Isoperimetric Constants of Infinite Plane Graphs*

Serge Lawrencenko,¹ Michael D. Plummer,² and Xiaoya Zha³

¹Department of Mathematics and Statistics, Rochester Institute of Technology,
Rochester, NY 14623, USA

²Department of Mathematics, Vanderbilt University,
Nashville, TN 37240, USA
michael.d.plummer@vanderbilt.edu

³Department of Mathematical Sciences, Middle Tennessee State University,
Murfreesboro, TN 37132, USA
xzha@mtsu.edu

Abstract. Let G be an infinite locally finite plane graph with one end and let H be a finite plane subgraph of G . Denote by $a(H)$ the number of finite faces of H and by $l(\partial H)$ the number of the edges of H that are on the boundary of the infinite face or a finite face not in H . Define the isoperimetric constant $h(G)$ to be $\inf_H l(\partial H)/a(H)$ and define the isoperimetric constant $h(\delta)$ to be $\inf_G h(G)$ where the infimum is taken over all infinite locally finite plane graphs G having minimum degree δ and exactly one end. We establish the following bounds on $h(\delta)$ for $\delta \geq 7$:

$$\frac{(\delta - 6)(\delta^2 - 8\delta + 15)}{(\delta - 4)(\delta^2 - 8\delta + 13)} \leq h(\delta) \leq \sqrt{\frac{\delta - 6}{\delta - 2}}.$$

1. Introduction

The subject matter of this paper is hyperbolic infinite triangulations, and, generally, infinite cell 2-complexes G with *one end*—that is, the carrier of G is homeomorphic to the plane or, in other words, all points in the plane are covered by G . We draw parallels between the metric properties of the cell 2-complexes and their continuous counterparts. This approach gives rise to several discrete extremal problems. The purpose of this paper is to establish an upper bound and new lower bounds for the isoperimetric constants of G .

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Let M^2 be a topological 2-manifold. Let G be a cell 2-complex homeomorphic to M^2 (more accurately, the carrier of G is homeomorphic to M^2). It is a well known fact that if M^2 is noncompact (compact, respectively), then G is infinite (finite). By $V(G)$, $E(G)$, and $F(G)$ we denote the sets of the vertices (0-cells), edges (1-cells), and faces (2-cells) of G , respectively. Especially, G is called a *triangulation* of M^2 when it is simplicial. The faces of a triangulation are also called *triangles*. The *degree* of $v \in V(G)$, denoted by $\deg_G(v)$, is the number of the edges of which v is a vertex. The *star* of $v \in V(G)$, denoted by $\text{st}(v)$, is the union of the cells of which v is a vertex. G is δ -*regular* if each vertex of G has degree δ .

The following is a classical geometric fact which provides background and motivation for the present research. Let M^2 be a simply connected Riemannian 2-manifold which is complete as a metric space. If the Gaussian curvature K is uniformly bounded above by a negative constant,

$$K(p) \leq -\varepsilon^2 < 0 \quad (1.1)$$

for each point $p \in M^2$, then for any piecewise smooth closed curve C bounding a 2-disk D ,

$$\frac{\text{length } C}{\text{area } D} \geq \varepsilon. \quad (1.2)$$

Letting C vary and taking the infimum of these ratios leads to a positive number which is an important type of so-called isoperimetric constant of M^2 (see [C], [G], and [Y]), which we refer to as the Cheeger–Gromov–Yau constant.

Let G be a cell 2-complex homeomorphic to a closed topological 2-manifold M^2 . For a vertex $v \in V(G)$, the *combinatorial curvature* $K_c(v)$ is defined to be 2π minus the sum of the angles around v , where each face of G is thought of as a Euclidean regular polygon. Thus G models a metric of M^2 by way of the combinatorial curvature K_c which is a discrete analogue of the Gaussian curvature K . In this interpretation, if G is δ -regular, restriction on the minimum degree δ of G becomes a restriction on the curvature of M^2 . The larger δ , the more curved (negatively) M^2 is. The hyperbolicity of M^2 is guaranteed by the condition $\delta \geq 7$. This condition is a discrete analogue of condition (1.1), while a 6-regular triangulation models a flat 2-manifold.

