Imbedded Circle Patterns with the Combinatorics of the Square Grid and Discrete Painlevé Equations*

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Abstract. A discrete analogue of the holomorphic maps z' and $\log(z)$ is studied. These maps are given by Schramm's circle pattern with the combinatorics of the square grid. It is shown that the corresponding circle patterns are imbedded and described by special separatrix solutions of discrete Painlevé equations. Global properties of these solutions, as well as of the discrete z' and $\log(z)$, are established.

1. Introduction

The theory of circle patterns is a fast developing field of research on the border of complex analysis and discrete geometry. Recent progress in this area has its origin in Thurston's idea [20] about approximating the Riemann mapping by circle packings. By now some aspects of the theory of circle patterns, such as discrete analogues of conformal mappings, are well understood, while others are still waiting to be clarified.

Classical circle packings comprised of disjoint open disks were later generalized to circle patterns where the disks may overlap (see for example [11]). Different underlying combinatorics were considered. Schramm introduced a class of circle patterns with the combinatorics of the square grid [18]; hexagonal circle patterns with constant multi-ratios were studied by Bobenko et al. in [5]; a new rich class of hexagonal patterns with constant intersection angles was investigated in [4].

The convergence of discrete conformal maps represented by circle packings was proven by Rodin and Sullivan [17]. This result was refined for prescribed regular combinatorics. He and Schramm [12] showed that for hexagonal packings the convergence is

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C^∞ . Uniform convergence for circle patterns with the combinatorics of the square grid and orthogonal neighboring circles was established by Schramm [18].

Other facts underlining the striking analogy between circle patterns and classical theory are the uniformization theorem concerning circle packing realizations of cell complexes of prescribed combinatorics [2], the discrete maximum principle, Schwarz's lemma and rigidity properties [15], [11], and the discrete Dirichlet principle [18].

It turned out that an effective approach to the description and the construction of circle patterns with overlapping circles is given by the theory of integrable systems (see [7], [4] and [5]). In particular, Schramm's circle patterns studied in this paper are governed by a difference equation which is the stationary Hirota equation (see [18]). This equation is an example of an integrable difference equation. It first appeared in a different branch of mathematics—the theory of integrable systems (see [21] for a good survey).

On the other hand, not very much is known about analogues of standard holomorphic functions, although computer experiments give evidence for their existence [9]. For circle packings with hexagonal combinatorics the only explicitly described examples are Doyle spirals, which are discrete analogues of exponential maps [8], and conformally symmetric packings, which are analogues of a quotient of Airy functions [3]. For patterns with overlapping circles more explicit examples are known: discrete versions of $\exp(z)$, $\operatorname{erf}(z)$ [18], z^γ , $\log(z)$ [1] are constructed for patterns with underlying combinatorics of the square grid; z^γ , $\log(z)$ are also described for hexagonal patterns with both multi-ratio [5] and constant angle [4] properties.

Whereas computer experiments reveal the regular behavior of circle patterns corresponding to discrete z^γ and $\log(z)$, only the local property of immersion was proved for Schramm's patterns [1]. This property turned out to be connected with special solutions of discrete Painlevé II equations, thus giving a geometrical interpretation thereof. The aim of this paper is to prove the global property of imbeddedness for the square grid circle patterns corresponding to z^γ and $\log(z)$ (which was conjectured in [1]).

To visualize the analogy between Schramm's circle patterns and conformal maps, consider regular patterns composed of unit circles and suppose that the radii are being deformed to preserve the orthogonality of neighboring circles and the tangency of half-neighboring ones. Discrete maps taking the intersection points of the unit circles of the standard regular patterns to the respective points of the deformed patterns mimic classical holomorphic functions, the deformed radii being analogous to $|f'(z)|$ (see Fig. 1).

It is easy to show that the lattice comprised of the centers of circles of Schramm's pattern and their intersection points is a special discrete conformal mapping (see Definition 1 below). The latter were introduced in [6] in the framework of discrete integrable geometry, originally without any relation to circle patterns.

Definition 1. A map $f: \mathbf{Z}^2 \rightarrow \mathbf{R}^2 = \mathbf{C}$ is called a discrete conformal map if all its elementary quadrilaterals are conformal squares, i.e., their cross-ratios are equal to -1 :

$$\begin{aligned} q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) &:= \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} \\ &= -1. \end{aligned} \quad (1)$$

Fig. 1. Schramm's circle patterns as a discrete conformal map. The discrete version of the holomorphic mapping $z^{3/2}$ is shown.

