex figures—the facets have automorphism group $\langle s_0, s_1, s_2 \rangle$, and the vertex figures have automorphism group $\langle s_1, s_2, s_3 \rangle$. This regular polytope is therefore locally projective. We then have the following.

Theorem 2.3. *Any polytope whose facets are cubes and whose vertex figures are hemi-icosahedra is a quotient $\mathcal{M}(W)/N$ of $\mathcal{M}(W)$ by some semispase subgroup N of W .*

Proof. This follows from the above considerations, and from Theorem 1 of [11], which shows that all such polytopes will be quotients of a “universal” regular such polytope. Alternatively, the proof could be made using Theorem 5.2 of [5], by pointing out that the group W must act on the flags of such a polytope via the flag action. $\square$

Therefore, to find such polytopes, it is enough to find conjugacy classes of subgroups of W satisfying the conditions mentioned in the theorem at the end of Section 1. Before this is done, some further properties of W will be noted.

Corollary 5.3 of [8] gives the structure of the group W . Firstly, since the hemi-icosahedron is neighbourly (every pair of vertices is joined by an edge), the group and polytope are finite. The group will have structure $C_2^6.H_0 \cong C_2^6.A_5$, that is, a wreath product of C_2 with the six-point permutation representation of A_5 .

GAP version 4 release 2 [3] found a permutation representation for W on 12 points, given by

$$\begin{aligned} s_0 &\mapsto (6, -6), \\ s_1 &\mapsto (1, 2)(5, 6)(-6, -5)(-2, -1), \\ s_2 &\mapsto (2, 3)(4, 5)(-5, -4)(-3, -2) \quad \text{and} \\ s_3 &\mapsto (1, 2)(3, 4)(-4, -3)(-2, -1). \end{aligned}$$

It is easily verified then that $v_1 \mapsto (1, 6, -1, -6)(2, 3)(-2, -3)$ and therefore has order 4. Likewise v_2 and v_3 have order 4 in W .

From the permutation representation, it may be discovered that W acts primitively on the blocks $\{i, -i\}$, for $i = 1, \dots, 6$. Each block defines a generator $(i, -i)$ of the C_2^6 in the expression $W = C_2^6.A_5$.

The elements of the group C_2^6 may be represented as the vertices $(0, 0, 0, 0, 0, 0)$, $(0, 0, 0, 0, 0, 1)$ and so on of a 6-cube. If we denote the natural basis of R^6 by $\{e_1, e_2, \dots, e_6\}$, the facets of the cube (equivalently, the vertices of its dual, the cross polytope) may be given by linear equations $e_1 = 0$, $e_1 = 1$, $e_2 = 0$ and so forth. The action of A_5 on the 6-cube defines naturally an action on these 12 linear equations, which are in

Table 1. Conjugacy classes of elements of W which may be elements of semispars subgroups.

Size	Representative
1	Identity
15	$(5, -5)(6, -6) = (s_0 s_1)^2$
10	$(4, -4)(5, -5)(6, -6) = (s_0 s_1 s_2)^3$
10	$(3, -3)(4, -4)(6, -6) = v_1 v_2^2 v_1^{-1} (s_0 s_1 s_2)^3$
15	$(3, -3)(4, -4)(5, -5)(6, -6) = v_1 v_2^2 v_1^{-1} s_2 (s_0 s_1)^2 s_2$
6	$(2, -2)(3, -3)(4, -4)(5, -5)(6, -6) = v_2 v_1^2 v_2 s_2 s_1 s_0 s_1 s_2$
1	$(1, -1)(2, -2)(3, -3)(4, -4)(5, -5)(6, -6) = (v_1 v_2)^2 v_1 s_1 s_2 s_1 s_0 s_1 s_2$
120	$(2, 3, -2, -3)(4, 5, -4, -5)(6, -6) = v_1 v_2^2 v_1^{-1} s_0 s_2$
60	$(1, 2, -1, -2)(3, 4, -3, -4)(5, -5)(6, -6) = v_2^2 v_3 s_0$
60	$(1, 2)(3, -3)(4, -4)(5, 6, -5, -6)(-2, -1) = v_1 v_2^2 v_1^{-1} s_0 s_1 s_2 s_1 s_0 s_1 s_2 s_0$

one-to-one correspondence with the 12 points of the permutation representation given above.

