The Characteristic Intersection Property of Line-Free Choquet Simplices in E^d

Valeriu Soltan

Department of Mathematical Sciences, George Mason University,
4400 University Drive, Fairfax, VA 22030-4444, USA
vsoltan@gmu.edu

Abstract. Generalizing the characteristic intersection property of Choquet simplices, it is proved that for line-free convex bodies B_1 and B_2 in E^d , the following conditions are equivalent: (i) there is a line-free convex body $B \subset E^d$ such that every nonempty intersection $B_1 \cap (v + B_2)$, $v \in E^d$, is a homothetic copy of B , (ii) both B_1 and B_2 are Choquet simplices and the nonempty intersections $B_1 \cap (v + B_2)$, $v \in E^d$, are homothetic copies of a Choquet simplex B . All such triplets B_1, B_2, B are described.

Introduction

In what follows, by a *convex body* we mean a closed convex set with nonempty interior in the Euclidean space E^d . We will distinguish *bounded* and *unbounded* convex bodies. A set in E^d is called *line-free* provided it contains no line. By a d -polyhedron (in particular, a d -polytope, a d -simplex, or a simplicial d -cone) we mean a convex polyhedral set of dimension d .

In 1956 Choquet [2] defined a *simplex* (afterwards called a *Choquet simplex*) as a convex set S in a linear space E such that for any two homothetic copies of S their intersection, if nonempty, is again a homothetic copy of S (possibly degenerated into a point):

$$(u + \lambda S) \cap (v + \mu S) = w + \nu S, \quad u, v, w \in E, \quad \lambda, \mu, \nu \geq 0.$$

From [1] and [6] it follows that a line-free convex body $B \subset E^d$ is a Choquet simplex if and only if B is either a usual d -simplex of the form $\text{conv}(x_0, x_1, \dots, x_d)$ or a simplicial d -cone of the form $B = \sum_{i=1}^d [x_0, x_i]$, where $x_0, x_1, \dots, x_d$ are affinely independent points in E^d and $[x, z]$ means the half-line through z with the endpoint x .

As was mentioned in [4], the above definition can be slightly modified in the following way: a line-free convex body $B \subset E^d$ is a Choquet simplex if and only if every nonempty

intersection of B and a translate of B is a homothetic copy of B (possibly degenerated into a point):

$$B \cap (v + B) = w + \lambda B, \quad v, w \in E^d, \quad \lambda \geq 0. \quad (1)$$

The following generalization of the characteristic intersection property of Choquet simplices was obtained in [7].

Theorem 1. *Let B_1 and B_2 be a pair of bounded convex bodies in E^d such that the nonempty intersections $B_1 \cap (v + B_2)$, $v \in E^d$, are homothetic copies (possibly degenerated into points) of a given bounded convex body $B \subset E^d$. Then B_1 , B_2 , and B are homothetic Choquet simplices, i.e., they are homothetic d -simplices.*

Our purpose here is to extend this result for the case of line-free convex bodies in E^d (see Theorem 2).

Let us recall that a boundary point x (or a boundary half-line l) of a convex body $B \subset E^d$ is called *exposed* provided there is a hyperplane H such that $B \cap H = \{x\}$ (respectively, $B \cap H = \{l\}$). The set of exposed points of a convex body B is denoted $\exp B$. It is well known (see [5] and [8]) that a bounded convex body in E^d is the closed convex hull of its exposed points, and an unbounded line-free convex body in E^d is the closed convex hull of the union of its exposed points and exposed half-lines. As a result, any line-free convex body in E^d has at least one exposed point.

A boundary point x of a convex body $B \subset E^d$ is called *regular* if there is a unique hyperplane supporting B at x . Denote by $N(B)$ the family of outward unit normals to the convex body B at its regular points. In particular, a convex body $B \subset E^d$ is a d -polyhedron if and only if the set $N(B)$ is finite.

The characteristic cone $C(B)$ of a convex body $B \subset E^d$ (also known as the recession cone) is defined by $C(B) = \{y \in E^d: x + \lambda y \in B \text{ for every } \lambda \geq 0 \text{ and } x \in B\}$. Finally, $U_r(a) = \{x \in E^d: \|x - a\| \leq r\}$ stands for the closed ball with radius r and center a .

Main Result

The main result of this paper is given in Theorem 2 below. For a boundary point x of a convex body $B \subset E^d$, we define the tangent cone $T_x(B)$ as the closure of the union of the half-lines $[x, z)$, where $z \in B \setminus \{x\}$. It is easily seen that $T_x(B)$ is a closed convex cone with apex x , distinct from E^d , and $T_x(B) = T_x(B \cap U_r(x))$ for any $r > 0$.

Theorem 2. *For a pair of line-free convex bodies B_1 and B_2 in E^d , conditions (1) and (2) below are equivalent.*

- (1) *There is a line-free convex body $B \subset E^d$ such that every nonempty intersection $B_1 \cap (v + B_2)$, $v \in E^d$, is a homothetic copy of B (possibly degenerated into a point).*
- (2) *There is a Choquet simplex B , i.e., B is a d -simplex of the form $B = \text{conv}(x_0, x_1, \dots, x_d)$ or a simplicial d -cone of the form $B = \sum_{i=1}^d [x_0, x_i)$, where $x_0, x_1, \dots$,*

x_d are affinely independent points, such that one of the following conditions holds:

- (a) Both B_1 and B_2 are homothetic copies of B .
- (b) One of B_1, B_2 , say B_1 , is a homothetic copy of B , and B_2 is a simplicial d -cone that is translate of a tangent cone $T_x(B)$, where $x \in \{x_0, x_1, \dots, x_d\}$.
- (c) Both B_1 and B_2 are simplicial d -cones that are translates of tangent cones $T_x(B)$ and $T_z(B)$, where x and z are distinct points in $\{x_0, x_1, \dots, x_d\}$ (this condition is possible only if B is a d -simplex).

