A Euclidean Ramsey Problem

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Abstract. Let C_n denote the set of points in R^n whose coordinates are all 0 or 1, i.e., the vertex set of the unit n -cube. Graham and Rothschild [2] proved that there exists an integer N such that for $n \geq N$, any 2-coloring of the edges of the complete graph on C_n contains a monochromatic plane K_4 . Let N^* be the minimum such N . They noted that N^* must be at least 6. Their upper bound on N^* has come to be known as *Graham's number*, often cited as the largest number that has ever been put to any practical use. In this note we show that N^* must be at least 11 and provide some experimental evidence suggesting that N^* is larger still.

1. Introduction

In [2] Graham and Rothschild prove a general Ramsey theorem, from which they derive many well-known Ramsey theorems. One consequence of their result is a Ramsey theorem for hypercubes in R^n . In this note we are interested in showing that the associated Ramsey number is larger, perhaps much larger, than previously thought.

Following the notation of [2], we let $C_n = \{(x_1, \dots, x_n) : x_i = 0 \text{ or } 1\}$ be the set of vertices of the unit n -cube in R^n . A subset S of C_n is called a k -subspace of C_n if $|S| = 2^k$ and S is contained in some k -dimensional Euclidean subspace of R^n . We use $K(C_n)$ to denote the embedded complete graph in R^n on vertex set C_n whose edges are straight line segments. In their Corollary 12, Graham and Rothschild prove the following:

Given integers k, l, r , there exists an integer $N(k, l, r)$ such that if $n \geq N(k, l, r)$ and the k -subspaces of C_n are r -colored, then there exists an l -subspace of C_n all of whose k -subspaces have one color.

Noting that the first nontrivial case is $N(1, 2, 2)$, they define N^* to be the minimum value that $N(1, 2, 2)$ can assume. They note that $N^* \geq 6$ and that their constructive proof

gives an upper bound for N^* . This upper bound has been the subject of much discussion since the publication of their paper in 1971 (see [1] for example). Special notations have to be used to specify the number concisely. One of the better known of these is Knuth's arrow notation [3], defined as follows. Let a be a positive integer. Define

$$a \uparrow a = a^a$$

and

$$a \underbrace{\uparrow \cdots \uparrow}_k a = a \overbrace{\uparrow \cdots \uparrow}^{k-1 \text{ arrows}} a$$

for $k > 1$. Then Graham's number is $3 \underbrace{\uparrow \cdots \uparrow}_{64 \text{ arrows}} 3$.

2. A Counting Theorem on $K(C_n)$

Recall that the *Hamming distance* between two vertices of the cube is the number of coordinates at which the two vertices differ. We call an edge a *k-edge* if the Hamming distance between the two vertices of the edge is k , and define the function $k(e)$ on the edge set by letting $k(e) = k$ if e is a k -edge. For a vertex v , we call the set of coordinate positions where v has a coordinate of 1, the *support* of v , denoted $S(v)$. We prove the following theorem.

Theorem 1. *If e is a k -edge in $K(C_n)$, then e is on*

$$2^{k-1} + 2^{n-k} - 2$$

plane K_4 's, and the total number of plane K_4 's in $K(C_n)$ is

$$2^{n-3}(3^n - 2^{n+1} + 1).$$

Proof. Let $e = uv$ be a k -edge. Without loss of generality we can take u to be $\mathbf{0}$ (the all-zero vector). The plane K_4 's containing e can be classified as either of two types: (1) those in which e is a diagonal, and (2) those in which e is a boundary edge. For those of type (1), the other two vertices of the K_4 must have supports which are nonempty and which together partition the support of v . Thus there are

$$\frac{2^k - 2}{2}$$

K_4 's of type (1).

In the plane K_4 's where e is a boundary edge, there is another boundary edge incident with $\mathbf{0}$ and some vertex w . Then $S(v)$ and $S(w)$ are disjoint and their union gives the support of the fourth vertex of the plane K_4 . Thus there are

$$2^{n-k} - 1$$

choices for w and hence that same number of plane K_4 's using e as a boundary edge. Now sum the expressions for the two types of plane K_4 's containing e , and we have proved the first assertion.