This definition is motivated by the following properties:

1. The definition is Möbius invariant.
2. A smooth map $f: D \subset \mathbf{C} \rightarrow \mathbf{C}$ is conformal (holomorphic or antiholomorphic) if and only if

$$\lim_{\varepsilon \rightarrow 0} q(f(x, y), f(x + \varepsilon, y), f(x + \varepsilon, y + \varepsilon), f(x, y + \varepsilon)) = -1$$

for all $(x, y) \in D$. For some examples see [6] and [13]. A naive method to construct a discrete analogue of the function $f(z) = z^\gamma$ is to start with $f_{n,0} = n^\gamma$, $n \geq 0$, $f_{0,m} = (im)^\gamma$, $m \geq 0$ and then compute $f_{n,m}$ for any $n, m > 0$ using (1). However, a map so determined has a behavior which is far from that of usual holomorphic maps. Different elementary quadrilaterals overlap (see the left lattice in Fig. 2).

Definition 2. A discrete conformal map $f_{n,m}$ is called imbedded if inner parts of different elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ do not intersect.

Fig. 2. Two discrete conformal maps with close initial data $n = 0, m = 0$. The second lattice describes a discrete version of the holomorphic mapping $z^{2/3}$.

The condition for a discrete conformal map to be imbedded can be relaxed as follows.

Definition 3. A discrete conformal map $f_{n,m}$ is called an immersion if inner parts of adjacent elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are disjoint.

To construct an imbedded discrete analogue of z^γ , which is the right lattice presented in Fig. 2, a more complicated approach is needed. Equation (1) can be supplemented with the following nonautonomous constraint:

$$\begin{aligned} \gamma f_{n,m} = & 2n \frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})} \\ & + 2m \frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}, \end{aligned} \quad (2)$$

which plays a crucial role in this paper. This constraint, as well as its compatibility with (1), is derived from some monodromy problem (see [1]). We assume $0 < \gamma < 2$ and denote $\mathbf{Z}_+^2 = \{(n, m) \in \mathbf{Z}^2: n, m \geq 0\}$. Motivated by the asymptotics of the constraint (2) at $n, m \rightarrow \infty$ and the properties

$$z^\gamma(\mathbf{R}_+) \in \mathbf{R}_+, \quad z^\gamma(i\mathbf{R}_+) \in e^{\gamma\pi i/2}\mathbf{R}_+$$

of the holomorphic mapping z^γ we use the following definition [7] of the “discrete” z^γ .

Definition 4. The discrete conformal map $Z^\gamma: \mathbf{Z}_+^2 \rightarrow \mathbf{C}$, $0 < \gamma < 2$, is the solution of (1), (2) with the initial conditions

$$Z^\gamma(0, 0) = 0, \quad Z^\gamma(1, 0) = 1, \quad Z^\gamma(0, 1) = e^{\gamma\pi i/2}. \quad (3)$$

Obviously, $Z^\gamma(n, 0) \in \mathbf{R}_+$ and $Z^\gamma(0, m) \in e^{\gamma\pi i/2}(\mathbf{R}_+)$ for any $n, m \in \mathbf{N}$.

A limit of rescaled Z^γ as γ approaches 2 gives discrete Z^2 . Discrete Log is defined as dual to Z^2 (see Section 4 for details). Figure 2 suggests that Z^γ is imbedded. The corresponding theorem is the main result of this paper.

Theorem 1. *The discrete maps Log and Z^γ for $0 < \gamma \leq 2$ are imbedded.*

The proof is based on the analysis of geometric and algebraic properties of the corresponding lattices. In Section 2 a brief review of the necessary results from [1] is given. It is shown that Z^γ corresponds to a circle pattern of Schramm’s type. (The circle pattern corresponding to $Z^{2/3}$ is presented in Fig. 3.) Next, analyzing the equations for radii of the circles we show that in order to prove that Z^γ is imbedded it is enough to establish a special property of the separatrix solutions $P_{N,M}$, $Q_{N,M}$ of the following infinite

Fig. 3. Schramm's circle pattern corresponding to $Z^{2/3}$.

sequence of systems of ordinary difference equations of Painlevé type (N labels the system):

$$Q_{N,M+1} = \frac{(M+N)P_{N,M}(P_{N,M} - Q_{N,M}^2) - (M-N)Q_{N,M}^2(1 + P_{N,M})}{Q_{N,M}((M+N)(Q_{N,M}^2 - P_{N,M}) - (M-N)P_{N,M}(1 + P_{N,M}))},$$

$$P_{N,M+1} = \frac{(2M+\gamma)P_{N,M} + (2N+\gamma)Q_{N,M}Q_{N,M+1}}{(2(N+1)-\gamma)P_{N,M} + (2(M+1)-\gamma)Q_{N,M}Q_{N,M+1}}.$$

Namely, it is shown that Z^γ is imbedded if the infinite sequence of solutions $Q_{N,M}$, $P_{N,M}$ with special initial data $Q_{N,N}$, $P_{N,N}$ is subject to

$$(\gamma - 1)(Q_{N,M}^2 - P_{N,M}) \geq 0, \quad Q_{N,M} > 0, \quad P_{N,M} > 0.$$

The existence and uniqueness of the corresponding initial data are given in Section 3. In Section 4 discrete Log and Z^2 are defined and the proof of Theorem 1 for these maps is completed.