Auxiliary Lemmas

We say that points $x \in \exp C$ and $z \in \exp D$ of convex bodies C and D in E^d correspond to each other provided the tangent cone $T_x(C)$ is a translate of the tangent cone $T_z(D)$. We will need the following lemma on tangent cones.

Lemma 1. *If a and b are distinct exposed points of a convex body $D \subset E^d$, then none of the cones $T_a(D)$, $(a - b) + T_b(D)$ lies in the other.*

Proof. Indeed, assume, for contradiction, that $(a - b) + T_b(D) \subset T_a(D)$ (the case $T_a(D) \subset (a - b) + T_b(D)$ is similar). Let H be a hyperplane with the property $D \cap H = \{a\}$. Clearly, H supports the cone $T_a(D)$. Then the hyperplane $(b - a) + H$ supports the cone $T_b(D)$, and, as a result, $(b - a) + H$ supports D at b . Two parallel hyperplanes, H and $(b - a) + H$, both supporting D from the same side, should coincide: $H = (b - a) + H$. Finally, $\{a\} = D \cap H = D \cap (b - a + H) = \{b\}$, a contradiction. $\square$

Lemma 2. *Under condition (1) of Theorem 2, any exposed point of B_1 (respectively, of B_2) corresponds to an exposed point of B . Moreover, if B_1 is bounded, then any exposed point of B corresponds to an exposed point of B_1 .*

Proof. Let $z \in \exp B_1$ (the case $z \in \exp B_2$ is similar). Choose a vector $v \in E^d$ such that $z \in \text{int}(v + B_2)$ and consider the intersection $C = B_1 \cap (v + B_2)$. Clearly, z is an exposed point of C and $T_z(C) = T_z(B_1)$. Since $C = w + \lambda B$ for some $w \in E^d$ and $\lambda > 0$ (by condition (1)), the exposed point z of B_1 corresponds to the exposed point $u = \lambda^{-1}(z - w)$ of B .

Now assume that B_1 is bounded, and let $z \in \exp B$. Denote by H a hyperplane in E^d with the property $B \cap H = \{z\}$, and let H_1 be a translate of H that supports B_1 at some point u such that B_1 lies on the same side of H_1 as B with respect to H (this is possible because B_1 is bounded). Choose a vector $v \in E^d$ such that $u \in \text{int}(v + B_2)$ and consider the intersection $C = B_1 \cap (v + B_2)$. Since C is homothetic to B (by condition (1)) and since H_1 supports C , the intersection $C \cap H_1$ consists of a single point. From the inclusion $u \in B_1 \cap H_1$ we get $C \cap H_1 = \{u\}$. Hence $u \in \exp C$ and $T_u(C)$ is a translate of $T_z(B)$. Clearly, $T_u(B_1) = T_u(C)$. Thus z corresponds to the exposed point u of B_1 . $\square$

Lemma 3. *Under condition (1) of Theorem 2, if B is bounded and B_2 is unbounded, then for any vector $v \in E^d$ the set $\text{int } B_1$ contains at most one exposed point of the body $v + B_2$.*

Proof. Assume, for contradiction, the existence of a vector $v \in E^d$ and distinct exposed points p, q of B_2 such that $\text{int } B_1$ contains both $p' = v + p$ and $q' = v + q$, which are exposed points of $v + B_2$. Denote by H_p and H_q some closed half-spaces in E^d with the property $B_2 \cap H_p = \{p\}$ and $B_2 \cap H_q = \{q\}$. Clearly,

$$(v + B_2) \cap (v + H_p) = \{p'\} \quad \text{and} \quad (v + B_2) \cap (v + H_q) = \{q'\}.$$

Since B_2 is unbounded, there is a half-line $l \subset B_2$ with the endpoint $\frac{1}{2}(p + q)$. Let a be the unit vector in the direction of l .

Put $C = B_1 \cap (v + B_2)$ and denote by m the intersection of C with the half-line $v + l$. Since C is homothetic to B , which is bounded, m is a line segment.

Now choose a real number $\varepsilon > 0$ so small that $\text{int } B_1$ still contains both exposed points $p'' = v + \varepsilon a + p$ and $q'' = v + \varepsilon a + q$ of the body $v + \varepsilon a + B_2$. Put $C' = B_1 \cap (v + \varepsilon a + B_2)$. From the fact that C' is homothetic to B and from the relations $\|p' - q'\| = \|p'' - q''\|$,

$$(v + \varepsilon a + B_2) \cap (v + \varepsilon a + H_p) = \{p''\}, \quad (v + \varepsilon a + B_2) \cap (v + \varepsilon a + H_q) = \{q''\}$$

we conclude that C' is a translate of C . At the same time, as is easily seen, the segment $m' = (v + \varepsilon a + l) \cap C'$ is shorter than m , a contradiction. $\square$

Lemma 4. *Let B, B_1 , and B_2 satisfy condition (1) of Theorem 2.*

- (i) *If B is a d -polyhedron, then both B_1 and B_2 are d -polyhedra and $N(B_1) \cup N(B_2) \subset N(B)$.*
- (ii) *If B_1 is a d -polytope, then B is a d -polytope with $N(B) = N(B_1)$.*

Proof. (i) Let x be a regular point of B_1 and let H be the hyperplane supporting B_1 at x . Denote by e the outward unit normal to B_1 at x . Choose a translate $v + B_2$ that contains x in its interior and consider the body $C = B_1 \cap (v + B_2)$. Clearly, x is a regular point of C and e is the outward unit normal to C at x . Since C is homothetic to B , we have $e \in N(C) = N(B)$. As a result, $N(B_1) \subset N(B)$. In particular, $N(B_1)$ is finite and B_1 is a d -polyhedron. Similarly, $N(B_2) \subset N(B)$ and B_2 is a d -polyhedron.