We deal with a specific situation in which M^2 is the plane and G is a *plane graph*—that is, an embedding of a planar locally finite graph (1-complex) in the plane. Thus a plane graph is a two-dimensional object; it is determined by the triple of sets $\{V(G), E(G), F(G)\}$. We restrict our attention to the case in which G has one end. This restriction, along with the condition $\delta \geq 7$ (or even $\delta \geq 6$), guarantees, by Corollary 6.4, that G corresponds to a *cell 2-complex* in which every face is bounded by a cycle (no repeated vertices), and two faces may intersect only at a single vertex or a single edge. A *plane cell 2-complex* is one homeomorphic to the plane. It is necessarily infinite and locally finite [M2]. The case of a single end is of particular interest thanks to the fact that some infinite locally finite plane graphs with one end are the universal coverings of finite graphs on closed compact surfaces. Furthermore, it has turned out [PZ] that such plane graphs also have a structure similar to that of graph embeddings on surfaces with large representativity and minimum degree at least 6.

A *finite plane subgraph* of G is a plane embedding of the graph determined by fixing finite subsets of $V(G)$ and $E(G)$, naturally inherited from G .

Let H be a finite plane subgraph of G . Denote by $a(H)$ the number of finite faces of H (the “area” of H), and by $l(\partial H)$ the number of the edges of H that are on the boundary of at most one finite face (the “perimeter” of H). By analogy with the continuous case, define the *discrete isoperimetric constant* $h(G)$ by

$$h(G) = \inf_H \frac{l(\partial H)}{a(H)}, \quad (1.3)$$

and define the δ -*isoperimetric constant* $h(\delta)$ by

$$h(\delta) = \inf_G h(G), \quad (1.4)$$

where the infimum is taken over all plane cell 2-complexes G with minimum degree δ . We show in Section 4 that we may assume that G is a triangulation when estimating (1.4) and that H is a *polygon* in G when estimating (1.3), in which case $l(\partial H) = |E(\partial H)|$ and $a(H) = |F(H)|$, where ∂H denotes the boundary of H . A (triangulated) polygon means a subcomplex of G , the carrier of which is a 2-disk, and a 2-disk means a topological space homeomorphic to the region determined by inequality $x^2 + y^2 \leq 1$ in the Euclidean (x, y) -plane. If G is a cell 2-complex homeomorphic to a closed 2-manifold, then $\text{st}(v)$ is a 2-disk for each $v \in V(G)$.

Remark. In the literature, two commonly used definitions of the isoperimetric constant of an arbitrary connected graph G are

$$(i) \quad i_p(G) = \inf_X \left\{ \frac{|d(X)|}{m(X)} \right\}$$

(see [M1], [W], [Do], [HS], and [Z]) and

$$(ii) \quad i(G) = \inf_X \left\{ \frac{|d(X)|}{|X|} \right\}$$

(see [HJL] and [M1]), where $d(X)$ denotes the set of edges having one end in X and the other in $V(G) - X$ and $m(X)$ denotes the sum of the degrees of the vertices in X . In each case the infimum is taken over all nonempty finite sets of vertices $X \subseteq V(G)$. When graph G is embedded in a surface, $h(G)$ and $i(G)$ are dual concepts and $h(G) = i(G^*)$ where G^* is the surface dual of G . When G is k -vertex-regular, then $i(G) = ki_p(G)$.

While the isoperimetric constants $i(G)$ and $i_p(G)$ are used in the study of the growth rate of graphs, random walks, difference Laplacians and percolation on graphs as well as some other problems, we prefer to use $h(G)$ here (also see [HS]) as the isoperimetric constant. It seems to us somewhat more “natural” to do so, because this definition captures more geometric meaning in the sense that it is a discrete analog of Cheeger’s isoperimetric inequality [C] (which is the quotient of the measure of the $(n-1)$ -dimensional geometric boundary of an n -dimensional submanifold and the measure of the submanifold itself; i.e., “area” over “volume”).