For $N = 0$ the system for $Q_{N,M}$, $P_{N,M}$ reduces to the special case of discrete Painlevé equation dPII:

$$(n+1)(x_n^2 - 1) \left(\frac{x_{n+1} - ix_n}{i + x_n x_{n+1}} \right) - n(x_n^2 + 1) \left(\frac{x_{n-1} + ix_n}{i + x_{n-1} x_n} \right) = \gamma x_n \quad (4)$$

(see [10] and [16] for more examples). In [1] this equation was the main tool to prove the immersion of Z^γ : it was shown that Z^γ is immersed if the unitary solution $x_n = e^{i\alpha_n}$ of this equation with $x_0 = e^{i\gamma\pi/4}$ lies in the sector $0 < \alpha_n < \pi/2$.

Similar problems have been studied in the framework of the isomonodromic deformation method [14]. In particular, connection formulas were derived. These formulas describe the asymptotics of solutions for $n \rightarrow \infty$ as a function of initial conditions (see in particular [10]). These methods seem to be insufficient for our purposes since we need to control the behavior of solutions for finite n 's as well. The geometric origin of our equations permits us to prove the abovementioned properties by purely geometric methods.

Fig. 4. Surface of glued quadrilaterals of an immersed but nonimbedded discrete map.

To illustrate the difference between the immersed and imbedded discrete conformal maps, imagine that the elementary quadrilaterals of the map are made of elastic inextensible material and glued along the corresponding edges to produce a surface with a border. If this surface is immersed it is locally flat. Being dropped down it will not have folds. At first sight it seems to be sufficient to guarantee imbeddedness, provided $Z_{n,0} \rightarrow \infty$ and $Z_{0,m} \rightarrow \infty$ as $n \rightarrow \infty$ (which follows from $Z_{n,0}^\gamma = (2c(\gamma)/\gamma)(n/2)^\gamma(1 + O(1/n^2))$, $n \rightarrow \infty$, see [1]). However, a surface with such properties may still have some limit curve with self-intersections thus giving overlapping quadrilaterals. A hypothetical example of such a surface is shown in Fig. 4.

2. Circle Patterns and Z^γ

We recall the necessary results from [1]. Z^γ of Definition 4 determines a special case of circle patterns with the combinatorics of the square grid as defined by Schramm in [18]. Indeed, discrete $f_{n,m}$ satisfying (1) and (2) with initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$ has the equidistant property

$$f_{2n,0} - f_{2n-1,0} = f_{2n+1,0} - f_{2n,0}, \quad f_{0,2m} - f_{0,2m-1} = f_{0,2m+1} - f_{0,2m}$$

for any $n \geq 1$, $m \geq 1$. Given initial $f_{0,0}$, $f_{0,1}$ and $f_{1,0}$ the constraint (2) allows one to compute $f_{n,0}$ and $f_{0,m}$ for all $n, m \geq 1$. Now using (1) one can successively compute $f_{n,m}$ for any $n, m \in \mathbf{N}$. Observe that if $|f_{n+1,m} - f_{n,m}| = |f_{n,m+1} - f_{n,m}|$, then the quadrilateral $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ is of kite form—it is inscribed in a circle and is symmetric with respect to the diameter of the circle $[f_{n,m}, f_{n+1,m+1}]$. If the angle at the vertex $f_{n,m}$ is $\pi/2$, then the quadrilateral $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ is of kite form too. In this case the quadrilateral is symmetric with respect to its diagonal $[f_{n,m+1}, f_{n+1,m}]$.

Proposition 1 [1]. *Let $f_{n,m}$ satisfy (1) and (2) in $\mathbf{Z}_+^2$ with initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$. Then all the elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are of kite form. All edges at the vertex $f_{n,m}$ with $n + m \equiv 0 \pmod{2}$ are of the same length*

$$|f_{n+1,m} - f_{n,m}| = |f_{n,m+1} - f_{n,m}| = |f_{n-1,m} - f_{n,m}| = |f_{n,m-1} - f_{n,m}|.$$

All angles between the neighboring edges at the vertex $f_{n,m}$ with $n + m = 1 \pmod{2}$ are equal to $\pi/2$.

Proposition 1 implies that for any n, m : $n + m = 0 \pmod{2}$ the points $f_{n+1,m}$, $f_{n,m+1}$, $f_{n-1,m}$, $f_{n,m-1}$ lie on a circle with the center $f_{n,m}$.

