Line Transversals to Unit Disks*

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Abstract. We show that if every three members of a finite disjoint family of unit disks in the plane have a line transversal, then there is a line transversal to all except at most 12 disks in the family. We derive an analogous result for translates of a general compact convex set, with the constant equal to 47.

1. Introduction

A prominent role in combinatorial geometry is played by the Helly theorem which states that if every $d + 1$ members of a family of compact convex subsets of $\mathbf{R}^d$ intersect, then the whole family is intersecting. This is commonly expressed by saying that the Helly number for intersections of compact convex sets in $\mathbf{R}^d$ is $d + 1$. Among the many results and questions inspired by the Helly theorem, there is a result of Tverberg on common transversals which will be our starting point.

A *line transversal* to a collection $\mathcal{F}$ of compact convex subsets of $\mathbf{R}^d$ is a line meeting every member of $\mathcal{F}$. It is natural to ask whether there is an analogue of the Helly theorem involving line transversals instead of common points. (This is a particular case of a question of Vincensini [14].) In general, the answer is in the negative, as shown, e.g. by examples in [7]. The situation is different if we restrict ourselves to translates of a compact convex set. Tverberg [12] proved the following conjecture of Grünbaum [6]:

Theorem 1.1 (Tverberg). *If every five members in a disjoint family $\mathcal{F}$ of translates of a compact convex set in the plane have a line transversal, then there is a line transversal to $\mathcal{F}$.*

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An account of the history of this result is given, e.g. in [3]. For a broader view of the context of Helly-type problems, consult [1], [3] or [5].

In 1980 Katchalski and Lewis [10] showed that much can be said even if we assume only that every three members of $\mathcal{F}$ have a line transversal. They proved:

Theorem 1.2. *There is a constant C such that if every three members in a finite disjoint family $\mathcal{F}$ of translates of a compact convex set in the plane have a line transversal, then there is a line meeting all but at most C members of $\mathcal{F}$.*

They established the result with $C = 192\pi$ and conjectured that in fact $C = 2$ would do (the Katchalski–Lewis conjecture). Examples constructed in [8] show, however, that C is at least 4. The upper bound on the smallest value of C was improved in [13] and [8], and the current best value of 22 was established in [9]. We remark that for families of axis-parallel unit squares, the question was completely settled in [9] by showing that the assertion of Theorem 1.2 holds with $C = 4$ but not $C = 3$.

Focusing on another special case where $\mathcal{F}$ consists of unit disks, we prove the following result.

Theorem 1.3. *If $\mathcal{F}$ is a finite disjoint family of unit disks in the plane such that every three of the disks have a line transversal, then there is a line intersecting all but at most 12 members of $\mathcal{F}$.*

In Section 4 we outline how the method used for disks can be adapted to obtain a proof of Theorem 1.2 with $C = 47$.

2. Preliminaries

It is well known that three pairwise disjoint convex sets in $\mathbf{R}^2$ have *no* line transversal if and only if each can be strictly separated from the other two by a line.

If $D = \{D_1, D_2, D_3\}$ is a set of three pairwise disjoint convex sets in $\mathbf{R}^2$ and if S is a triple of lines such that each D_i is strictly separated from both the D_j 's with $j \neq i$ by a line in S , then we call S a (strictly) *separating triple* for D . (See Fig. 1.) A related notion is that of a *weakly separating triple*, where we do not require the separating lines to be strictly separating (so they may touch the sets D_i).

Fig. 1. A strictly separating triple.

Let $\mathcal{F}$ be a finite family of pairwise disjoint unit disks in $\mathbf{R}^2$ (that is, disks of unit radius).

We may assume that no three disks in $\mathcal{F}$ have a common tangent which weakly separates one of them. To achieve this, inflate all the disks very slightly, preserving the (non-)existence of a transversal to any subset of $\mathcal{F}$. The inflation can be done so that all common tangents of the above type are eliminated. Finally, rescale the family so that the disks are of unit radius again.