In this paper we deal with the problem of estimating $h(G)$ and $h(\delta)$ for plane cell 2-complexes G with $\delta \geq 7$. It can be easily seen that $h(G) = 0$ if G is a 6-regular plane

triangulation. A number of people [Do], [BMST], [M1], [M3], [M4], [MW], [HJL], [HS], [Z], [S], [W] have looked at the isoperimetric constant $h(G)$, sometimes in the form of the plane dual. Here for the sake of uniformity and completeness, we restate some of these results in terms of $h(\delta)$. The first is a constant bound obtained by Dodziuk [Do]:

$$h(\delta) \geq \frac{1}{26} \quad (\delta \geq 7). \quad (1.5)$$

Historically, this bound appears to be the first discrete analogue of the isoperimetric inequality (1.2). (A similar inequality also appears, in another context, in [De].) The Dodziuk bound (1.5) coupled with the uniform boundedness of the degree of a vertex from above ensures the transience of random walks on G . Bound (1.5) was significantly improved by Mohar [M4] who showed that

$$h(\delta) \geq \frac{\delta - 6}{\delta - 4} \quad (\delta \geq 7). \quad (1.6)$$

The primary aim of this paper is to derive a still better lower bound for $h(\delta)$:

$$h(\delta) \geq \frac{(\delta - 6)(\delta^2 - 8\delta + 15)}{(\delta - 4)(\delta^2 - 8\delta + 13)} \quad (\delta \geq 7). \quad (1.7)$$

Note that bound (1.7) is for infinite, locally finite planar graphs with one end and minimum degree at least 7. The proof given in Sections 5 and 7 is combinatorial. In fact, bound (1.7) follows from our bounds for $h(G)$ when G is a δ -regular plane triangulation (Theorem 5.2) and when G is a plane cell 2-complex with minimum degree δ (Theorem 7.1).

Recently, Häggström et al. [HJL] and, independently, Higuchi and Shirai [HS] have shown that in the special case when G is a tessellation of the hyperbolic plane which is both d -vertex-regular and f -face-regular (with $1/d + 1/f < \frac{1}{2}$), then

$$h(G) = (f - 2) \sqrt{1 - \frac{4}{(d - 2)(f - 2)}}. \quad (1.8)$$

In the case when the graph G is both d -vertex-regular and f -face-regular, the value in (1.8) is better than the bound in (1.7). In fact, we believe that the value in (1.8) is the exact value for $h(\delta)$.

It is worth mentioning that Mohar [M4] studied infinite triangulations in a more general setting: he triangulated open subsets of surfaces with non-negative Euler characteristic, possibly with multiple ends. Most of his bounds are tight. However, in the case of infinite plane graphs with one end, his bound (1.6) is not tight and we improve his bound to that given in (1.7).

In Section 2 we obtain another lower bound for $h(G)$ where G is a δ -regular triangulation of the plane:

$$h(G) \geq \frac{\pi(\delta - 6)}{2\delta} \cdot \frac{1}{\ln(D + \sqrt{D^2 - 1})} \quad (\delta \geq 7), \quad (1.9)$$

where $D = (2 \sin(\pi/\delta))^{-1}$. This lower bound is geometric; it is derived from (1.2) by a purely geometric argument. Although it is far from being tight, the method we use

to obtain it demonstrates that the isoperimetric constant (1.3) is indeed an analogue of the Cheeger–Gromov–Yau constant. Surprisingly, we were unable to find any account of (1.9) in the literature. For $\delta = 7$, this geometric bound is considerably tighter than the Mohar bound (1.6), but is weaker than our bound (1.7). However, as $\delta \rightarrow \infty$, the geometric bound (1.9) implies only the trivial inequality $h(\delta) \geq 0$.