Corollary 1 [1]. *The circumscribed circles of the quadrilaterals $(f_{n-1,m}, f_{n,m-1}, f_{n+1,m}, f_{n,m+1})$ with $n + m = 0 \pmod{2}$ form a circle pattern of Schramm type (see [18]), i.e. the circles of neighboring quadrilaterals intersect orthogonally and the circles of half-neighboring quadrilaterals with a common vertex are tangent (see Fig. 3).*

Consider the sublattice $\{n, m: n + m = 0 \pmod{2}\}$ and denote by $\mathbf{V}$ its quadrant

$$\mathbf{V} = \{z = N + iM: N, M \in \mathbf{Z}^2, M \geq |N|\},$$

where

$$N = (n - m)/2, \quad M = (n + m)/2.$$

We use complex labels $z = N + iM$ for this sublattice. Denote by $C(z)$ the circle of the radius

$$R(z) = |f_{n,m} - f_{n+1,m}| = |f_{n,m} - f_{n,m+1}| = |f_{n,m} - f_{n-1,m}| = |f_{n,m} - f_{n,m-1}| \quad (5)$$

with the center at $f_{N+M, M-N} = f_{n,m}$. From Proposition 1 it follows that any two circles $C(z)$, $C(z')$ with $|z - z'| = 1$ intersect orthogonally and any two circles $C(z)$, $C(z')$ with $|z - z'| = \sqrt{2}$ are tangent.

Let $\{C(z)\}$, $z \in \mathbf{V}$, be a circle pattern of Schramm type on the complex plane. Define $f_{n,m}: \mathbf{Z}_+^2 \rightarrow \mathbf{C}$ as follows:

- (a) if $n + m = 0 \pmod{2}$, then $f_{n,m}$ is the center of $C((n - m)/2 + i((n + m)/2))$,
- (b) if $n + m = 1 \pmod{2}$, then

$$\begin{aligned} f_{n,m} &:= C\left(\frac{n - m - 1}{2} + i\frac{n + m - 1}{2}\right) \cap C\left(\frac{n - m + 1}{2} + i\frac{n + m + 1}{2}\right) \\ &= C\left(\frac{n - m + 1}{2} + i\frac{n + m - 1}{2}\right) \cap C\left(\frac{n - m - 1}{2} + i\frac{n + m + 1}{2}\right). \end{aligned}$$

Since all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are of kite form (1) is satisfied automatically. In what follows the function $f_{n,m}$, defined as above by (a) and (b), is called a *discrete conformal map corresponding to the circle pattern $\{C(z)\}$* .

Theorem 2 [1]. *Let $f_{n,m}$ satisfying (1) and (2) with initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$ be an immersion, then $R(z)$ defined by (5) satisfies the following equations:*

$$\begin{aligned} R(z)R(z+1)(-2M - \gamma) + R(z+1)R(z+1+i)(2(N+1) - \gamma) \\ + R(z+1+i)R(z+i)(2(M+1) - \gamma) + R(z+i)R(z)(-2N - \gamma) = 0, \quad (6) \end{aligned}$$

for $z \in \mathbf{V}_l := \mathbf{V} \cup \{-N + i(N-1) | N \in \mathbf{N}\}$ and

$$(N+M)(R(z)^2 - R(z+1)R(z-i))(R(z+i) + R(z+1)) \\ + (M-N)(R(z)^2 - R(z+i)R(z+1))(R(z+1) + R(z-i)) = 0, \quad (7)$$

for $z \in \mathbf{V}_{\text{rint}} := \mathbf{V} \setminus \{\pm N + iN | N \in \mathbf{N}\}$.

Conversely let $R(z): \mathbf{V} \rightarrow \mathbf{R}_+$ satisfy (6) for $z \in \mathbf{V}_l$ and (7) for $z \in \mathbf{V}_{\text{rint}}$. Then $R(z)$ defines an immersed circle packing with the combinatorics of the square grid. The corresponding discrete conformal map $f_{n,m}$ is an immersion and satisfies (2).

From the initial condition (3) we have

$$R(0) = 1, \quad R(i) = \tan \frac{\gamma\pi}{4}. \quad (8)$$

Equation (6) at $z = N + iN$ and $z = -N + i(N-1)$, $N \in \mathbf{N}$, reads as

$$R(\pm(N+1) + i(N+1)) = \frac{2N + \gamma}{2(N+1) - \gamma} R(\pm N + iN) \quad (9)$$

and defines $R(\pm N + iN)$ for all $N \in \mathbf{N}$. Now (6) and (7) determine $R(z)$ for any other $z \in \mathbf{V}$. Besides $R(z)$ satisfies (7) at $z = -N + iN$, $N \in \mathbf{N}$, which reads

$$R(-N + 1 + iN)R(-N + i(N+1)) = R^2(-N + iN).$$

By symmetry one gets

$$R(N - 1 + iN)R(N + i(N+1)) = R^2(N + iN).$$

This equation allows us to compute $R(N + i(N+1))$. (Moreover, it implies that the center O of $C(N + iN)$ and the points $A = C(N + iN) \cap C(N - 1 + iN) \setminus C(N - 1 + iN) \cap C(N + i(N+1))$ and $B = C(N + iN) \cap C(N + i(N+1)) \setminus C(N - 1 + iN) \cap C(N + i(N+1))$ are collinear so the points $f_{n,0}$ lie on a straight line.) To prove Theorem 1 we consider the more general initial condition:

