A Helly-Type Theorem for Line Transversals to Disjoint Unit Balls*

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Abstract. Let $\mathcal{F}$ be a family of disjoint unit balls in $\mathbb{R}^3$. We prove that there is a Helly-number $n_0 \leq 46$, such that if every n_0 members of $\mathcal{F}$ ($|\mathcal{F}| \geq n_0$) have a line transversal, then $\mathcal{F}$ has a line transversal. In order to prove this we prove that if the members of $\mathcal{F}$ can be ordered in a way such that every 12 members of $\mathcal{F}$ are met by a line consistent with the ordering, then $\mathcal{F}$ has a line transversal. The proof also uses the recent result on geometric permutations for disjoint unit balls by Katchalski, Suri, and Zhou.

1. Introduction

Let $\mathcal{F}$ be a family of disjoint convex sets in $\mathbb{R}^d$. A line intersecting every member of $\mathcal{F}$ is called a *transversal* to $\mathcal{F}$, and we say that $\mathcal{F}$ has T . If every n or fewer members of $\mathcal{F}$ have a transversal we say that $\mathcal{F}$ has $T(n)$. A transversal induces an ordering on the sets corresponding to the order in which the sets are intersected, and this ordering is called a *geometric permutation*.

Much effort has been devoted to finding conditions for when a given family of convex sets has a transversal, and for a survey on the recent progress made in the field of geometric transversal theory, the reader should consult Wenger's survey [8]. The main result of our paper is the following:

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Theorem 1. *There exists a positive integer $n_0 \leq 46$, such that $T(n_0)$ implies T for any family of pairwise disjoint unit balls in $\mathbb{R}^3$.*

In Section 2 we prove an analogue to Hadwiger's Transversal Theorem [5] which is an important part of the proof of Theorem 1. Another important result that we need is the recent result on geometric permutations for families of unit balls by Katchalski et al. [7].

2. A Hadwiger-Type Result

We start this section by introducing some new notions and proving a technical lemma. Let A, B, C be disjoint convex sets in $\mathbb{R}^3$. Consider the set of rays (directed half-lines) intersecting the balls in the same order, say, A, B, C , and translate each such ray so that its starting point is in the origin. The translated rays form a cone which we call $\mathcal{K}(ABC)$. We can now state our technical lemma.

Lemma 1. *If A, B, C are disjoint unit balls in $\mathbb{R}^3$, then $\mathcal{K}(ABC)$ is convex.*

Proof. Let A and B be disjoint unit balls in $\mathbb{R}^3$ with centers in $(0, 0, 0)$ and $(d, 0, b)$, respectively, with $d > 0$ and $0 \leq b \leq 2$. Let $H(z)$ be the plane parallel to the (x, y) -plane at height z . For $b - 1 \leq z \leq 1$ the intersections $H(z) \cap A$ and $H(z) \cap B$ are two discs of radius $R(z)$ and $R(z - b)$, respectively, where

$$R(z) = \sqrt{1 - z^2}.$$

Consider the directed lines in $H(z)$ intersecting A before B . For each such directed transversal t in $H(z)$ let $\alpha \in (-\pi/2, \pi/2)$ be the angle between t and the x -axis in $H(z)$. Letting z traverse from $b - 1$ to 1 the directed transversals give us a bounded region in the (z, α) -plane. Call this region $\mathcal{Q}(AB)$. Note that the shape of $\mathcal{Q}(AB)$ is only dependent on the relative position of the balls and the normal vector of $H(z)$. Keeping A, B , and the normal vector of $H(z)$ fixed, but changing the coordinate axes results in a translation of $\mathcal{Q}(AB)$ in the (z, α) -plane.