Therefore, the above permutation representation may be obtained by considering the natural action of the hemi-icosahedral group on the vertices of the six-dimensional cross polytope. This fact was guessed at by the anonymous referee.

Elements of semispars subgroups must not be conjugates of any element of $s_i H_i$. Of the 3840 elements of W , only 810 satisfy that condition. All these have order either 1, 2, 4, 6 or 10. Evidently, if a semispars subgroup contains an element of order 6 or 10, it must also contain one of order 3 or 5, hence elements of order 6 or 10 cannot be elements of semispars subgroups of W . Eliminating the order 6 and 10 elements leaves a set Σ of only 298 elements of W which may be elements of semispars subgroups. The 57 order 2 elements are all elements of $\langle(i, -i), i = 1, \dots, 6\rangle$ as are the squares of the 240 order 4 elements.

The elements of Σ fall into a number of conjugacy classes as listed in Table 2. From this, it would be possible to deduce, manually, conjugacy classes of subgroups satisfying condition one for the group to be semispars. The groups derived could be checked against the other two conditions, and so the quotients classified. However instructive this approach would be, it is easier simply to ask a computer to find *all* the conjugacy classes of subgroups of W , and then ask it to check the conditions directly itself. In the end, this easier approach was chosen (see Section 3 for more details).

The polytope $\mathcal{M}$ itself may be considered as a “tiling” of some unusual “space” by cubes. The action of v_1 on $\mathcal{M}$ moves a flag from one cube to the corresponding flag of an adjacent cube. The actions of v_2 and v_3 are similar, but in orthogonal directions. It is for this reason it is useful to write $W = VH_3$ as in Lemma 2.1. In this “space” there are five cubes around each edge, and ten around each vertex. Furthermore, the action of v_i repeated four times successively brings one back to the original cube one started from (and with the same orientation). The (anonymous) referee pointed out that this “space” may be realized as a subcomplex of the boundary complex of the 6-cube (see Section 8D of [9]).

Since $|W| = 3840$, it follows that $\mathcal{M}$ has 3840 flags. Each (cubic) facet of $\mathcal{M}$ is a member of 48 of those flags, so that there are a total of 80 cubes in the polytope. They are

Table 2. Elements of V which canonically represent the cubes of $\mathcal{M}$. ($\chi \in \{0, \pm 1\}$.)

v_3^χ
$v_1^{\pm 1} v_3^\chi, v_2^{\pm 1} v_3^\chi$
$v_1 v_2^{\pm 1} v_3^\chi, v_1^{-1} v_2, v_2 v_1^{\pm 1} v_3^\chi, v_2^{-1} v_1^{\pm 1}$
$v_1^2 v_3^\chi, v_2^2 v_3^\chi$
$v_1 v_2 v_1 v_3^\chi, v_1 v_2^{-1} v_1, v_1^{-1} v_2 v_1^{-1} v_3^\chi, v_1^{-1} v_2^{-1} v_1^{-1}, v_2 v_1 v_2 v_3^\chi, v_2 v_1^{-1} v_2, v_2^{-1} v_1 v_2^{-1} v_3^\chi, v_2^{-1} v_1^{-1} v_2^{-1}$
$v_1^2 v_2^{\pm 1}, v_2^2 v_1^{\pm 1}$
$v_1^{\pm 1} v_2^2 v_3^\chi, v_2^{\pm 1} v_1^2 v_3^\chi$
$v_1^2 v_2^2 v_3^\chi, (v_1 v_2^{\pm 1})^2, v_1 v_2^2 v_1^{\pm 1}, v_2 v_1^2 v_2, (v_2 v_1^{\pm 1})^2$
$v_1 v_2 v_1 v_2 v_1$

listed in Table 2. That is, Table 2 gives a canonical list of elements of $V = \langle v_1, v_2, v_3 \rangle$ which map some flag in some “base cube” to the other cubes of the polytope.