(ii) Let x be a regular point of B and let H be the hyperplane supporting B at x . Denote by H_1 the hyperplane parallel to H and supporting B_1 such that B_1 lies on the same side of H_1 as B does with respect to H . Choose points $x_1 \in B_1 \cap H_1$ and $v \in E^d$ with the property $x_1 \in \text{int}(v + B_2)$ and consider the intersection $C = B_1 \cap (v + B_2)$. Clearly, H_1 supports C . Since C is homothetic to B and H_1 is parallel to H , there is a regular point of C that belongs to H_1 . Obviously, this is possible only if z is a regular point of B_1 . Hence z lies in the relative interior of a facet of B_1 .

As a result, every hyperplane supporting B at a regular point is parallel to one of the B_1 facets. This implies that B is the intersection of finitely many closed half-spaces. Since B is bounded (being homothetic to C), we conclude that B is a d -polytope and $N(B) \subset N(B_1)$. From here and (i) above it follows that $N(B) = N(B_1)$. $\square$

Proof of Theorem 2

While proving Theorem 2, we consider the following four cases.

- I. Both B_1 and B_2 are bounded.
- II. B_1 is bounded and B_2 is unbounded (the case when B_2 is bounded and B_1 is unbounded is similar).
- III. Both B_1 and B_2 are unbounded and B is bounded.
- IV. B_1 , B_2 , and B are unbounded.

Case I: Both B_1 and B_2 are bounded. Obviously, bounded convex bodies B_1 and B_2 satisfying condition (2) are homothetic copies of B , which has to be a d -simplex. Hence B_1 and B_2 satisfy condition (2a). In this case, as proved in [7], conditions (1) and (2a) are equivalent.

Case II: B_1 is bounded and B_2 is unbounded.

(2) $\Rightarrow$ (1) Since B_1 is bounded, the body B from condition (2) is a bounded Choquet simplex, whence it is a d -simplex. We can represent B in a suitable coordinate system of E^d as

$$B = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq 0, \dots, \xi_d \geq 0, \xi_0 + \xi_1 + \dots + \xi_d \leq 1\}.$$

Obviously, B_1 and B_2 satisfy condition (2b), i.e., B_1 is homothetic to B and B_2 is a translate of a tangent cone $T_x(B)$, where x is a vertex of B . Let $B_1 = a + B$ and $B_2 = b + T_x(B)$ for some vectors $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d) \in E^d$. Without loss of generality, we may assume that $x = (0, \dots, 0)$. Then

$$B_1 = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq a_1, \dots, \xi_d \geq a_d, \xi_1 + \dots + \xi_d \leq 1 + a_1 + \dots + a_d\},$$

$$B_2 = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq b_1, \dots, \xi_d \geq b_d\}.$$

For an arbitrary vector $v = (v_1, \dots, v_d) \in E^d$, one has

$$B_1 \cap (v + B_2) = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq \max\{a_1, b_1 + v_1\}, \dots, \xi_d \geq \max\{a_d, b_d + v_d\}, \xi_1 + \dots + \xi_d \leq 1 + a_1 + \dots + a_d\}.$$

Clearly, the intersection $B_1 \cap (v + B_2)$ is nonempty if and only if

$$1 + a_1 + \dots + a_d \geq \max\{a_1, b_1 + v_1\} + \dots + \max\{a_d, b_d + v_d\}.$$

In this case $B_1 \cap (v + B_2) = w + \lambda B$, $\lambda \geq 0$, with

$$w = (\max\{a_1, b_1 + v_1\}, \dots, \max\{a_d, b_d + v_d\}),$$

$$\lambda = 1 + a_1 + \dots + a_d - (\max\{a_1, b_1 + v_1\} + \dots + \max\{a_d, b_d + v_d\}).$$

(1) $\Rightarrow$ (2) This part of the proof is divided into Lemmas 5–7.

Lemma 5. *Under condition (1), if there is a vector $v \in E^d$ such that $\text{int}(v + B_2)$ contains two distinct exposed points of B_1 , then $B_1 \subset (v + B_2)$.*

Proof. Assume for a moment that for a certain vector $v \in E^d$, the set $\text{int}(v + B_2)$ contains two distinct points $p, q \in \exp B_1$ and $v + B_2$ does not contain B_1 . Choose a point $s \in B_1 \setminus (v + B_2)$. Let H_p and H_q be some hyperplanes in E^d satisfying the conditions $B_1 \cap H_p = \{p\}$ and $B_1 \cap H_q = \{q\}$. One can find a number $\varepsilon > 0$ so small that the new translate $v + \varepsilon(p - s) + B_2$ still contains both points p and q in its interior.

Now consider the bodies $C = B_1 \cap (v + B_2)$ and $C' = B_1 \cap (v + \varepsilon(p - s) + B_2)$. By condition (1), both C and C' are homothetic to B . From here and the relations

$$C \cap H_p = C' \cap H_p = \{p\}, \quad C \cap H_q = C' \cap H_q = \{q\}$$

it follows that $C = C'$. The last is in contradiction with the fact that the segment $[s, p] \cap C$ is longer than the segment $[s, p] \cap C'$. $\square$

Lemma 6. *Under condition (1), both B and B_1 are a d -polytopes and B_2 is a d -polyhedron.*

Proof. Assume, for contradiction, that B_1 is not a d -polytope. Then the set $\exp B_1$ is infinite and has an accumulation point, a . Choose a vector $v \in E^d$ such that $a \in \text{int}(v + B_2)$. One can find a pair x, z of exposed points of B_1 so close to a that both x and z belong to $\text{int}(v + B_2)$. By Lemma 5, $B_1 \subset v + B_2$. From here and condition (1) we conclude that $B_1 = B_1 \cap (v + B_2)$ is homothetic to B .