We now show the following: *The region $\mathcal{Q}(AB)$ in the (z, α) -plane is convex.* First some simple observations. The region $\mathcal{Q}(AB)$ is symmetric about the lines $z = b/2$ and $\alpha = 0$, and the extremal values of α are obtained exactly when $z = b/2$. Thus we need to verify that the region bounded by $b/2 \leq z \leq 1$, $\alpha \geq 0$, and the upper boundary of $\mathcal{Q}(AB)$ is convex. The upper boundary of $\mathcal{Q}(AB)$ is given by the function

$$G(z) = \sin^{-1}(f(z)),$$

where

$$f(z) = \frac{R(z) + R(z - b)}{d}.$$

To this end we must verify that $G''(z) \leq 0$ for $b/2 \leq z \leq 1$.

It is easily verified that $f(z) < 1$, and that

$$\begin{aligned} f'(z) &= \frac{R'(z) + R'(z-b)}{d}, & f''(z) &= \frac{R''(z) + R''(z-b)}{d}, \\ f'''(z) &= \frac{R'''(z) + R'''(z-b)}{d}, \end{aligned}$$

where

$$R'(z) = \frac{-z}{(1-z^2)^{1/2}}, \quad R''(z) = \frac{-1}{(1-z^2)^{3/2}}, \quad R'''(z) = \frac{-3z}{(1-z^2)^{5/2}}.$$

(Note that $R'(z)$ and $R'''(z)$ are decreasing functions on the interval $(-1, 1)$.) Further we have

$$G''(z) = \frac{f''(z)(1 - (f(z))^2) + f(z)(f'(z))^2}{(1 - (f(z))^2)^{3/2}} = \frac{g(z)}{(1 - (f(z))^2)^{3/2}}.$$

The sign of $G''(z)$ is determined by $g(z)$, and it is easily seen that $g(b/2) < 0$ and thus $G''(b/2) < 0$. Furthermore,

$$g'(z) = f'''(z)(1 - (f(z))^2) + (f'(z))^3.$$

Now $g'(b/2) = f'''(b/2) = f'(b/2) = 0$, and since $R'(z)$ and $R'''(z)$ are decreasing functions it follows that $f'(z)$ and $f'''(z)$ are also decreasing on the interval $(b/2, 1)$. Thus $g'(z) \leq 0$. This implies that $g(z) < 0$ and therefore also $G''(z) < 0$. This proves that $\mathcal{Q}(AB)$ is convex.

Now let A, B, C be disjoint unit balls in $\mathbb{R}^3$. We now show that $\mathcal{K}(ABC)$ is convex. Let l_1 and l_2 be directed transversals intersecting the sets in order A, B, C , and let l'_i be the translation of l_i to the origin, for $i = 1, 2$. If $l'_1 = l'_2$ there is nothing to prove so assume $l'_1 \neq l'_2$ and let H be the plane spanned by l'_1 and l'_2 . Using planes parallel to H we get regions $\mathcal{Q}(AB)$, $\mathcal{Q}(BC)$, and $\mathcal{Q}(AC)$ in the (z, α) -plane. The representatives of l_1 and l_2 in the (z, α) -plane must lie in the intersection $\mathcal{Q}(AB) \cap \mathcal{Q}(BC) \cap \mathcal{Q}(AC)$. Now the reader will easily verify the following fact: the range of the α values of $\mathcal{Q}(AB) \cup \mathcal{Q}(BC) \cup \mathcal{Q}(AC)$ is less than 2π . It therefore follows that the intersection is convex, and this implies the convexity of $\mathcal{K}(ABC)$. $\square$

The following are considered classical theorems of convexity theory. The reader not familiar with these results should consult, e.g., Eckhoff's survey [4] for additional references and related results.

Helly's theorem states that if $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$, where $\mathcal{F}$ is finite or each member of $\mathcal{F}$ is compact, and if every $d + 1$ or fewer members of $\mathcal{F}$ have a common point, then there is a point in common to all members of $\mathcal{F}$.

A subset X of the d -sphere S^d is called *convex* if it contains, with every two nonantipodal points, the small arc of the great circle containing them. If, in addition, X does not contain antipodal points, then X is called *strongly convex*.