In Section 3 we establish the following upper bound:

$$h(\delta) \leq \sqrt{\frac{\delta - 6}{\delta - 2}} \quad (\delta \geq 7). \quad (1.10)$$

The difference between (1.10) and our bound (1.7) is $O(\delta^{-6})$, while the difference between (1.10) and the Mohar bound (1.6) is $O(\delta^{-2})$ (as $\delta \rightarrow \infty$).

Conjecture 1.1. *The upper bound (1.10) is the exact value for $h(\delta)$ ($\delta \geq 7$).*

2. The Geometric Lower Bound

In this section we establish inequality (1.9). Let G be a δ -regular triangulation of the plane with $\delta \geq 7$. (Note that a δ -regular triangulation of the plane with one end is uniquely determined.) We exploit the idea of Grünbaum and Shephard [GS] to realize G geometrically, and refer the reader to [R] for foundations of hyperbolic geometry. We use the standard Poincaré model of the hyperbolic plane H^2 with constant negative curvature $K = -1$. Draw an equilateral triangle Δ with angles of size $\alpha = 2\pi/\delta$, which has one of its vertices, v_0 , in the center of the Poincaré disk. Then tile the whole of H^2 by repeatedly reflecting that triangle about its sides. The infinite tessellation of H^2 thus obtained is a geometric realization of G . Remarkably, given a polygon Q in G , the perimeter and area of Q in H^2 are multiples of the combinatorial quantities $l(\partial Q) = |E(\partial Q)|$ and $a(Q) = |F(Q)|$, respectively. More precisely, we need to multiply $l(\partial Q)$ by $L(\Delta)$, the length of the side of Δ , to get the perimeter of Q and multiply $a(Q)$ by $A(\Delta)$, the area of Δ , to get the area of Q . It is a well known fact that $L(\Delta)$ and $A(\Delta)$ are functions of α and are determined from the following equations:

$$\cosh \frac{L(\Delta)}{2} \cdot \sin \frac{\alpha}{2} = \frac{1}{2}, \quad A(\Delta) = \pi - 3\alpha. \quad (2.1)$$

Now apply inequality (1.2) with $\varepsilon = 1$ to the piecewise smooth curve ∂Q to get (1.9).

Above we have used the standard model of H^2 with $K = -1$, but even if we varied K , we could not do better than (1.9), using this geometric approach. We could assign to K any negative constant value $-\varepsilon^2$, modifying (2.1) accordingly, but ε would cancel out from the bound and we would obtain as the bound an expression involving only α . However, α is determined to be $2\pi/\delta$ by the δ -regularity of G .

From the geometric construction it follows that G has a so-called lens structure. More precisely, G is the limit of the *lens sequence* $\{L_n\}_{n=0}^\infty$ centered at v_0 , defined by recursion on n as follows: $L_0 = \{v_0\}$, and, for $n \geq 1$, L_n is obtained from L_{n-1} by adjoining the triangles of G incident to ∂L_{n-1} that do not lie in L_{n-1} . (L_n may be thought of as representing one side of an optical hyperbolic lens.) We chose v_0 as the center of

the Poincaré disk. In fact, *any* vertex $v_0 \in V(G)$ could be chosen as the center of a lens sequence. By a theorem of Grünbaum and Shephard [GS], the lens sequence is independent of the choice of the center v_0 , up to homeomorphism of H^2 .

3. The Upper Bound

Suppose $\delta \geq 7$ and let G be a δ -regular triangulation of the plane, described geometrically in the preceding section. In this section we show that the bound (1.10) is approached from above by the lens sequence $\{L_n\}$.