$$R(0) = 1, \quad R(i) = a \tan \frac{\gamma\pi}{4}. \quad (10)$$

Now the radii $R(N + iN)$, $R(N + i(N+1))$ can be represented in terms of the Γ -function:

$$R(N + iN) = c(\gamma) \frac{\Gamma(N + \gamma/2)}{\Gamma(N + 1 - \gamma/2)}, \quad \text{where } c(\gamma) = \frac{\gamma \Gamma(1 - \gamma/2)}{2\Gamma(1 + \gamma/2)}, \quad (11)$$

$$R(N + i(N+1)) = \left(a \tan \frac{\gamma\pi}{4} \right)^{(-1)^N} \\ \times \left(\frac{(2(N-1)+\gamma)(2(N-3)+\gamma)(2(N-5)+\gamma) \cdots}{(2N-\gamma)(2(N-2)-\gamma)(2(N-4)-\gamma) \cdots} \right)^2. \quad (12)$$

Theorem 2 allows us to reformulate the property of the circle lattice to be immersed completely in terms of the system (6), (7). Namely, to prove that Z' is an immersion one should show that the solution of the system (6), (7) with initial data (8) is positive for all $z \in \mathbf{V}$ (see [1]). To prove Theorem 1 one needs a more subtle property of this solution.

Theorem 3. *If for a solution $R(z)$ of (6), (7) with $\gamma \neq 1$ and initial conditions (8) it holds that*

$$R(z) > 0, \quad (\gamma - 1)(R(z)^2 - R(z - i)R(z + 1)) \geq 0 \quad (13)$$

in V_{int} , then the corresponding discrete conformal map (DCM) is imbedded.

Proof. Since $R(z) > 0$ the corresponding DCM is immersion (see [18] and [1]). Consider a piecewise linear curve Γ_n formed by segments $[f_{n,m}, f_{n,m+1}]$ where $n > 0$ and $0 \leq m \leq n-1$ and the vector $\mathbf{v}_n(m) = (\mathbf{f}_{n,m}, \mathbf{f}_{n,m+1})$ along this curve. Due to Proposition 1 this vector rotates only in vertices with $n + m = 0 \pmod{2}$ as m increases along the curve. The sign of the rotation angle $\theta_n(m)$, where $-\pi < \theta_n(m) < \pi$, $0 < m < n$, is defined by the sign of the expression $R(z)^2 - R(z + i)R(z + 1)$ (note that there is no rotation if this expression vanishes), where $z = (n - m)/2 + i(n + m)/2$ is a label for the circle with the center in $f_{n,m}$. If $n + m = 1 \pmod{2}$, define $\theta_n(m) = 0$.

Now the theorem hypothesis and (7) imply that the vector $\mathbf{v}_n(m)$ rotates with increasing m in the same direction for all n , and, namely, clockwise for $\gamma < 1$ and counterclockwise for $\gamma > 1$.

Consider the sector $B := \{z = re^{i\varphi} : r \geq 0, 0 \leq \varphi \leq \gamma\pi/4\}$. The terminal points of the curves Γ_n lie on the sector border.

Lemma 1. *For the curve Γ_n it holds that*

$$\left| \sum_{m=1}^{n-1} \theta_n(m) \right| < \frac{\pi}{4}(1 + |\gamma|). \quad (14)$$

Proof of Lemma 1. We prove inequality (14) for $1 < \gamma < 2$ by induction for n . For $n = 1$ the inequality is obviously true since the curve Γ_1 is a segment perpendicular to $\mathbf{R}_+$. Define the angle $\alpha_n(m)$ between $i\mathbf{R}_+$ and the vector $\mathbf{v}_n(m)$ by $f_{n,m+1} - f_{n,m} = e^{i(\alpha_n(m) + \pi/2)}|f_{n,m+1} - f_{n,m}|$, where $0 \leq \alpha_n(m) < 2\pi$, $0 \leq m < n$. Then $\sum_{m=1}^l \theta_n(m) = \alpha_n(l) - \alpha_n(0) + 2\pi k_n(l)$ for some positive integer $k_n(l)$ increasing with l . Note that $\alpha_n(0) < \pi/2$, which easily follows from (11), (12), and $\alpha_n(n-1) < (\gamma\pi/4 + \pi/2) - \pi/2 = \gamma\pi/4$, since for immersed Z^γ the angle between the vector $\mathbf{v}_n(n-1)$ and $e^{i\gamma\pi/4}\mathbf{R}_+$ is less than $\pi/2$. Let (14) hold for $n > 1$, then $\left| \sum_{m=1}^{n-1} \theta_n(m) \right| = \sum_{m=1}^{n-1} \theta_n(m) < \gamma\pi/4$ (all $\theta_n(m)$ are positive for $1 < \gamma < 2$). That implies that $k_n(l) = 0$, since $k_n(l) = (\sum_{m=1}^l \theta_n(m) - \alpha_n(l) + \alpha_n(0))/2\pi \leq (\sum_{m=1}^l \theta_n(m) + |\alpha_n(l)| + |\alpha_n(0)|)/2\pi < (\gamma\pi/4 + \gamma\pi/4 + \pi/2)/2\pi < 1$ and $k_n(l)$ is an integer. Let $\alpha_{n+1}(l) = \alpha_n(l) + \sigma_n(l)$. All elementary quadrilaterals are of kite form therefore $|\sigma_n(l)| < \pi/2$.