Helly's theorem easily implies the following *Spherical Helly Theorem*: Let $\mathcal{F}$ be a family of strongly convex sets in S^d , where $\mathcal{F}$ is finite or each member of $\mathcal{F}$ is compact.

If every $d + 2$ or fewer members of $\mathcal{F}$ have a common point, then there is a point in common to all members of $\mathcal{F}$.

We now can prove the following Hadwiger-type result.

Theorem 2. *Let $\mathcal{F}$ be a family of disjoint unit balls in $\mathbb{R}^3$. If there is a linear ordering of $\mathcal{F}$ such that every 12 members are met by a directed line consistently with that ordering, then $\mathcal{F}$ admits a line transversal.*

Proof. For each triple $X \prec Y \prec Z$ of the ordering of $\mathcal{F}$ we get the associated cone, $\mathcal{K}(XYZ)$. The assumption that each 12 members of $\mathcal{F}$ are met by a directed line consistent with the ordering implies that every four cones contain a common ray. By Lemma 1, and since a cone obviously cannot contain opposite rays, we find (by the Spherical Helly Theorem) that there is a ray common to every cone. Taking the projection of $\mathcal{F}$ in the direction of the common ray onto a suitable plane we get a family of discs, each three intersecting, thus there is a point in common to each disc (by Helly's theorem in the plane). The line corresponding to this point is a transversal for $\mathcal{F}$. $\square$

3. Proof of the Theorem

First some new notions. Let $\mathcal{C}(n)$ denote the collection of all families of n disjoint unit balls in $\mathbb{R}^3$ that have a common transversal. For each $\mathcal{F} \in \mathcal{C}(n)$ let $d(\mathcal{F})$ denote the greatest distance between the centers of two members of $\mathcal{F}$. Finally, we define $\mathcal{D}(n)$ as

$$\mathcal{D}(n) = \inf\{d(\mathcal{F}); \mathcal{F} \in \mathcal{C}(n)\}.$$

A simple volume argument shows that $\mathcal{D}(n) = \Omega(n)$ (for details see [7]). We can now state a recent result by Katchalski et al. [7]. Our version of this result is not the exact statement in Theorem 3.4 of [7], but a closer inspection of their proof yields the following:

Theorem 3 (Katchalski–Suri–Zhou). *There exists a positive integer k_0 such that the following always holds. Let $\mathcal{F}$ be a family of disjoint unit balls in $\mathbb{R}^3$ where $d(\mathcal{F}) \geq \mathcal{D}(k_0)$. Then $\mathcal{F}$ admits at most four distinct geometric permutations, which can be expressed as*

$$\begin{aligned} (B_1 \cdots B_i UVXY B_{i+5} \cdots B_j), \quad & (B_1 \cdots B_i VUXY B_{i+5} \cdots B_j), \\ (B_1 \cdots B_i UVYX B_{i+5} \cdots B_j), \quad & (B_1 \cdots B_i VUYX B_{i+5} \cdots B_j). \end{aligned}$$

Since our attention is restricted to $\mathbb{R}^3$, we have only stated Theorem 3 in this setting, but the reader should be aware that the statement holds in $\mathbb{R}^d$, for arbitrary d , with the constant k_0 depending only on d . By a rough analysis of the Katchalski–Suri–Zhou proof for the case $d = 3$ we found that $k_0 \leq 31$. As in their proof, we first showed that for any family of at least 31 disjoint unit balls, any two switched pairs must be disjoint; that is, any member can appear in at most one switched pair. To see this assume that there is a member X contained in two switched pairs, XY and XZ . Next consider the

orthogonal projection of the family into the plane spanned by the centers of X , Y , and Z . Since there are at least 28 other members of the family, we can show that, in the projection, some member must be disjoint from X , Y , and Z . However, in [1] it was shown that for any family of at least four disjoint unit discs in the plane, there can be at most one switched pair, contradicting the existence of the switched pairs XY and XZ . Then, following the Katchalski–Suri–Zhou proof, we showed that there can be at most two switched pairs.