It is convenient to describe G also in graph-theoretic language. For a fixed vertex $v_0 \in V(G)$, the vertices at distance n from v_0 , $n = 0, 1, \dots$, determine a cycle $C_n = \partial L_n$. Thus C_n are concentric cycles centered at v_0 . All the edges of G are either on C_n or have one endvertex on C_n and the other on C_{n-1} or C_{n+1} , for some $n \geq 1$. Furthermore, each vertex $v \in V(C_n)$ is adjacent to one or two vertices on C_{n-1} , and is said to be of *Type I* or *Type II* accordingly. Note that each Type I vertex on C_n is thus adjacent to some $\delta - 3$ vertices on C_{n+1} and that each Type II vertex is adjacent to some $\delta - 4$ vertices on C_{n+1} . Type II vertices occur as early as on C_2 . Let l_n denote $|E(C_n)|$, the *length* of C_n , and let a_n denote $|F(L_n)|$.

The lemmas in this section are related to the δ -regular ($\delta \geq 7$) triangulation G of the plane and assume the notation of the preceding paragraph.

Lemma 3.1. *For the lens sequence $\{L_n\}$ we have*

$$l_0 = 1, \quad l_n = \frac{\delta}{\sqrt{(\delta-2)(\delta-6)}}(\alpha^n - \beta^n) \quad (n \geq 1),$$

where

$$\alpha = \frac{1}{2}(\delta - 4 + \sqrt{(\delta-2)(\delta-6)}), \quad \beta = \frac{1}{2}(\delta - 4 - \sqrt{(\delta-2)(\delta-6)}). \quad (3.1)$$

Proof. To write a recurrence relation for l_n , observe that C_n (for $n \geq 2$) has l_{n-1} vertices of degree 4 and $l_n - l_{n-1}$ vertices of degree 3 in L_n , so that

$$l_1 = \delta, \quad l_2 = \delta(\delta - 4), \quad l_n = (\delta - 4)l_{n-1} - l_{n-2} \quad (n \geq 3).$$

Solving the characteristic equation of this recurrence relation, $x^2 - (\delta - 4)x + 1 = 0$, we write the general solution,

$$l_n = D_1 \alpha^n + D_2 \beta^n \quad (n \geq 1) \quad (3.2)$$

and the values $D_1 = -D_2 = \delta / \sqrt{(\delta-6)(\delta-2)}$ are determined using the initial conditions. $\square$

Lemma 3.2. *$\{l_n / l_{n-1}\}_{n \geq 1}$ is a strictly decreasing sequence.*

Proof. Since α and β are solutions of the characteristic equation, we have $\alpha\beta = 1$, and $\alpha - \beta = \sqrt{(\delta - 2)(\delta - 6)}$. Using Lemma 3.1 along with these identities, we obtain

$$\begin{aligned} l_n^2 - l_{n-1}l_{n+1} &= \frac{\delta^2}{(\delta - 2)(\delta - 6)} [(\alpha^n - \beta^n)^2 - (\alpha^{n-1} - \beta^{n-1})(\alpha^{n+1} - \beta^{n+1})] \\ &= \frac{\delta^2}{(\delta - 2)(\delta - 6)} (\alpha^2 - 2 + \beta^2) = \frac{\delta^2}{(\delta - 2)(\delta - 6)} (\alpha - \beta)^2 = \delta^2. \end{aligned}$$

Therefore the sequence $\{l_n\}_{n \geq 0}$ is, in fact, strictly log-concave down and the lemma follows. $\square$

Lemma 3.3. *For the lens sequence $\{L_n\}$ the following two statements hold:*

- (1) $\{l_n/a_n\}_{n \geq 1}$ is a strictly decreasing sequence, and
- (2) $\lim_{n \rightarrow \infty} l_n/a_n = \sqrt{(\delta - 6)/(\delta - 2)}$.