We prove that $k_{n+1}(m) = 0$ for $0 \leq m \leq n+1$. Obviously $k_{n+1}(0) = 0$. Assume $k_{n+1}(l) = 0$ but $k_{n+1}(l+1) > 0$. The increment of the left-hand side of

$$\sum_{m=1}^l \theta_{n+1}(m) = \alpha_{n+1}(l) - \alpha_{n+1}(0) + 2\pi k_{n+1}(l)$$

as $l \rightarrow l+1$ is $\theta_{n+1}(l+1) < \pi$. The increment of the right-hand side is no less than $2\pi + \alpha_{n+1}(l+1) - \alpha_{n+1}(l) \geq 2\pi - \alpha_{n+1}(l) \geq 2\pi - \alpha_n(l) - |\sigma_n(l)| > 2\pi -$

$\gamma\pi/4 - \pi/2 > \pi$. The obtained contradiction gives $k_{n+1}(l) = 0$ and $\sum_{m=1}^n \theta_{n+1}(m) = \alpha_{n+1}(n) - \alpha_{n+1}(0) \leq \alpha_{n+1}(n) < \gamma\pi/4$. Lemma 1 is proved. $\square$

The obvious corollary of Lemma 1 is that the curve Γ_n has no self-intersection and lies in the sector B since the rotation of the vector $\mathbf{v}_n(m)$ along the curve is less than $\gamma\pi/4 < \pi/2$. Each such curve cuts the sector B into a finite part and an infinite part. Since the curve Γ_n is convex and the borders of all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ for imbedded Z^γ have positive orientation the segments of the curve Γ_{n+1} lie in the infinite part. Now induction in n completes the proof of Theorem 3 for $1 < \gamma < 2$. The proof for $0 < \gamma < 1$ is similar. The differences are that $\theta_n(m)$ is not positive, the angle α is naturally defined as negative: $-2\pi < \alpha_n(m) < 0$, so that $-\pi/2 < \alpha_n(0) \leq 0$ and $(\gamma\pi/4)(2 - \gamma) < \alpha_n(n - 1) < 0$. Details are left to the reader. $\square$

Corollary 2.

$$\lim_{n \rightarrow \infty} Z_{n,m}^\gamma = \infty, \quad \lim_{m \rightarrow \infty} Z_{n,m}^\gamma = \infty.$$

Since the terminal points of the curves Γ_n lie on the sector border the proof easily follows from convexity of the curves Γ_n , inequality (14) and from

$$\lim_{n \rightarrow \infty} Z_{n,0}^\gamma = \infty.$$

3. Z^γ and Discrete Painlevé Equations

Let $R(z)$ be a solution of (6), (7) with initial condition (8). For $z \in \mathbf{V}_{\text{int}}$ define $P_{N,M} = P(z) = R(z+1)/R(z-i)$, $Q_{N,M} = Q(z) = R(z)/R(z-i)$. Then (6) and (7) are rewritten as follows:

$$Q_{N,M+1} = \frac{(M+N)P_{N,M}(P_{N,M} - Q_{N,M}^2) - (M-N)Q_{N,M}^2(1 + P_{N,M})}{Q_{N,M}((M+N)(Q_{N,M}^2 - P_{N,M}) - (M-N)P_{N,M}(1 + P_{N,M}))}, \quad (15)$$

$$P_{N,M+1} = \frac{(2M+\gamma)P_{N,M} + (2N+\gamma)Q_{N,M}Q_{N,M+1}}{(2(N+1)-\gamma)P_{N,M} + (2(M+1)-\gamma)Q_{N,M}Q_{N,M+1}}. \quad (16)$$

Property (13) for (15), (16) reads as

$$(\gamma-1)(Q_{N,M}^2 - P_{N,M}) \geq 0, \quad Q_{N,M} > 0, \quad P_{N,M} > 0. \quad (17)$$

Equations (15) and (16) can be considered as a dynamical system for variable M , and the expressions (11) and (12) defining initial conditions $P_{N,N+1}$, $Q_{N,N+1}$ for (15) and (16) as functions of a , N , only $Q_{N,N+1}$ being dependent on a .

Theorem 4. *There exists $a > 0$ such that for the solutions $R(z)$ of (6), (7) with initial conditions (10), (13) holds.*

Proof. Due to the following lemma it is sufficient to prove (13) only for $0 < \gamma < 1$.

Lemma 2. *If $R(z)$ is a solution of (6), (7) for γ , then $1/R(z)$ is a solution of (6), (7) for $\tilde{\gamma} = 2 - \gamma$.*

The lemma is proved by straightforward computation.