Proof of Theorem 1. First we may assume that the family is finite, as the infinite case is well known (see, e.g., [3]). Let $n_0 = 46$ and let $\mathcal{F} = \{B_1, \dots, B_{n+1}\}$, with $n \geq n_0$ being a family of $n + 1$ disjoint unit balls in $\mathbb{R}^3$ that has $T(n)$. We shall show that $\mathcal{F}$ admits a line transversal, which will imply Theorem 1.

Assume without loss of generality that the center of B_1 and B_{n+1} is the pair of greatest distance. Let $\mathcal{T}$ be the set of directed lines intersecting B_1 before B_{n+1} . If for a given pair $B_i, B_j \in \mathcal{F}$ there exists transversals $t, t' \in \mathcal{T}$ such that t intersects B_i before B_j and t' intersects B_j before B_i we call $\{B_i, B_j\}$ a *switched pair*.

We now notice that if there are no switched pairs we automatically get a linear ordering, $<$, on $\mathcal{F}$ as follows: for any pair $B_i, B_j \in \mathcal{F}$ and any transversal $t \in \mathcal{T}$ intersecting B_i and B_j , t must intersect B_i and B_j in some order, say, B_i before B_j . Since there are no switched pairs this order is independent of the choice of t , and we let $B_i < B_j$. It is easy to see that this ordering is transitive: assume $B_i < B_j$ and $B_j < B_k$. Any transversal $t \in \mathcal{T}$ for B_i, B_j, B_k must meet B_i before B_j and B_j before B_k , but this transversal must also meet B_i before B_k , thus $B_i < B_k$. Now since we have $T(n)$, where $n \geq n_0 > 12$, each 12 members of $\mathcal{F}$ are met by a transversal $t \in \mathcal{T}$ which is consistent with the ordering. By Theorem 2, $\mathcal{F}$ admits a common line transversal. This takes care of the case where there are no switched pairs.

Now we assume that there are switched pairs. Let $\mathcal{P}_1$ be a switched pair and let $\mathcal{T}_1 \subset \mathcal{T}$ be the set of directed lines meeting $\mathcal{P}_1$. Define the switched pairs inductively as follows: $\mathcal{P}_{i+1}$ is a switched pair if there are transversals $t, t' \in \mathcal{T}_i$ such that t and t' meet the members of $\mathcal{P}_{i+1}$ in opposite order, and let $\mathcal{T}_{i+1} \subset \mathcal{T}_i$ be the set of directed lines meeting $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_{i+1}$.

Since $|\mathcal{F}| \geq n_0 + 1 > 31$, it follows from Theorem 3 that there can be at most two distinct switched pairs, and they must be disjoint as well. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the switched pairs. (From this end we are assuming there are two switched pairs, but the argument works the same if there is only one.)

We now show that we can fix an ordering on the switched pairs. By this we mean that there exists a pair of orderings of $\mathcal{P}_1$ and $\mathcal{P}_2$, such that for any set $\mathcal{N} \subset \mathcal{F}$, of cardinality at most 10, there exist transversals $t \in \mathcal{T}_2$ meeting $\mathcal{N}$ such that t meets $\mathcal{P}_1$ and $\mathcal{P}_2$ consistently with this pair of orderings. To see this, assume the contrary: for a given pair of orderings of $\mathcal{P}_1$ and $\mathcal{P}_2$ there must exist a set $\mathcal{N} \subset \mathcal{F}$, of cardinality at most 10, such that any transversal $t \in \mathcal{T}_2$, meeting $\mathcal{N}$, induces a pair of orderings on $\mathcal{P}_1$ and $\mathcal{P}_2$ which is different from the given pair of orderings. Doing this for each of the four different pairs of orderings of $\mathcal{P}_1$ and $\mathcal{P}_2$ we get sets $\mathcal{N}_1, \dots, \mathcal{N}_4$, each of cardinality at most 10. By the way these sets are defined, there is no transversal $t \in \mathcal{T}_2$ intersecting $\mathcal{N}_1 \cup \dots \cup \mathcal{N}_4$. However, since $|\mathcal{N}_1 \cup \dots \cup \mathcal{N}_4| \leq 40$ and $n \geq 46$, this contradicts the assumption $T(n)$.