The list of elements in Table 2 may be expressed in other ways using the following relations: $s_0 v_1 s_0 = v_1^{-1}, s_1 v_1 s_1 = v_2, s_2 v_1 s_2 = v_1, s_0 v_2 s_0 = v_2, s_1 v_2 s_1 = v_1, s_2 v_2 s_2 = v_3, s_0 v_3 s_0 = v_3, s_1 v_3 s_1 = v_3, s_2 v_3 s_2 = v_2, v_3 v_1 = v_2 v_1 v_3 s_1 s_2 s_1 s_0 s_1, v_3 v_2 = v_1 v_2 v_3 s_2 s_1 s_0, v_3^2 = v_1 v_2^2 v_1^{-1} (s_1 s_0 s_1 s_2)^2$ and $(v_1 s_0 s_1)^5 = 1$.

3. The Quotient Polytopes

GAP version 3 [10] was used to find the conjugacy classes of subgroups of W , then to discover which conjugacy classes yield quotient polytopes, and finally to analyze the polytopes discovered.

W itself has 963 conjugacy classes of subgroups. Of these, only 77 satisfy the first condition for the groups to be semisparse, and of these, all but seven satisfy the second and third conditions. Therefore, the universal polytope discussed earlier has 70 quotient polytopes. Of these, three are regular, namely $\mathcal{M}(W)$ itself, a regular polytope of type $\{2, 3, 5\}$ (which has exactly two vertices, and whose vertex figures are hemi-icosahedra) and one other, a quotient of $\mathcal{M}(W)$ by a normal subgroup N of W of order 2, generated by $v_1 v_2 v_1 v_2 v_1 s_1 s_2 s_1 s_0 s_1 s_2$. The latter is the element of W of maximum length in the generators s_i , and is $(1, -1)(2, -2)(3, -3)(4, -4)(5, -5)(6, -6)$ in the permutation representation given in Section 2.

The three polytopes which are regular are of course section regular also (Theorem 4.2 of [6]). In fact, of the 70 quotient polytopes, there are 11 which have only cubes for facets, and are therefore locally projective section regular polytopes of type $\{4, 3, 5\}$. Some information about these is shown in Table 3. The numbering of these polytopes is in line with the order in which they were found by GAP. The group $\Gamma(\mathcal{Q})$ is here defined as the minimal string C-group which acts on the flags of the polytope via the flag action. This indicates that the only section regular quotient of $\{4, \{3, 5\}_5\}$ (of type $\{4, 3, 5\}$) is $\{4, \{3, 5\}_5\}$ itself. Only for regular polytopes is $|\text{Aut}(\mathcal{Q})|$ equal to $|\Gamma(\mathcal{Q})|$. For all these section regular polytopes, the numbers of vertices, edges, 2-faces and facets are respectively $64/|N|$, $192/|N|$, $240/|N|$ and $80/|N|$, where N is the subgroup of W used to obtain the quotient.

Table 3. Section regular (and locally projective) quotients.