Now let b be an exposed point of B_2 and let $b_1, b_2, \dots$ be a sequence of interior points of B_2 tending to b . Clearly, every translate $(a - b_k) + B_2$, $k = 1, 2, \dots$, contains a as an interior point, and, as shown above, B_1 lies in every convex body $(a - b_k) + B_2$. By a continuity argument, $(a - b) + B_2$ also contains B_1 . Since a is an exposed point of $(a - b) + B_2$, there is a hyperplane H such that $H \cap (a - b + B_2) = \{a\}$. Hence $H \cap B_1 = \{a\}$ and a is an exposed point of B_1 .

Next chose any point $p \in \text{int } B_1$ and consider the intersection $C = B_1 \cap (p - b + B_2)$. Clearly, the hyperplane $H' = (p - a) + H$ supports C such that $C \cap H' = \{p\}$, i.e., p is an exposed point of C correspondent to a in B_1 (due to the fact that B_1 is homothetic to B). Since $p \in \text{int } B_1$, there is a ball $U_r(p)$ that lies in $\text{int } B_1$, whence the boundary of C in the neighborhood $U_r(p)$ is a part of the boundary of the body $(p - b) + B_2$. By Lemma 3, p is the only exposed point of $(p - b) + B_2$ in $\text{int } B_1$. This implies that p is the only exposed point of C in $U_r(p)$. On the other hand, since C is homothetic to B_1 and since a is an accumulation point of B_1 , the correspondent point $p \in \exp C$ has to be an accumulation point of the set $\exp C$. The obtained contradiction shows that B_1 is a d -polytope.

Being homothetic to B_1 , the body B is also a d -polytope. From Lemma 4 it follows that B_2 is a d -polyhedron. $\square$

Lemma 7. *Under condition (1), B is a d -simplex and B_1, B_2 satisfy condition (2b).*

Proof. We will prove this statement by induction on $d = \dim E^d$. The case $d = 1$ trivially holds. Assume that the statement of Lemma 7 is true for all $d \leq n - 1$ ($n \geq 2$),

and let B, B_1, B_2 be convex bodies in E^n satisfying condition (1) such that B and B_1 are bounded and B_2 is unbounded.

By Lemmas 4 and 6, both B and B_1 are n -polytopes with $N(B) = N(B_1)$ and B_2 is an unbounded n -polyhedron with $N(B_2) \subset N(B)$. Let F_2 be any unbounded facet of B_2 , and denote by e the outward unit normal to B_2 at F_2 . Denote by F and F_1 the facets of B and B_1 , respectively, both having the same outward unit normal e .

We are going to show that F_2 is a simplicial $(n-1)$ -cone and F, F_1 are homothetic $(n-1)$ -simplices both satisfying condition (2b) for $d = n-1$. Indeed, denote by H_1 the hyperplane containing F_1 , and consider the family of translates $v + B_2$, $v \in E^d$, all having the facets $v + F_2$ in H_1 . Clearly, $F_1 \cap (v + F_2) \neq \emptyset$ implies $B_1 \cap (v + B_2) \neq \emptyset$. From condition (1) it follows that the intersections $B_1 \cap (v + B_2)$ are homothetic copies of B , and, as a result, their facets $F_1 \cap (v + F_2)$ are homothetic copies of F . By the inductive assumption, F and F_1 are homothetic $(n-1)$ -simplices and F_2 is a simplicial $(n-1)$ -cone.

From the fact that every unbounded facet of the n -polyhedron B_2 is a simplicial $(n-1)$ -cone we easily conclude that B_2 is a convex n -cone.

Our next step is to show that B and B_1 are homothetic n -simplices. In the notation above, let H'_1 be the hyperplane parallel to H_1 and supporting B_1 such that B_1 lies between H_1 and H'_1 . Denote by F'_1 the intersection of B_1 and H'_1 , and let x be any point in F'_1 . Choose a point $z \in \text{rint } F_2$ and consider the intersection $B' = B_1 \cap (x - z + B_2)$. Clearly, B' lies in H' . Since B' is a degenerate homothetic copy of B , we have from condition (1) that B' is a point. The last is possible only if F'_1 consists of a single point, i.e., $F'_1 = \{x\}$.

By a continuity argument, one can find a number $\varepsilon \in]0, 1[$ so close to 1 that x is the only vertex of B_1 contained in the body $\varepsilon(x - z) + B_2$. In this case the intersection $C = B_1 \cap (\varepsilon(x - z) + B_2)$ is a convex bounded n -cone with apex x and base $F = B_1 \cap (\varepsilon(x - z) + F_2)$. Since C is homothetic to B , the base F is an $(n-1)$ -simplex. Therefore C is an n -simplex. As a result, B is an n -simplex.

Since B_2 is a convex d -cone and B_1 is bounded, we can choose a vector $w \in E^n$ such that $B_1 \subset w + B_2$. From here one gets $B_1 = B_1 \cap (w + B_2)$, i.e., B_1 is homothetic to B . Hence B_1 is an n -simplex.

It remains to show that B_2 satisfies condition (2b). Indeed, from the fact that B_2 is a line-free convex n -cone, it follows that $N(B_2)$ contains at least n vectors. On the other hand, since $N(B_2)$ is a proper subset of $N(B)$ and $N(B)$ consists of $n+1$ vectors, the set $N(B_2)$ has exactly n vectors. Hence B_2 is a simplicial n -cone of the form $b + T_e(B)$, where e is a vertex of B . $\square$

Case III: B is bounded and both B_1, B_2 are unbounded.