Lemma 2.1. *There is no weakly separating triple for any subset $\{D_1, D_2, D_3\}$ of $\mathcal{F}$.*

Proof. This follows from our assumption on $\mathcal{F}$: as long as a separating line does not touch all three disks D_i , it can be made strictly separating by a small perturbation. However, this would yield a strictly separating triple for the D_i 's. $\square$

3. Proof of the Theorem

We now prove Theorem 1.3. Let $\mathcal{F}$ be the family in question. By a result of Eckhoff [2] (see also [13]), the fact that every three disks in $\mathcal{F}$ have a common line transversal implies that some parallel strip Z of width 2 is intersected by all disks in $\mathcal{F}$. Without loss of generality, we assume that the center line of Z is the horizontal axis, as in Fig. 2.

We may also assume that some two disks in $\mathcal{F}$ are strictly separated by a vertical line. If this is not true, then the projections of all the disks to the horizontal axis have a common point, and the vertical line through this point is a transversal for $\mathcal{F}$.

Next, we introduce some terminology.

If p is an oriented line, then the *positive closed halfplane* of p , H_p^+ , is the closure of the set of points lying to the right of p if one looks in the direction given by the orientation of p . The *negative closed halfplane*, H_p^- , of p is defined symmetrically.

Let $J = [-\pi/2, \pi/2]$. For $\alpha \in J$, we define an oriented line $p(\alpha)$ as follows. Take an oriented line p such that the angle between the x axis and p (in this order) is α , and the whole of $\mathcal{F}$ is contained in H_p^- . Translate p to the first position where H_p^+ contains some disk from $\mathcal{F}$. The resulting line is $p(\alpha)$. Note that $H_{p(\alpha)}^+$ may contain more than one disk.

Let $\ell(\alpha)$ be the leftmost vertical tangent to a disk contained in $H_{p(\alpha)}^+$. Similarly, $r(\alpha)$ is defined as the rightmost such tangent. They are both oriented upwards. Refer to Fig. 3 for an illustration.

Fig. 2. A parallel strip of width 2 intersected by all disks.

Fig. 3. The left and right regions of α .

We use the term *leftmost (rightmost) disk touching $p(\alpha)$* to denote the disk contained in $H_{p(\alpha)}^+$ which is first (last) in the ordering induced by the orientation of $p(\alpha)$. Together, these disks are referred to as the *extremal disks touching $p(\alpha)$* .

Define the *left region* $L(\alpha)$ and the *right region* $R(\alpha)$ of α by

$$L(\alpha) = H_{p(\alpha)}^- \cap H_{\ell(\alpha)}^- \quad \text{and} \quad R(\alpha) = H_{p(\alpha)}^- \cap H_{r(\alpha)}^+.$$

(See Fig. 3.) Note that $L(-\pi/2)$ and $R(\pi/2)$ are both empty.

Let L be the subset of J consisting of points α such that the interior of $L(\alpha)$ contains a member of $\mathcal{F}$. Define $R \subset J$ similarly, with $R(\alpha)$ in place of $L(\alpha)$.

Proposition 3.1. *L and R are nonempty disjoint open subsets of J .*

Proof. To see that L and R are disjoint, consider a point α in their intersection. There is a disk D_1 in $\text{int } L(\alpha)$ and a disk D_2 in $\text{int } R(\alpha)$ (where int stands for the interior). Take any disk C in $H_{p_\alpha}^+$. Observe that $p(\alpha)$, $\ell(\alpha)$ and $r(\alpha)$ form a weakly separating triple for D_1 , D_2 and C (see Fig. 4). Lemma 2.1 provides the contradiction.

To show that L is nonempty, take two disks in $\mathcal{F}$ that are separated by a vertical line (they are assumed to exist), and note that at least one of them must be contained in $L(\pi/2)$. Thus $\pi/2 \in L$. Similarly, $-\pi/2 \in R$.

We prove that L (say) is open in J . We need to show that for any point $\alpha \in L$, L contains a neighborhood of α .

Fig. 4. A weakly separating triple showing that $L \cap R = \emptyset$.

Assume that $\alpha \in L$, with a disk D contained in $L(\alpha)$. Let C be the rightmost disk touching $p(\alpha)$. For a small $\varepsilon > 0$ and any $\beta \in (\alpha, \alpha + \varepsilon)$, $H_{p(\beta)}^+$ contains C as well, so $\ell(\beta) = \ell(\alpha)$, and it is not hard to see that $L(\beta) \supset L(\alpha)$ and hence $\beta \in L$.