Since the number of k -edges is $\binom{n}{k}2^{n-1}$, the total number of plane K_4 's is

$$\sum_{k=1}^n \binom{n}{k} 2^{n-1} \left(\frac{2^k - 2}{2} + 2^{n-k} - 1 \right).$$

Using the fact that

$$\sum_{k=1}^n \binom{n}{k} 2^k = 3^n,$$

the above expression easily reduces to the desired quantity. $\square$

3. A New Lower Bound

To improve the lower bound we used a surprisingly simple deterministic algorithm. In fact, it is hard to imagine a simpler algorithm than the one described below.

Theorem 2. $N^* \geq 11$.

Proof. A list of edges is in dictionary order if $u_1v_1 < u_2v_2$ if and only if $u_1 < u_2$ or $u_1 = u_2$ and $v_1 < v_2$. The following algorithm colors $K(C_n)$, for $n \leq 10$:

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create a fixed list of edges in dictionary order
color all edges red
while there are monochromatic plane  $K_4$ 's
  for each edge  $e$  in the list
    if  $e$  can be recolored without creating plane  $K_4$ 's then
      recolor  $e$ 
    end if
  end for
end while

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The colorings produced for $n = 10$ would be difficult to display here, since there are 523,776 edges whose color needs to be specified. Instead we have made colorings for $n \leq 10$, as well as some colorings discussed in the next section, available on the World Wide Web at

<http://isu.indstate.edu/ge/GEOMETRY>.

In the first step of the algorithm, it appears that almost any ordering of the edges will do. For example, when $n = 7$ we generated over 10,000 random edge orderings and they all led to a good colorings. $\square$

4. Experimental Evidence

A case can be made that the true value of N^* is likely to be substantially larger than 11. However, due the computational limitations, both in running time and in memory use, we were unable to mount a serious attempt at coloring $K(C_{11})$.

The running time for each case increased rapidly with n . For $n = 10$, it took over 30 hours on a 1.4 GHz CPU to produce a good coloring. Based on the rate of increase, we estimate that it might take over 1000 hours of processor time to complete the coloring. Furthermore, in order to allow the program to run even that fast we need some large data structures. The largest of these is a list of plane K_4 's containing each edge. Since each plane K_4 contains six edges, Theorem 1 implies that this data structure has roughly $6 \times 2^{n-3}(3^n - 2^{n+1} + 1)$ entries, which means that for $n = 11$, the data structure would occupy over $\frac{1}{2}$ gigabyte of memory. These requirements put the case $n = 11$ just out of range of this writer's computing resources.

The fact that so simple an algorithm was able to produce a good coloring seems to support the view that the true value of N^* is much larger. In fact, we tried a variety of coloring algorithms, and nearly all of them worked, the one described above is merely the simplest.

In order to attempt to convince the skeptical reader that N^* is much larger than 11, we devised the following experiment. We begin with one of our good colorings of $K(C_{10})$ and attempt to recolor some of the edges with two colors. When an edge is assigned two colors, we mean that it may be part of a plane K_4 in either color. So the presence of bicolored edges should make it harder to complete the coloring. Our feeling was that if we are able to color $K(C_{10})$ with many bicolored edges, we have further evidence suggesting that N^* is greater than 11. So we wrote a simple optimization algorithm that attempted to maximize the number of bicolored edge, and used it on known good colorings for $n < 11$. The results of the experiment are summarized in Table 1. In the column labeled *bicolored* we indicate that maximum number of bicolored edges we were able to obtain in a good coloring of $K(C_{10})$. The total number of edges and the fraction of edges that were bicolored are also given. The results for $n = 8$ appear to be an anomaly for which we have no explanation.

The interested reader can obtain computer representations of the colorings that give the results in the table from the aforementioned web site.

We conclude by offering the opinion that were $N^* = 11$, our ability to bicolor nearly one-third of the edges for $n = 10$ would be very surprising.

Table 1. Results of the experiment.

n	Bicolored	$ E(Q_n) $	Ratio
3	20	28	0.714
4	74	120	0.617
5	294	496	0.593
6	973	2,016	0.483
7	3,786	8,128	0.466
8	10,327	32,640	0.316
9	52,002	130,816	0.398
10	170,404	523,776	0.325

References

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