	$ N $	Generators	$ \text{Aut}(\mathcal{Q}) $	$ \Gamma(\mathcal{Q}) $
$\{4, \{3, 5\}_5\}.\text{I}$	1	None	3840	3840
$\{4, \{3, 5\}_5\}.\text{II}$	2	$(v_1 v_2)^2 v_1 s_1 s_2 s_1 s_0 s_1 s_2$	1920	1920
$\{4, \{3, 5\}_5\}.\text{III}$	2	$v_2 v_1^2 v_2 s_2 s_1 s_0 s_1 s_2$	320	3840
$\{4, \{3, 5\}_5\}.\text{V}$	2	$v_3^2 s_0$	192	3840
$\{4, \{3, 5\}_5\}.\text{VII}$	2	$v_3^2 (s_0 s_1)^2$	128	3840
$\{4, \{3, 5\}_5\}.\text{VIII}$	4	$v_3^2 (s_0 s_1)^2, v_1^2 v_2^2$	192	3840
$\{4, \{3, 5\}_5\}.\text{X}$	4	$v_3^2 (s_0 s_1)^2, v_1^2 v_2^2 (s_1 s_0 s_1 s_2)^2$	96	3840
$\{4, \{3, 5\}_5\}.\text{XXIII}$	4	$v_3^2 (s_0 s_1)^2, (v_1 v_3^{-1})^2 s_0 s_1 s_2 s_1 s_0 s_1$	32	3840
$\{4, \{3, 5\}_5\}.\text{XXV}$	4	$v_3^2 s_0, v_2^2 s_0$	32	3840
$\{4, \{3, 5\}_5\}.\text{XXIX}$	4	$v_3^2 s_0, v_1^2 v_2^2$	32	3840
$\{4, \{3, 5\}_5\}.\text{XXXIII}$	8	$v_3^2 s_0, v_2^2 s_0, v_1^2 (s_1 s_0 s_1 s_2)^2$	96	3840

There are a further eight which are of type $\{4, 3, 5\}$, but which have a mixture of cubes and halfcubes for facets, and are therefore not section regular. These are shown in Table 4. Each group has the generator $(s_0 s_1 s_2)^3$. This is what ensures the existence of halfcubes amongst the facets of the polytope. (See Section 2 of [7] for a detailed analysis of the quotients of the cube.) Note once again that only one of these polytopes is a quotient of $\{4, \{3, 5\}_5\}.\text{II}$. The last column of Table 4 shows the number of cubes and halfcubes amongst the facets of the polytope. An entry of the form $C^c H^h$ indicates that there are h halfcubes and c cubes. The total number of facets, 2-faces, edges and vertices may be calculated respectively as $[c + h]$, $\frac{1}{2}[6c + 3h]$, $\frac{1}{5}[12c + 6h]$ and $\frac{1}{5}[4c + 2h]$.

Tables 5 and 6 show the remaining quotients of the original regular polytope $\{4, \{3, 5\}_5\}$. The two tables respectively show those polytopes which do not, and which do, have some “degenerate” facets of type $\{2, 3\}$. Again, the generators of the polytopes are given, and the types of the facets is also given. An entry of the form $C^c H^h P^p D^d$ indicates that the polytope has, as its facets, c cubes, h hemicubes, p “digonal prisms” and d degenerate polytopes. (A digonal prism is a prism whose base is a digon. The degenerate polytope here has two vertices linked by three edges and three digons—somewhat like a banana.) Again, the reader is referred to [7] for a classification of quotients of the cube.

Table 4. Quotients of type $\{4, 3, 5\}$.

	$ N $	Generators other than $(s_0 s_1 s_2)^3$	$ \text{Aut}(\mathcal{Q}) $	$ \Gamma(\mathcal{Q}) $	Facets
$\{4, \{3, 5\}_5\}.\text{IV}$	2	None	192	3840	$C^{36} H^8$
$\{4, \{3, 5\}_5\}.\text{XI}$	4	$(v_1 v_2)^2 v_1 s_1 s_2 s_1 s_0 s_1 s_2$	96	1920	$C^{18} H^4$
$\{4, \{3, 5\}_5\}.\text{XXIV}$	4	$v_2 v_1^2 v_2 s_0$	32	3840	$C^{16} H^8$
$\{4, \{3, 5\}_5\}.\text{XXVII}$	4	$(v_1 v_2)^2 v_1 s_0 s_1$	32	3840	$C^{18} H^4$
$\{4, \{3, 5\}_5\}.\text{XXVIII}$	4	$v_2 v_1^2 v_2 s_1 s_0 s_1$	32	3840	$C^{18} H^4$
$\{4, \{3, 5\}_5\}.\text{XXXII}$	8	$v_2 v_1^2 v_2 s_0, v_1^2 v_3^2 (s_1 s_0 s_1 s_2)^2$	96	3840	$C^6 H^8$
$\{4, \{3, 5\}_5\}.\text{XXXV}$	8	$v_2 v_1^2 v_2 s_1 s_0 s_1, v_1^2 v_3^2 (s_1 s_0 s_1 s_2)^2$	48	3840	$C^7 H^6$
$\{4, \{3, 5\}_5\}.\text{XXXIX}$	8	$v_2 v_1^2 v_2 s_1 s_0 s_1, v_1^2 v_3^2 (s_0 s_1 s_2)^3$	48	3840	$C^9 H^2$