Finally, take $\beta \in (\alpha - \varepsilon, \alpha)$. If ε is small enough, $p(\beta)$ touches the leftmost disk touching α , and so, again, $\ell(\beta) = \ell(\alpha)$. Furthermore, ε may be taken small enough so that $p(\beta)$ does not intersect D . Therefore, D is in the interior of $L(\beta)$, implying $\beta \in L$ as required.

The statement that R is open can be proved in a symmetric way. $\square$

Proposition 3.1 implies that the connected space J cannot be covered by L and R . In other words, there is a point $\alpha_0 \in J$ such that neither $\text{int } L(\alpha_0)$ nor $\text{int } R(\alpha_0)$ contains a member of $\mathcal{F}$. From now on, we denote $p(\alpha_0)$ simply as p .

Let C_1, C_2 be the leftmost and rightmost disks touching p , respectively. Define the vertical closed strip S_i ($i = 1, 2$) to be the union of all vertical lines intersecting C_i . Clearly, S_i is of width 2.

Proposition 3.2. *Every disk in $\mathcal{F}$ which is not met by p must intersect $S_1 \cup S_2$.*

Proof. Let $D \in \mathcal{F}$ avoid both p and $S_1 \cup S_2$. By the definition of $p(\alpha)$, D must lie in H_p^- . If H_p^+ contains a single disk (i.e. $S_1 = S_2$), we are done, since $S_1, L(\alpha_0)$ and $R(\alpha_0)$ together cover all of H_p^- , so there is no choice for D but to intersect S_1 .

It remains to discuss the case where H_p^+ contains more than one disk. The set

$$T = \text{int } H_p^- - (L(\alpha_0) \cup S_1 \cup S_2 \cup R(\alpha_0))$$

is then nonempty. However, assuming that the disk D is contained in T , it is easy to find a separating triple for C_1, C_2 and D : just take the two vertical lines bounding T , together with p as in Fig. 5. This concludes the proof. $\square$

Before proving the main result, we need the following technical lemma.

Lemma 3.3. *Let D be a unit disk centered at the origin. Let Q be an axis-parallel square with sides of length 6, and with opposite corners at $(-3, -1)$ and $(3, 5)$. For any*

Fig. 5. A weakly separating triple (solid lines) showing that no disk can be contained in T .

Fig. 6. The region Q_s .

oriented line s with $D \subset H_s^+$, the region

$$Q_s = Q \cap H_s^-$$

can contain at most six pairwise disjoint unit disks.

Proof. The situation is pictured in Fig. 6.

It is sufficient to consider the lines s which are tangent to D . Observe that the area $A(Q_s)$ is maximized when s is a horizontal tangent to D , oriented to the right, in which case $A(Q_s) = 24$.

By [4, p. 67], the area of the convex hull of any $n > 1$ disjoint unit disks is at least $n\sqrt{12}$. Since Q_s is a convex set for any s , it follows that at most $\lfloor 24/\sqrt{12} \rfloor = 6$ disks can be packed in Q_s . $\square$

The proof of Theorem 1.3 will be complete when we prove the following proposition.

Proposition 3.4. *The line p intersects all but at most 12 members of $\mathcal{F}$.*

Proof. Every disk $D \in \mathcal{F}$ is contained in the closed horizontal strip Z' of width 6, whose center line is the x axis. On the other hand, if D is not intersected by p , then, by Proposition 3.2, D must be contained in one of the closed vertical strips S'_1, S'_2 of width 6 whose center lines coincide with those of S_1 and S_2 , respectively.