Let $0 < \gamma < 1$ and $(P_{N,M}, Q_{N,M})$ correspond to the solution of (6), (7) defined by initial conditions (10). Define the real function $F(P)$ for $P \in \mathbf{R}_+$:

$$F(P) = \sqrt{P} \quad \text{for } 0 \leq P \leq 1, \quad F(P) = 1 \quad \text{for } 1 \leq P.$$

Designate

$$\begin{aligned} D_u &:= \{(P, Q) : P > 0, Q > F(P)\}, & D_d &:= \{(P, Q) : Q < 0\}, \\ D_0 &:= \{(P, Q) : P > 0, 0 \leq Q \leq F(P)\}, & D_f &:= \{(P, Q) : P \leq 0, Q \geq 0\}. \end{aligned}$$

Now define the infinite sequences $\{q_n\}, \{p_n\}, n \in \mathbf{N}$, as follows:

$$\begin{aligned} \{q_n(a)\} &:= \{Q_{0,1}, Q_{0,2}, Q_{1,2}, Q_{0,3}, Q_{1,3}, Q_{2,3}, \dots, Q_{0,M}, Q_{1,M}, \dots, Q_{M-1,M}, \dots\}, \\ \{p_n(a)\} &:= \{P_{0,1}, P_{0,2}, P_{1,2}, P_{0,3}, P_{1,3}, P_{2,3}, \dots, P_{0,M}, P_{1,M}, \dots, P_{M-1,M}, \dots\}, \end{aligned}$$

and the sets

$$\begin{aligned} A_u(n) &:= \{a \in \mathbf{R}_+ : (p_n(a), q_n(a)) \in D_u, (p_k(a), q_k(a)) \in D_0, \forall 0 < k < n\}, \\ A_d(n) &:= \{a \in \mathbf{R}_+ : (p_n(a), q_n(a)) \in D_d, (p_k(a), q_k(a)) \in D_0, \forall 0 < k < n\}. \end{aligned}$$

$A_u(n)$ and $A_d(n)$ are open sets since the denominators of (15) and (16) do not vanish in D_0 . Indeed, the curve $Q^2 = P + ((M - N)/(M + N))P(P + 1)$ lies outside D_0 . Moreover, direct computation shows that $A_u(1) \neq \emptyset$ and $A_d(2) \neq \emptyset$, therefore the sets

$$A_u := \bigcup A_u(k), \quad A_d := \bigcup A_d(k)$$

are not empty. Finally, define

$$A_0 := \{a \in \mathbf{R}_+ : (p_n(a), q_n(a)) \in D_0, \forall n \in \mathbf{N}\}.$$

Note that A_0, A_u, A_d are mutually disjoint and the sequence $\{p_n\}, \{q_n\}$ is constructed so that

$$\mathbf{R}_+ = A_0 \cup A_u \cup A_d. \quad (18)$$

Indeed $(P_{N,M}, Q_{N,M})$ cannot jump from D_0 into D_f in one step $M \rightarrow M + 1$ since $P_{N,M+1}$ is positive for positive $P_{N,M}, Q_{N,M}, Q_{N,M+1}$. The relation (18) would be impossible for $A_0 = \emptyset$, since the connected set $\mathbf{R}_+$ cannot be covered by two open disjoint nonempty subsets A_u and A_d , therefore $A_0 \neq \emptyset$. $\square$

Proposition 2. *The set A_0 consists of only one element, namely, $A_0 = \{1\}$.*

Proof. A positive solution $R(z)$ of (6), (7) provides an immersed DCM corresponding to the circle patterns with radii $R(z)$. It was proven in [1] that the initial value (10) for immersed Z^γ consists of only one element $a = 1$. $\square$

Proof of Theorem 1 for Z^γ with $0 < \gamma < 2$. Combining Theorems 3 and 4 with Proposition 2 one easily deduces Theorem 1. $\square$

Remark. For $N = 0$, the system (15), (16) for $Q_{N,M}$, $P_{N,M}$ reduces to the special case (4) of discrete Painlevé equation dPII. For imbedded Z^γ there corresponds the unitary solution $x_n = e^{i\alpha_n}$ of this equation with $x_0 = e^{i\gamma\pi/4}$ in the sector $0 < \alpha_n < \pi/2$. Surprisingly, condition (13) does not imply that α_n is monotonous. Since the angle α_m is related to the radii by $\tan \alpha_m = R(i(m+1))/R(im)$, in terms of radii condition (13) is reformulated as

$$(\gamma - 1)(R(i(m+1)) - R(i(m-1))) \geq 0.$$

4. The Discrete Maps Z^2 and Log. Duality

This section begins with the requisite material from [1]. Definition 4 was given for $0 < \gamma < 2$. For $\gamma < 0$ or $\gamma > 2$, the radius $R(1+i) = \gamma/(2-\gamma)$ of the corresponding circle patterns becomes negative and some elementary quadrilaterals around $f_{0,0}$ intersect. However, for $\gamma = 2$, one can renormalize the initial values of f so that the corresponding map remains an immersion. We consider Z^γ , with $0 < \gamma < 2$, and make the following renormalization for the corresponding radii: $R \rightarrow ((2-\gamma)/\gamma)R$. Then as $\gamma \rightarrow 2-0$ from below we have

$$R(0) = \frac{2-\gamma}{\gamma} \rightarrow +0, \quad R(1+i) = 1, \quad R(i) = \frac{2-\gamma}{\gamma} \tan \frac{\gamma\pi}{4} \rightarrow \frac{2}{\pi}.$$