Table 5. Quotients with prismoid facets.

	Generators other than $(s_0s_1)^2$	Facets
$\{4, \{3, 5\}_5\}$. VI	None	$C^{32}P^{16}$
$\{4, \{3, 5\}_5\}$. IX	$v_2^2(s_0s_1)^2$	C^8P^{24}
$\{4, \{3, 5\}_5\}$. XIII	$v_1^2v_2^2s_1s_0s_1$	$C^{12}H^8P^8$
$\{4, \{3, 5\}_5\}$. XIV	$v_3^2s_0$	$C^{16}P^8$
$\{4, \{3, 5\}_5\}$. XV	$(v_1v_2)^2v_1s_0s_1s_2(s_0s_1)^2s_2$	$C^{16}P^8$
$\{4, \{3, 5\}_5\}$. XVI	$(v_1v_2)^2v_1s_0s_1s_2s_1s_0s_1s_2$	$C^{16}P^8$
$\{4, \{3, 5\}_5\}$. XVII	v_3^2	$C^{12}P^{16}$
$\{4, \{3, 5\}_5\}$. XVIII	$v_2v_1^2v_2(s_1s_0s_1s_2)^2$	$C^{16}P^8$
$\{4, \{3, 5\}_5\}$. XIX	$v_2v_1^2v_2(s_0s_1s_2)^3$	$C^{14}H^4P^8$
$\{4, \{3, 5\}_5\}$. XX	$v_2^2(s_1s_0s_1s_2)^2$	$C^{12}P^{16}$
$\{4, \{3, 5\}_5\}$. XXI	$v_1^2v_2^2(s_0s_1s_2)^3$	$C^{16}P^8$
$\{4, \{3, 5\}_5\}$. XXII	$v_1^2v_2^2(s_1s_0s_1s_2)^2$	$C^{16}P^8$
$\{4, \{3, 5\}_5\}$. XXVI	$v_3^2s_0s_1$	$C^{16}P^8$
$\{4, \{3, 5\}_5\}$. XXX	$v_2^2(s_0s_1s_2)^3$	$C^{14}H^4P^8$
$\{4, \{3, 5\}_5\}$. XXXI	$v_3^2, v_1^2v_2^2(s_0s_1)^2$	C^4P^{12}
$\{4, \{3, 5\}_5\}$. XXXVI	$v_2^2(s_1s_0s_1s_2)^2, v_1^2v_3^2s_2(s_0s_1)^2s_2$	C^4P^{12}
$\{4, \{3, 5\}_5\}$. XXXVII	$v_2^2(s_0s_1)^2, v_1^2v_3^2$	C^4P^{12}
$\{4, \{3, 5\}_5\}$. XXXVIII	$v_2^2(s_0s_1)^2, v_1^2v_3^2s_0$	$C^3H^2P^{12}$
$\{4, \{3, 5\}_5\}$. XLII	$v_3^2, v_1^2v_2^2(s_1s_0s_1s_2)^2$	C^6P^8
$\{4, \{3, 5\}_5\}$. XLIII	$v_3^2s_0, v_1^2v_2^2s_1s_0s_1$	$C^6H^4P^4$