Now that we have fixed an ordering on the switched pairs, we are provided with an ordering of $\mathcal{F}$. To see this assume $B_i < B_j$ and $B_j < B_k$. We must show that the ordering is transitive. If none of the members here are switched pairs this follows as before. Note also that by the disjointness of the switched pairs there can only be one switched pair among B_i, B_j, B_k . If $\{B_i, B_k\}$ is a switched pair, we assume by contradiction that the fixed ordering of this pair is $B_k < B_i$ (the other cases are treated similarly). Then there must be a transversal $t \in \mathcal{T}_2$ meeting B_k before B_i , B_i before B_j , and B_j before B_k , which is impossible.

Finally we must show that for any 12 members $B_{i_1}, \dots, B_{i_{12}} \in \mathcal{F}$ there is a transversal intersecting them consistently with the ordering. If there are no switched pairs among the B_{i_j} 's, this follows trivially. If there are any switched pairs there are at most 10 of the B_{i_j} 's that are not in any switched pair, so there is a transversal $t \in \mathcal{T}$, meeting each B_{i_j} , which meets the switched pairs consistently with the fixed orderings as well as the ordering of the other sets. Again, by Theorem 2, $\mathcal{F}$ admits a line transversal. $\square$

4. Final Remarks

We conclude with some final remarks on how Theorem 1 can be improved.

1. There are clearly improvements to be made in our proofs that will reduce the value of n_0 . In particular, the value 12 in Theorem 2 can be reduced to 9, if one can show that the maximal angles of the cones $\mathcal{K}(ABC)$ are less than $2\pi/3$. This would reduce the upper bound of n_0 to 36. Determining a good bound for the value of k_0 in Theorem 3 is clearly necessary in order to reduce the upper bound for the Helly-number. Our bound $k_0 \leq 31$ is obviously much too big. Furthermore, it is quite possible that a family of disjoint balls can admit only one switched pair. This would further reduce the value of n_0 to $\max\{18, k_0\}$.
2. Lemma 1 might very well hold for line transversals in higher dimensions. If so, Theorem 2 will also generalize. This, in turn, will imply that Theorem 1 generalizes to higher dimensions as well. We have left this as an open problem.
3. Lemma 1 cannot in general be extended to families of tetrahedra which the following example illustrates. Let S be the convex hull of the points $(0, 0, 0)$, $(0, -2, 1)$, $(0, -2, -4)$, and $(-1, 1, -3)$. Let $A = S + (-1, 1, 0)$, $B = S + (0, 2, 2)$, $C = S + (2, 0, 4)$. The reader will easily verify that the following lines intersect the sets in the order A, B, C : $(t, t, 1)$ and $(t, -t, 0)$ for $t \in \mathbb{R}$. (see Fig. 1). However, there is no transversal parallel to the x -axis since the projections of A , B , and C on the (y, z) -plane do not have a common point (see Fig. 2).
4. In light of the previous construction it is also worth noting that Holmsen and Matoušek [6] recently showed that no Helly theorem exists for line transversals to translates in $\mathbb{R}^3$. In particular, they showed that for every $n > 2$ there exists a convex body $K = K(n) \subset \mathbb{R}^3$ and a family $\mathcal{F}$ of disjoint translates of K such that any $n - 1$ members of $\mathcal{F}$ admit a line transversal, but $\mathcal{F}$ has no line transversal. Their construction also refutes the existence of a Hadwiger-type result for families of disjoint translates in $\mathbb{R}^3$.

Fig. 1. Cutting the family of tetrahedra by the planes $z = 0$ and $z = 1$.

Fig. 2. Projection of the family of tetrahedra into the yz -plane.

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