(2) $\Rightarrow$ (1) Obviously, a bounded body B satisfying condition (2) is a d -simplex. We can represent B in a suitable coordinate system of E^d as

$$B = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq 0, \dots, \xi_d \geq 0, \xi_0 + \xi_1 + \dots + \xi_d \leq 1\}.$$

Since both B_1 and B_2 are unbounded, they must satisfy condition (2c), i.e., B_1 and B_2 are translates of tangent cones $T_x(B)$ and $T_z(B)$, where x and z are distinct vertices of B . Let $B_1 = a + T_x(B)$, $B_2 = b + T_z(B)$ for some vectors $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in E^d$. Without loss of generality, we may assume that $x = (1, 0, \dots, 0)$

and $z = (0, 0, \dots, 0)$. Then

$$B_1 = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_2 \geq a_2, \dots, \xi_d \geq a_d, \xi_1 + \dots + \xi_d \leq 1 + a_1 + \dots + a_d\},$$

$$B_2 = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq b_1, \dots, \xi_d \geq b_d\}.$$

For an arbitrary vector $v = (v_1, \dots, v_d) \in E^d$, one has

$$B_1 \cap (v + B_2) = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq b_1 + v_1, \xi_2 \geq \max\{a_2, b_2 + v_2\}, \dots, \\ \xi_d \geq \max\{a_d, b_d + v_d\}, \xi_1 + \dots + \xi_d \leq 1 + a_1 + \dots + a_d\}.$$

Clearly, the intersection $B_1 \cap (v + B_2)$ is nonempty if and only if

$$1 + a_1 + \dots + a_d \geq b_1 + v_1 + \max\{a_2, b_2 + v_2\} + \dots + \max\{a_d, b_d + v_d\}.$$

In this case $B_1 \cap (v + B_2) = w + \lambda B$, $\lambda \geq 0$, with

$$w = (b_1 + v_1, \max\{a_2, b_2 + v_2\}, \dots, \max\{a_d, b_d + v_d\}),$$

$$\lambda = 1 + a_1 + \dots + a_d - (b_1 + v_1 + \max\{a_2, b_2 + v_2\} + \dots + \max\{a_d, b_d + v_d\}).$$

(1) $\Rightarrow$ (2) This part of the proof is divided into Lemmas 8–9.

Lemma 8. *Under condition (1), both B_1 and B_2 are convex d -cones.*

Proof. Since any nonempty intersection $B_1 \cap (v + B_2)$ is a bounded set, the characteristic cones $C(B_1)$ and $C(B_2)$ satisfy the equality $C(B_1) \cap C(B_2) = \{0\}$. In this case we can choose a vector $x \in E^d$ such that B_1 and $x + B_2$ are disjoint. Without loss of generality, we may assume that B_1 and B_2 are disjoint themselves. Then there is a pair of parallel hyperplanes, H_1 and H_2 , each separating B_1 and B_2 such that $H_1 \cap B_1$ and $H_2 \cap B_2$ are exposed points of B_1 and B_2 , respectively (see [3]). Let $H_1 \cap B_1 = \{p\}$ and $H_2 \cap B_2 = \{q\}$. From Lemma 3 it follows that there is a real number $r > 0$ such that q is the only exposed point of B_2 in $U_r(q)$. From here we conclude that B_2 is locally conic at q , i.e., there is a number $\mu > 0$ such that for any point $x \in \text{bd } B_2 \cap U_\mu(q)$ the line segment $[q, x]$ lies in $\text{bd } B_2$. We are going to prove that $B_2 = T_q(B_2)$, i.e., that B_2 is a d -cone with apex q .

Assume, for contradiction, that B_2 is not a d -cone with apex q . Then we can choose an exposed half-line l of the cone $T_q(B_2)$ that does not lie in B_2 . Since B_2 is locally conic at q , the intersection $l \cap B_2$ is a nontrivial exposed segment, $[q, t]$, of B_2 . Choose an arbitrary point $v \in \text{int } B_1$ and consider the body $B'_2 = (v - q) + B_2$. By the above, B'_2 is locally conic at v and the segment $[v, w] = (v - q) + [q, t]$ is an exposed segment of B'_2 . Let H be a hyperplane in E^d with the property $H \cap B'_2 = [v, w]$.

For any real number $\varepsilon \geq 0$, put

$$B'_2(\varepsilon) = \varepsilon(v - w) + B'_2, \quad L(\varepsilon) = \varepsilon(v - w) + [v, w], \\ v(\varepsilon) = \varepsilon(v - w) + v, \quad w(\varepsilon) = \varepsilon(v - w) + w.$$

Clearly, $L(\varepsilon)$ is an exposed line segment of $B'_2(\varepsilon)$ such that $B'_2(\varepsilon) \cap H = L(\varepsilon)$. Moreover, $v(\varepsilon)$ and $w(\varepsilon)$ are the endpoints of $L(\varepsilon)$.

We claim that the segment $L(0) = [v, w]$ does not lie in B_1 . Indeed, assume for a moment that $L(0) \subset B_1$. Since $v(0) = v \in \text{int } B_1$, one can choose a number $\varepsilon > 0$ so small that $L(\varepsilon)$ lies in the interior of B_1 . In this case, by an argument similar to that from the proof of Lemma 3, we get a contradiction.

Hence, by a continuity argument, there is a real number $\varepsilon_1 > 0$ such that $w(\varepsilon_1)$ belongs to the boundary of B_1 and $L(\varepsilon_1)$ intersects $\text{int } B_1$. As above, $L(\varepsilon_1)$ does not lie in B_1 ; whence its endpoint $v(\varepsilon_1)$ is not in B_1 . Continuously decreasing the value of ε , we get a real number ε_0 such that $v(\varepsilon_0)$ belongs to $\text{bd } B_1$ and $L(\varepsilon_0)$ intersects $\text{int } B_1$. Similarly, continuously increasing the number $\varepsilon \geq \varepsilon_1$, we get the value ε_2 such that $B_1 \cap L(\varepsilon_2) = w(\varepsilon_2)$. As a result, the line through v and w intersects the boundary of B_1 at two points, v^* and w^* , such that $v^* = v(\varepsilon_0) = w(\varepsilon_2)$ and $w^* = w(\varepsilon_1)$.