Proof. We prove part (1) first. Observe that $a_{k+1} - a_k = l_k + l_{k+1}$ for each $k = 1, 2, \dots$. Hence,

$$a_n = a_1 + \sum_{k=1}^{n-1} (a_{k+1} - a_k) = l_1 + \sum_{k=1}^{n-1} (l_k + l_{k+1}) = 2 \left(\sum_{k=1}^n l_k \right) - l_n, \quad (3.3)$$

and therefore

$$\begin{aligned} a_{n+1}l_n - a_nl_{n+1} &= \left[2 \left(\sum_{k=1}^{n+1} l_k \right) - l_{n+1} \right] l_n - \left[2 \left(\sum_{k=1}^n l_k \right) - l_n \right] l_{n+1} \\ &= 2l_1l_n + 2 \sum_{k=2}^n (l_kl_n - l_{k-1}l_{n+1}) \\ &= 2l_1l_n + 2 \sum_{k=2}^n \left(\frac{l_k}{l_{k-1}} - \frac{l_{n+1}}{l_n} \right) l_{k-1}l_n. \end{aligned}$$

By Lemma 3.2, $l_k/l_{k-1} - l_{n+1}/l_n > 0$, since $k \leq n$. Hence $a_{n+1}l_n - a_nl_{n+1} > 0$ and part (1) follows.

We proceed to prove part (2). From (3.3) we derive

$$\frac{a_n}{l_n} = \frac{2}{l_n} \left(\sum_{k=1}^n l_k \right) - 1. \quad (3.4)$$

Now, using (3.4), (3.2), the obvious double inequality $0 < \beta < 1 < \alpha$, and routine algebraic manipulations with the second equation of (3.1), we obtain

$$\lim_{n \rightarrow \infty} \frac{a_n}{l_n} = \frac{2C_1\alpha}{1-\alpha} \left(\lim_{n \rightarrow \infty} \frac{1-\alpha^n}{C_1\alpha^n + C_2\beta^n} \right) - 1 = \frac{\alpha+1}{\alpha-1} = \sqrt{\frac{\delta-2}{\delta-6}}.$$

The proof of part (2) is complete. $\square$

Theorem 3.4. *Bound (1.10) holds for $\delta \geq 7$.*

Proof. By Lemma 3.3, parts (1) and (2), we have

$$h(G) \leq \inf \frac{l_n}{a_n} = \sqrt{\frac{\delta - 6}{\delta - 2}}$$

when G is a δ -regular plane triangulation, and the theorem follows. $\square$

4. Reductions

Lemma 4.1. *Let m_i, n_i ($i = 1, 2$), and b be positive numbers. If $m_1/n_1 > b$ and $m_2/n_2 \geq b$, then $(m_1 + m_2)/(n_1 + n_2) > b$.*

Proof. The proof is a straightforward verification. $\square$

Let G be a plane cell 2-complex. For a plane subgraph H , “singular” vertices and edges are those where the carrier of H fails to be a 2-manifold. More precisely, we define a *singular vertex* of H as one of a small neighborhood of the carrier which, in H , is disconnected by the removal of v . A *singular edge* of H is one not contained in any face of H .

Lemma 4.2. *When determining $h(\delta)$ in (1.4) and $h(G)$ in (1.3) we may assume that G is a triangulation and H is a polygon, respectively, and therefore $l(\partial H) = |E(\partial H)|$ and $a(H) = |F(H)|$.*

Proof. G can be changed into a triangulation G' by appropriately triangulating the nontriangular faces (if any) without producing multiple edges. This procedure increases the value of the denominator in (1.3) and therefore a lower bound for $h(G')$ is a lower bound for $h(G)$.

By Lemma 4.1, we may assume that H is connected. Next, we may assume that H has no singular edge. For, suppose H has a singular edge e . Let $H_1 = H \setminus \{e\}$. Then $l(\partial H_1) = l(\partial H) - 1$, $a(H) = a(H_1)$, and $l(\partial H)/a(H) > l(\partial H_1)/a(H_1)$. Hence a strict lower bound for $l(\partial H_1)/a(H_1)$ certainly is a strict lower bound for $l(\partial H)/a(H)$. Finally, if H contains a singular vertex but no singular edge, then it can be easily seen that H is a union of edge-disjoint polygons, possibly with holes. We then close all holes with new faces and note that this process only decreases the ratio (1.3) for each polygon with holes. Therefore we may assume that H is a union of edge-disjoint polygons. Lemma 4.1 guarantees that the smallest among the ratios (1.3) for those polygons provides a lower bound on the ratio for the whole H . Therefore we may assume that H is connected and contains no singular vertex and no holes, which means that H is a single polygon. $\square$

5. The Lower Bound for Triangulations

In this section we prove that the bound in (1.7) is a lower bound for $h(G)$ when G is a δ -regular plane triangulation. As noted in the Introduction, for δ -regular triangulations, the result in (1.8) which appeared in the recent works [HJL] and [HS] is sharp and therefore better than the bound in (1.7). However, we use an approach different from that used by these authors because it subsequently extends to *non*-vertex-regular triangulations.