$\{4, \{3, 5\}_5\}$. XLIV	$v_3^2, v_1^2v_2^2(s_0s_1s_2)^3$	C^6P^8
$\{4, \{3, 5\}_5\}$. XLV	$v_3^2, (v_1v_3^{-1})^2s_0s_1s_2s_1s_0s_1$	C^6P^8
$\{4, \{3, 5\}_5\}$. XLVI	$v_3^2, (v_1v_3^{-1})^2s_0s_1s_2$	C^6P^8
$\{4, \{3, 5\}_5\}$. XLVII	$v_3^2, v_1^2v_2^2s_1s_0s_1$	$C^4H^4P^8$
$\{4, \{3, 5\}_5\}$. XLVIII	$v_2^2(s_0s_1s_2)^3, v_1^2v_3^2s_2s_1s_0s_1s_2$	$C^6H^4P^4$
$\{4, \{3, 5\}_5\}$. XLIX	$v_3^2s_0, v_1^2v_2^2(s_0s_1s_2)^3$	C^8P^4
$\{4, \{3, 5\}_5\}$. L	$v_3^2, v_2^2(s_0s_1s_2)^3$	$C^4H^4P^8$
$\{4, \{3, 5\}_5\}$. LIII	$v_2^2(s_1s_0s_1s_2)^2, v_1^2v_3^2$	C^6P^8
$\{4, \{3, 5\}_5\}$. LIV	$v_3^2s_0, v_2^2(s_0s_1)^2$	$C^3H^2P^{12}$
$\{4, \{3, 5\}_5\}$. LV	$v_3^2s_0, v_2^2(s_1s_0s_1s_2)^2$	$C^5H^2P^8$
$\{4, \{3, 5\}_5\}$. LVI	$v_3^2, v_2^2(s_0s_1)^2$	C^2P^{16}
$\{4, \{3, 5\}_5\}$. LVII	$v_2^2(s_1s_0s_1s_2)^2, v_1^2s_2s_1s_0s_1s_2$	$C^3H^6P^8$
$\{4, \{3, 5\}_5\}$. LVIII	$v_2v_1^2v_2(s_0s_1s_2)^3, v_1^2v_3^2$	$C^5H^6P^4$
$\{4, \{3, 5\}_5\}$. LIX	$v_2^2(s_0s_1s_2)^3, v_1^2v_3^2$	$C^7H^2P^4$
$\{4, \{3, 5\}_5\}$. LX	$v_2^2(s_0s_1s_2)^3, v_1^2v_3^2s_0$	$C^6H^4P^4$
$\{4, \{3, 5\}_5\}$. LXI	$v_2^2(s_1s_0s_1s_2)^2, v_1^2v_3^2s_2s_1s_0s_1s_2$	$C^5H^2P^8$
$\{4, \{3, 5\}_5\}$. LXII	$v_3^2, v_2^2(s_0s_1s_2)^3, v_1^2s_2s_1s_0s_1s_2$	H^4P^6
$\{4, \{3, 5\}_5\}$. LXIV	$v_3^2s_0, v_2^2(s_1s_0s_1s_2)^2, v_1^2s_2s_1s_0s_1s_2$	H^4P^6
$\{4, \{3, 5\}_5\}$. LXVIII	$v_3^2, v_2^2(s_0s_1)^2, v_1^2s_2s_1s_0s_1s_2$	H^2P^8