Now consider the intersections $C(\varepsilon) = B_1 \cap B'_2(\varepsilon)$, $0 \leq \varepsilon < \varepsilon_2$. By condition (1), every $C(\varepsilon)$ is a convex body homothetic to B , and $T(\varepsilon) = H \cap C(\varepsilon)$ is an exposed line segment of $C(\varepsilon)$. Clearly, $T(\varepsilon) = L(\varepsilon) \cap C(\varepsilon)$ and

$$T(\varepsilon) = \begin{cases} [v(\varepsilon), w^*] & \text{if } 0 \leq \varepsilon \leq \varepsilon_0, \\ [v^*, w^*] & \text{if } \varepsilon_0 \leq \varepsilon \leq \varepsilon_1, \\ [v^*, w(\varepsilon)] & \text{if } \varepsilon_1 \leq \varepsilon < \varepsilon_2. \end{cases}$$

Choose a half-line m with the endpoint 0, which lies in the characteristic cone $C(B_2)$ and put

$$M(\varepsilon) = \begin{cases} B_1 \cap (v(\varepsilon) + m) & \text{if } 0 \leq \varepsilon \leq \varepsilon_0, \\ B_1 \cap (v^* + m) & \text{if } \varepsilon_0 \leq \varepsilon < \varepsilon_2. \end{cases}$$

Since $C(B_1) \cap C(B_2) = \{0\}$, each $M(\varepsilon)$ is a line segment. It is easily seen that $M(\varepsilon)$ is a chord of $C(\varepsilon)$ in the direction m and its position is uniquely determined by $T(\varepsilon)$. Moreover, $M(\varepsilon)$ and $T(\varepsilon)$ have a common endpoint: it is $v(\varepsilon)$ if $0 \leq \varepsilon \leq \varepsilon_0$, and it is v^* if $\varepsilon_0 \leq \varepsilon < \varepsilon_2$. Since $v(0) = v \in \text{int } B_1$, the length of $M(0)$ is positive.

Now, consider the size of $C(\varepsilon)$ when ε increases monotonously from 0 to ε_2 . Since all the bodies $C(\varepsilon)$ are homothetic to B , the size of $C(\varepsilon)$ is proportional to the length of $T(\varepsilon)$. Hence the size of $C(\varepsilon)$ increases when ε increases from 0 to ε_0 ; it remains constant when ε increases from ε_0 to ε_1 ; and it decreases to 0 when ε increases from ε_1 to ε_2 .

From the definition of the chord $M(\varepsilon)$ it follows that its length should also be proportional to the length of $T(\varepsilon)$. This is not the case since the length of $M(\varepsilon)$ increases to the length of $M(\varepsilon_0)$ when ε increases from 0 to ε_0 and then remains constant when ε increases from ε_0 to ε_2 . The obtained contradiction shows that our assumption on B_2 is not true, whence B_2 is a cone with apex q . In a similar way, B_1 is a convex d -cone with apex q . $\square$

Lemma 9. *Under condition (1), both B_1 and B_2 are simplicial d -cones and B is a d -simplex.*

Proof. By Lemma 8, both B_1 and B_2 are convex d -cones. Without loss of generality, we may assume that 0 is their common apex. Since any nonempty intersection $B_1 \cap (v + B_2)$, $v \in E^d$, is bounded, we have $B_1 \cap B_2 = \{0\}$. Let H be a hyperplane separating B_1 and

B_2 such that $B_1 \cap H = B_2 \cap H = \{0\}$. We divide the proof of Lemma 9 into a sequence of claims below. $\square$

Claim 1. *There exists a line through 0 that lies in $B_1 \cup B_2$.*

Proof of Claim 1. Assume, for contradiction, that no line through 0 lies in $B_1 \cup B_2$. Then $B_1 \cap (-B_2) = \{0\}$. Let G be a hyperplane separating B_1 and $-B_2$ such that $B_1 \cap G = (-B_2) \cap G = \{0\}$. Then both B_1 and B_2 lie in the same closed half-plane determined by G and $B_1 \cap G = B_2 \cap G = \{0\}$.

Choose a point $v \in \text{int } B_1$ and consider the intersection $C_1 = B_1 \cap (v + B_2)$. Clearly, the tangent cone $T_v(C_1)$ equals $v + B_2$. Similarly, for a given point $w \in \text{int } B_2$ and the intersection $C_2 = B_1 \cap (B_2 - w)$, the tangent cone $T_w(C_2)$ equals B_1 . Due to condition (1), the bodies C_1 and C_2 are homothetic. Then from the definition of C_1 and C_2 it follows that $v \in \exp C_1$ and $0 \in \exp C_2$ correspond to each other. As a result, $T_v(C_1)$ is a translate of $T_w(C_2)$. The last is impossible since B_1 is not a translate of B_2 . $\square$

Claim 2. *No line through 0 that lies in $B_1 \cup B_2$ meets $\text{int } B_1 \cup \text{int } B_2$.*

Proof of Claim 2. Assume, for contradiction, the existence of a line l through 0 that lies in $B_1 \cup B_2$ and meets $\text{int } B_1 \cup \text{int } B_2$. Let, for example, $l \cap \text{int } B_1 \neq \emptyset$. Then l intersects $\text{int } B_1 \cap (-B_2)$, and, by a continuity argument, there is a line l' through 0 that intersects $\text{int } B_1 \cap (-\text{int } B_2)$.

Now choose unit vectors $v \in l \cap \text{int } B_1$, $v' \in l' \cap \text{int } B_1$ and consider the intersections $C = B_1 \cap (v + B_2)$, $C' = B_1 \cap (v' + B_2)$. Since $0 \in C_1 \cap C_2$, both C_1 and C_2 contain more than a single point, and, by condition (1), C and C' are convex bodies homothetic to B .