This would suffice to prove that D is contained in one of two squares $T_i = Z' \cap S'_i$ ($i = 1, 2$) of area 36 each. We can improve this by observing that D must actually be contained in the intersection of these squares with H_p^- . In each square T_i , the situation is precisely as in Lemma 3.3. The lemma implies that each $T_i \cap H_p^-$ can contain no more than six disks. The total number of disks not intersected by p is thus at most 12 as claimed. $\square$

4. A Generalization to Translates

The method used above for disks can be adapted to work for translates of a compact convex set, thus providing an upper bound on the constant in the Katchalski–Lewis theorem:

Theorem 4.1. *If every three members of a finite disjoint family $\mathcal{F}$ of translates of a compact convex set C in $\mathbf{R}^2$ have a line transversal, then there is a line meeting all but 47 members of $\mathcal{F}$.*

There is an unsatisfactory gap between the value of the constant here and in Theorem 1.3. The proof of the more general statement involves some rather crude estimates which we have not been able to avoid so far. As it stands, Theorem 4.1 is inferior to the result of Holmsen [9], mentioned in the Introduction, who proved the same statement with 47 replaced by 22 using different methods. For this reason, we only sketch the necessary modifications to the proof of Theorem 1.3 and omit the precise details, hoping that the interested reader will be able to reconstruct the argument.

Let $\mathcal{F}$ be a finite family of translates of a compact convex set C . We may assume that C has an interior point, since, by [11], if C is a straight line segment, then the assumption in Theorem 4.1 implies the existence of a transversal for the whole family $\mathcal{F}$.

We begin with the following easy observation:

Lemma 4.2. *Any compact convex set C can be inscribed in a rectangle whose area is at most twice that of C .*

Proof. (sketch) Take a segment XY which is contained in C and whose length is equal to the diameter of C . The rectangle bounded by four lines touching C , two of them parallel to XY and the other two perpendicular, has the desired property. $\square$

Inscribe C in a rectangle R as in Lemma 4.2. We may in fact assume that the rectangle is a square, performing the obvious affine transformation of the plane if necessary. The transformation preserves the relevant properties of $\mathcal{F}$ and keeps the ratio between the area of R and that of C unchanged. By the same token, it can be assumed that the diameter of C is 2.

Let a be the length of a side of the square R . Since the diameter of C (which is 2) is at most the length of the diagonal of R (which is $a\sqrt{2}$), we conclude that the area $A(C) \geq A(R)/2 = a^2/2 \geq 1$.

The result of Eckhoff [2] used at the beginning of Section 3 holds for translates of an arbitrary compact convex set K . It guarantees the existence of a parallel strip Z , intersected by all $D \in \mathcal{F}$, and such that the width of Z equals the width of K in the direction of Z (i.e. the distance between the two lines in this direction touching K). In our situation, the width of Z will be at most 2, the diameter of C .

We proceed as in Section 3 and obtain the line $p = p(\alpha_0)$ and two strips S_1, S_2 as in Proposition 3.2. These strips are intersected by any member of $\mathcal{F}$ which is not met by p . The width of the strips is at most 2.

Note that we may assume that the width of both Z and (say) S_1 is exactly 2. (The appropriate affine transformation of the plane preserves the property $A(C) \geq 1$ since it can only increase the area.)

It follows that every member of $\mathcal{F}$ not intersected by p is contained in one of two squares T_1, T_2 of side 6. The area of T_i ($i = 1, 2$) is $A(T_i) = 36$. Similarly as in Lemma 3.3, we can gain by taking into account the fact that a part of $T_1 \cup T_2$ is cut off by p . In fact, a simple argument shows that $A(H_p^- \cap (T_1 \cup T_2)) \leq 48$. Since $A(C) \geq 1$, at most 47 translates of C can be packed in $(\text{int } H_p^-) \cap (T_1 \cup T_2)$. Theorem 4.1 follows.

5. Open Problems

An obvious open problem is to determine the smallest value of the constant in the Katchalski–Lewis theorem, even for disjoint unit disks.

Another question is whether an analogue of Theorem 1.2 might hold in the 3-space, say for disjoint unit balls:

Question 5.1. Is there a constant C such that for any finite disjoint family of unit balls in $\mathbf{R}^3$ in which every three members have a line transversal, there is a line intersecting all except at most C of the balls?

This question is even more interesting in view of the fact that it is not known whether there exists a finite Helly number for line transversals to disjoint unit balls in $\mathbf{R}^3$ (that is, whether an analogue of Theorem 1.1 holds in this case, with a suitable constant).

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Note added in proof. Question 5.1 has recently been answered in the negative by A. Holmsen. The problem mentioned in the paragraph following Question 5.1 has been solved by A. Holmsen, M. Katchalski and T. Lewis, who proved that the Helly number in question is at most 46.