The numbers of facets, 2-faces, edges and vertices of each polytope may be calculated respectively as $[c + h + p + d]$, $\frac{1}{2}[6c + 3h + 4p + 3d]$, $\frac{1}{3}[12c + 6h + 6p + 3d]$ and $\frac{1}{5}[4c + 2h + 2p + d]$.

Of all the quotients of $\{4, \{3, 5\}_5\}$, only **II** itself, **XI**, **XV**, **XXXI**, **XXXIV**, **XXXVI**, **XXXVII**, **XLIII**, **XLVIII**, **LXII**, **LXIV**, **LXVI** and **LXX** may be regarded as quotients

Table 6. Quotients with degenerate facets.

	Generators other than $(s_0s_1)^2$ and $s_2(s_0s_1)^2s_2$	Facets
$\{4, \{3, 5\}_5\}.\mathbf{XII}$	None	$C^{12}P^{12}D^8$
$\{4, \{3, 5\}_5\}.\mathbf{XXXIV}$	$(v_1v_2)^2v_1s_0s_1$	$C^6P^6D^4$
$\{4, \{3, 5\}_5\}.\mathbf{XL}$	$(v_1v_2)^2v_1s_0s_1s_0$	$C^6P^6D^4$
$\{4, \{3, 5\}_5\}.\mathbf{XLI}$	$v_3^2s_2(s_0s_1)^2s_2$	$C^2P^{12}D^8$
$\{4, \{3, 5\}_5\}.\mathbf{LI}$	$v_2v_1^2v_2(s_0s_1)^2$	$C^4P^{10}D^4$
$\{4, \{3, 5\}_5\}.\mathbf{LII}$	$v_2v_1^2v_2s_1s_0s_1$	$C^4H^4P^6D^4$
$\{4, \{3, 5\}_5\}.\mathbf{LXIII}$	$v_3^2s_2(s_0s_1)^2s_2, v_2^2(s_0s_1)^2$	P^5D^{10}
$\{4, \{3, 5\}_5\}.\mathbf{LXV}$	$v_2v_1^2v_2(s_0s_1)^2, v_1^2v_3^2s_2(s_0s_1)^2s_2$	P^9D^2
$\{4, \{3, 5\}_5\}.\mathbf{LXVI}$	$v_3^2s_2(s_0s_1)^2s_2, v_1^2v_2^2(s_0s_1)^2$	P^8D^4
$\{4, \{3, 5\}_5\}.\mathbf{LXVII}$	$v_3^2s_2(s_0s_1)^2s_2, v_1^2v_2^2s_1s_0s_1$	$H^2P^6D^4$
$\{4, \{3, 5\}_5\}.\mathbf{LXIX}$	$v_2v_1^2v_2(s_0s_1)^2, v_1^2v_3^2s_2s_1s_0s_1s_2$	$H^4P^5D^2$
$\{4, \{3, 5\}_5\}.\mathbf{LXX}$	$v_3^2s_2(s_0s_1)^2s_2, v_2^2(s_0s_1)^2, v_1^2$	D^{10}

of $\{4, \{3, 5\}_5\}.\mathbf{II}$. The polytope $\{4, \{3, 5\}_5\}.\mathbf{LXX}$ is itself a regular polytope, of type $\{2, 3, 5\}$.

4. Other Examples

Coxeter, in [2], and Grünbaum, in [4], give a locally projective polytope of type $\{3, 5, 3\}$. This regular polytope is self-dual, and has 11 hemi-icosahedra for its facets. Similarly, in [1], Coxeter exhibits a locally projective self-dual regular polytope of type $\{5, 3, 5\}$, with 57 hemi-dodecahedral facets.

Since the hemi-icosahedron and the hemi-dodecahedron have no proper quotients, any quotient of these polytopes by a group N must be section regular with, respectively, $11/|N|$ hemi-icosahedral or $57/|N|$ hemi-dodecahedral facets.

The fact that 11 is prime immediately dashes any hope of nontrivial quotients of the former. Therefore, there is only one polytope of type $\{3, 5, 3\}$ whose facets and vertex figures are all projective, namely the one given by Coxeter and Grünbaum. However, the fact that $57 = 3 \times 19$ makes it worth checking whether or not nontrivial quotients of the latter exist.

When the techniques of Section 3 are applied to the group of the latter, it is found that the group (of order 3420) has 19 conjugacy classes of subgroups. Of these, only one, the trivial subgroup, is semispars. It follows that Coxeter's $\{5, 3, 5\}$ is, after all, the only polytope of type $\{5, 3, 5\}$ whose facets and vertex figures are all projective.

Acknowledgments

The author thanks Cheryl Praeger and Greg Gamble for helpful advice regarding GAP and its use. The anonymous referee gave a great many useful comments and remarks which have enriched this paper no end.

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Received January 20, 2002, and in revised form May 29, 2002. Online publication January 24, 2003.