Clearly, C lies between the hyperplanes H and $v + H$ such that $C \cap H = \{0\}$ and $C \cap (v + H) = \{v\}$. Similarly, C' lies between H and $v' + H$ such that $C' \cap H = \{0\}$ and $C' \cap (v' + H) = \{v'\}$. Since C and C' are homothetic, one should have that the line segments $[v, 0]$ and $[v', 0]$ are parallel. This is not the case by the choice of the lines l and l' . The obtained contradiction proves Claim 2. $\square$

Let l be a line through 0 that lies in $B_1 \cup B_2$. By Claim 2, $l \subset \text{bd } B_1 \cup \text{bd } B_2$. Denote by S_i the part of $\text{bd } B_i$ that is X -rayed by l , $i = 1, 2$. Namely, S_i is the set of points $x \in \text{bd } B_i$ such that the line $x + l$ intersects $\text{int } B_i$. In Claim 3 we denote by n_z the outward unit normal to B_i at a regular point $z \in S_i$.

Claim 3. *Each set S_i , $i = 1, 2$, lies in a hyperplane through 0.*

Proof of Claim 3. Assume, for contradiction, that S_1 does not lie in a hyperplane. Then one can choose two regular points $v, w \in S_1$ with distinct outward unit normals n_v and n_w , respectively. Let $r > 0$ be a real number such that the balls $U_r(n_v)$ and $U_r(n_w)$ are disjoint. Furthermore, let $\mu > 0$ be a real number such that $n_x \in U_r(n_v)$ for any point $x \in S \cap U_\mu(v)$ and $n_x \in U_r(n_w)$ for any point $x \in S \cap U_\mu(w)$.

Since $B_1 \cap (v + B_2) = \{v\}$ and $B_1 \cap (w + B_2) = \{w\}$, we can choose two points $v', w' \in \text{int } B_1$ so close to v, w , respectively, that $B_1 \cap (v' + B_2) \subset U_\mu(v)$ and $B_1 \cap (w' + B_2) \subset U_\mu(w)$. Put $C_1 = B_1 \cap (v' + B_2)$ and $C_2 = B_1 \cap (w' + B_2)$. From the consideration above it follows that v is a regular point of C_1 and $n_x \in U_r(n_v)$ for any regular point x of B_1 that lies in the boundary of C_1 . Moreover, any regular point x of C_1 whose outward unit normal n_x to C_1 satisfies the condition $n_x \in U_r(n_v)$ is a regular point of B_1 .

By the same argument, w is a regular point of C_2 and $n_x \in U_r(n_w)$ for any regular point x of B_1 that lies in the boundary of C_2 . Also, any regular point x of C_2 whose outward unit normal n_x to C_2 satisfies the condition $n_x \in U_r(n_w)$ is a regular point of B_1 .

As a result $U_r(v) \cap U_r(w) = \emptyset$. This is in contradiction with the fact that both C_1 and C_2 are homothetic to B . Hence S_1 lies in a hyperplane through 0. Similarly, S_2 lies in a hyperplane through 0. $\square$

Claim 4. *The closure $T_i = \text{cl } S_i$ of S_i is a convex $(d - 1)$ -cone with apex 0. Moreover, there is a hyperplane D supporting both B_1 and B_2 such that $B_1 \cup B_2$ lies in a closed half-space determined by D and the intersection $D \cap \text{bd } B_i$ is an exposed half-line of B_i , $i = 1, 2$.*

Proof of Claim 4. From the definition of S_i it easily follows that $0 \in T_i$. Since B_i is a cone with apex 0, the set T_i is also a cone with apex 0. From Claim 3 we conclude that T_i is a convex $(d - 1)$ -cone.

Denote by M the hyperplane through 0 orthogonal to the line l . If the orthogonal projection R_i of B_i onto M contained a line, then T_i would also contain a line. The last is impossible because B_i is line-free. Hence R_i is a line-free convex cone and its closure $\text{cl } R_i$ is a convex acute cone in M .

By the above, there is a hyperplane D containing l such that $\text{cl } R_1 \cap D = \{0\}$ and, as a result, $B_1 \cap D = \{l_1\}$. In other words, l_1 is an exposed half-line of B_1 . It remains to show that $D \cap B_2$ is an exposed half-line of B_2 . For this, choose any points $v \in \text{int } B_1$, $w \in \text{int } B_2$ and consider the intersections $C' = B_1 \cap (v + B_2)$, $C'' = B_1 \cap (-w + B_2)$. Clearly, the hyperplane $D - w$ supports C'' such that $(D - w) \cap C''$ is an exposed line segment of C'' parallel to l . Since C' and C'' are homothetic bodies, $v + D$ supports C' along an exposed line segment parallel to l . Hence $D \cap B_2$ is an exposed half-line of B_2 . $\square$

Claim 5. $B_i = \text{conv}(l_i \cup T_i)$, $i = 1, 2$.

Proof of Claim 5. Choose any points $v \in \text{int } B_1$ and $w \in \text{int } B_2$ so close to the facets T_1 and T_2 , respectively, that $v + B_2$ intersects the boundary of B_1 within $\text{rint } T_1$ and B_1 intersects the boundary of $B_2 - w$ within $\text{rint } T_2 - w$. Now consider the intersections $C' = B_1 \cap (v + B_2)$ and $C'' = B_1 \cap (B_2 - w)$. From the choice of v and w it follows that C' is a bounded convex cone with apex w and base $T_1 \cap (v + B_2)$. Similarly, C'' is a bounded convex cone with apex w and base $(T_2 - w) \cap B_1$. Due to condition (1), C' and C'' are homothetic. From here we conclude that any exposed half-line of B_1 is in

$l_1 \cup T_1$. Similarly, any exposed half-line of B_2 is in $l_2 \cup T_2$. Hence $B_i = \text{conv}(l_i \cup T_i)$, $i = 1, 2$. $\square$

Claim 6. *Both B_1 and B_2 are simplicial d -cones.*

Proof of Claim 6. Denote by L the hyperplane containing the facet T_1 of B_1 , and let B'_2 be any $(d - 1)$ -dimensional intersection of the form $T_1 \cap (v + B_2)$, $v \in E^d$. Since $0 \in L$, we have $w + B'_2 = L \cap (v + w + B_2)$ for any vector $w \in L$.