Definition 5 [1]. $Z^2: \mathbf{Z}_+^2 \rightarrow \mathbf{R}^2 = \mathbf{C}$ is the solution of (1), (2) with $\gamma = 2$ and the initial conditions

$$\begin{aligned} Z^2(0, 0) &= Z^2(1, 0) = Z^2(0, 1) = 0, \\ Z^2(2, 0) &= 1, \quad Z^2(0, 2) = -1, \quad Z^2(1, 1) = i\frac{2}{\pi}. \end{aligned}$$

In this definition, (1) are (2) are understood to be regularized through multiplication by their denominators. Note that for the radii on the border one has $R(N+iN) = N$.

Lemma 2 reflects the fact that for any discrete conformal map f there is a *dual discrete conformal map* f^* defined by (see [7])

$$f_{n+1,m}^* - f_{n,m}^* = -\frac{1}{f_{n+1,m} - f_{n,m}}, \quad f_{n,m+1}^* - f_{n,m}^* = \frac{1}{f_{n,m+1} - f_{n,m}}. \quad (19)$$

Obviously this transformation preserves the kite form of elementary quadrilaterals and therefore is well-defined for Schramm's circle patterns. The smooth limit of the duality (19) is

$$(f^*)' = -\frac{1}{f'}.$$

The dual of $f(z) = z^2$ is, up to a constant, $f^*(z) = \log z$. Motivated by this observation, we define the discrete logarithm as the discrete map dual to Z^2 , i.e. the map corresponding to the circle pattern with radii

$$R_{\text{Log}}(z) = 1/R_{Z^2}(z),$$

where R_{Z^2} are the radii of the circles for Z^2 . Here one has $R_{\text{Log}}(0) = \infty$, i.e. the corresponding circle is a straight line. The corresponding constraint (2) can be also derived as a limit. Indeed, consider the map $g = ((2 - \gamma)/\gamma)Z^\gamma - (2 - \gamma)/\gamma$. This map satisfies (1) and the constraint

$$\begin{aligned} \gamma \left(g_{n,m} + \frac{2 - \gamma}{\gamma} \right) &= 2n \frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} \\ &\quad + 2m \frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}. \end{aligned}$$

Keeping in mind the limit procedure used to determine Z^2 , it is natural to define the discrete analogue of $\log z$ as the limit of g as $\gamma \rightarrow +0$. The corresponding constraint becomes

$$1 = n \frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} + m \frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}. \quad (20)$$

Definition 6 [1]. Log is the map $\text{Log}: \mathbf{Z}_+^2 \rightarrow \mathbf{R}^2 = \overline{\mathbf{C}}$ satisfying (1) and (20) with the initial conditions

$$\begin{aligned} \text{Log}(0, 0) &= \infty, & \text{Log}(1, 0) &= 0, & \text{Log}(0, 1) &= i\pi, \\ \text{Log}(2, 0) &= 1, & \text{Log}(0, 2) &= 1 + i\pi, & \text{Log}(1, 1) &= i\frac{\pi}{2}. \end{aligned}$$

The circle patterns corresponding to the discrete conformal mappings Z^2 and Log (Figs. 5 and 6) were conjectured by Schramm and Kenyon (see [19]), in [1] it was proved that they are immersed. The following proposition completes the proof of Theorem 1.

Proposition 3. *Discrete conformal maps Z^2 and Log are imbedded.*

Proof. The circle radii for Z^2 and Log are subject to (6), (7) with $\gamma = 2$ and $\gamma = 0$, respectively. For these values of γ Theorem 3 is true: the proof is the same since Z^2 and Log are immersed. Due to Lemma 2 it suffices to prove property (13) only for Z^2 .

Consider the discrete conformal map $((2 - \gamma)/\gamma)Z^\gamma$ with $0 < \gamma < 2$. The corresponding solution $R(z)$ of (6), (7) is a continuous function of γ . So there is a limit, as $\gamma \rightarrow 2 - 0$, of this solution with property (13), which is violated only for $z = 0$ since $R(0) = 0$ (see [1] for details). $\square$

Fig. 5. Discrete Z^2 .

Fig. 6. Discrete Log.

5. Concluding Remarks

Slightly modified, the approach suggested in this paper can be applied to establish the global properties of the discrete Z^K proposed in [1] for natural $K > 2$. Details will be discussed elsewhere.

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