If T_1 and $w + B'_2$ have nonempty intersection, then B_1 and $v + w + B_2$ have nonempty intersection. Since $B_1 \cap (v + w + B_2)$ is homothetic to B , we conclude that all nonempty intersections of the form $T_1 \cap (w + B'_2)$, $w \in L$, are homothetic to B'_2 . According to Case II proved above, B'_2 is a $(d - 1)$ -simplex and T_1 is a simplicial $(d - 1)$ -cone. From here and Claim 5 we conclude that B_1 is a simplicial d -cone. Similarly, B_2 is a simplicial d -cone. $\square$

Claim 7. *B is a d -simplex.*

Proof of Claim 7. Choosing a point $v \in \text{int } B_1$ so close to T_1 that $v + B_2$ intersects the boundary of B_1 within T_1 we get that the intersection $B_1 \cap (v + B_2)$ is a bounded convex cone with apex v and base $W = T_1 \cap (v + B_2)$. As proved in Claim 6, W is a $(d - 1)$ -simplex. Hence the intersection $B_1 \cap (v + B_2)$ is a d -simplex. Finally, B is a d -simplex. $\square$

Case IV: The bodies B , B_1 , and B_2 are unbounded.

(2) $\Rightarrow$ (1) If unbounded bodies B , B_1 , B_2 satisfy condition (2), then B is a simplicial d -cone and both B_1 , B_2 are translates of B (this derives from either of the conditions (2a) or (2b)). We can represent B in a suitable coordinate system of E^d as

$$B = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq 0, \dots, \xi_d \geq 0\}.$$

If $B_1 = a + B$ and $B_2 = b + B$, with $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in E^d$, then for any vector $v = (v_1, \dots, v_d)$,

$$B_1 \cap (v + B_2) = \{(\xi_1, \dots, \xi_d) \in E^d: \xi_1 \geq \max\{a_1, b_1 + v_1\}, \dots, \xi_d \geq \max\{a_d, b_d + v_d\}\}.$$

Equivalently, $B_1 \cap B_2 = w + B$, where $w = (\max\{a_1, b_1 + v_1\}, \dots, \max\{a_d, b_d + v_d\})$.

(1) $\Rightarrow$ (2) Since B is unbounded, its characteristic cone contains a half-line, l . Obviously, the characteristic cones of both B_1 and B_2 contain l ; so does the characteristic cone of any nonempty intersection $B_1 \cap (v + B_2)$, $v \in E^d$. From here and condition (1) we conclude that any nonempty intersection $B_1 \cap (v + B_2)$, $v \in E^d$, is d -dimensional. The remaining part of the proof is divided into Lemmas 10 and 11.

Lemma 10. *Under condition (1), both B_1 and B_2 are translates of the same convex d -cone.*

Proof. Assume, for contradiction, that the conclusion of Lemma 10 is false. Then one can choose exposed points $v \in \text{exp } B_1$ and $w \in \text{exp } B_2$ that do not correspond

to each other. From Lemma 2 it follows that B has distinct exposed points, x and z , corresponding to the points v and w , respectively. By Lemma 1, none of the tangent cones $T_x(B)$, $(x - z) + T_z(B)$ lies in the other. Then none of the tangent cones $T_v(B_1)$, $(v - w) + T_w(B_2)$ lies in the other.

Consider the intersection $C = B_1 \cap (v - w + B_2)$. Since C is unbounded by the assumption on B , it is a nondegenerate homothetic copy of B . Obviously, v is an exposed point of C and $T_v(C) = T_v(B_1) \cap (v - w + T_w(B_2))$. By the above, the tangent cone $T_v(C)$ is a proper part of each of the cones $T_v(B_1)$, $(v - w) + T_w(B_2)$. From here and the fact that B and C are homothetic we conclude that B contains an exposed point, y , distinct from x, z and corresponding to the exposed point $v \in \exp C$. Moreover, $T_v(B) \subset (v - x) + T_x(B)$, a contradiction with Lemma 1. $\square$

Lemma 11. *Under condition (1), B is a simplicial d -cone and both B_1 and B_2 are translates of B .*

Proof. By Lemma 10, both B_1 and B_2 are translates of the same line-free convex cone. Let z_1 and z_2 be the apexes of B_1 and B_2 , respectively. Then the body $C = B_1 \cap (z_1 - z_2 + B_2)$ equals B_1 , and, by condition (1), it is homothetic to B . Hence B_1 is a translate of B . Similarly, B_2 is a translate of B . Summing up, we get that B satisfies equation (1), i.e., B is a Choquet simplex. From [1] it follows that B is a simplicial d -cone. $\square$

References

1. J. Bair and R. Fourneau, A characterization of unbounded Choquet simplices, *Geom. Dedicata* **6** (1977), 95–98.
2. G. Choquet, Unicité des représentations intégrales au moyen de points extrémaux dans les cônes convexes réticulés, *C. R. Acad. Sci. Paris* **243** (1956), 555–557.
3. M. De Wilde, Some properties of the exposed points of finite-dimensional convex sets, *J. Math. Anal. Appl.* **99** (1984), 257–264.
4. H. G. Eggleston, B. Grünbaum, and V. Klee, Some semicontinuity theorems for convex polytopes and cell-complexes, *Comment. Math. Helv.* **39** (1964), 165–188.
5. V. Klee, Extremal structure of convex sets, II, *Math. Z.* **69** (1958), 90–104.
6. C. A. Rogers and G. C. Shephard, The difference body of a convex body, *Arch. Math. (Basel)* **8** (1957), 220–233.
7. V. Soltan, A characterization of homothetic simplices, *Discrete Comput. Geom.* **22** (1999), 193–200.
8. S. Straszewicz, Über exponierte Punkte abgeschlossener Punktmengen, *Fund. Math.* **24** (1935), 139–143.

Received June 10, 2002, and in revised form October 11, 2002. Online publication January 30, 2003.