Suppose u, v are two consecutive vertices on $C_n = \partial L_n$, where u precedes v in clockwise order. Denote by $u_1, \dots, u_\lambda$ the neighbors of u , and by $v_1 (= u_\lambda), \dots, v_\tau$ the neighbors of v , on C_{n+1} in clockwise order. The vertex u_1 (v_1 , respectively) is the *first neighbor* of u (v), and u_λ (v_τ) is the *last neighbor* of u (v). A *forward section* $R[uu_\lambda v_\tau v]$ with supporting edge uv is defined to be the union of the triangle $uu_\lambda v$ and the triangles incident to v which have an edge on C_{n+1} . Similarly, a *backward section* $R[uu_1 v_\tau v]$ is the union of the triangle $uu_\lambda v$ and the triangles incident to u which have an edge on C_{n+1} . A forward section $R[uu_\lambda v_\tau v]$ is said to be of Type I (Type II, respectively) if v is a Type I (Type II) vertex. Similarly, the Type (I or II) of a backward section $R[uu_1 v_\tau v]$ is determined by the Type of vertex u .

Let A_n denote the closure of $L_{n+1} \setminus L_n$. A *segment* of A_n is a polygon in A_n . Let S be a segment of A_n . We use U to represent a Type I section and W to represent a Type II section. Then A_n is a union of U 's and W 's:

$$A_n = \dots UUUWU \dots UWU \dots UWUUU \dots$$

Note that there are never two consecutive Type II sections. If U is a Type I section and W is a Type II section, then

$$|E(U \cap C_n)| = |E(W \cap C_n)| = 1, \quad |E(U \cap C_{n+1})| = \delta - 4, \quad |E(W \cap C_{n+1})| = \delta - 5.$$

Let W_1 and W_2 be two closest Type II forward (or both backward, respectively) sections in A_n , $n \geq 2$, with supporting edges $x_1 y_1$ and $x_2 y_2$, respectively. Then both y_1 and y_2 (x_1 and x_2) are Type II vertices, and y_1 (x_1) is the first neighbor of the *apex vertex* u on C_{n-1} , and y_2 (x_2) is its last neighbor. (Figure 5.1 illustrates the situation in which $\delta = 8$ and the shaded sections W_1 and W_2 are Type II forward sections.) If u is a Type I vertex, there are $\delta - 5$ Type I forward (backward) sections between W_1 and W_2 . The union of W_1 (W_2) and those $\delta - 5$ Type I sections is called a *forward (backward) Type I subdivision supported by the path* $P[x_1 x_2]$ ($P[y_1, y_2]$). By $P[xy]$ we denote the path from vertex x to vertex y on C_n taken in clockwise order. If u is a Type II vertex, then a *Type II subdivision* is defined similarly, replacing “ $\delta - 6$ ” with “ $\delta - 5$.” W_1 (W_2) is called the *leading section* of the forward (backward) subdivision. If $\mathcal{U}$ is a Type I subdivision and $\mathcal{W}$ is a Type II subdivision, then

$$\begin{aligned} |E(\mathcal{U} \cap C_n)| &= \delta - 4, \\ |E(\mathcal{U} \cap C_{n+1})| &= (\delta - 4)(\delta - 5) + (\delta - 5) = (\delta - 3)(\delta - 5), \\ |E(\mathcal{W} \cap C_n)| &= \delta - 5, \\ |E(\mathcal{W} \cap C_{n+1})| &= (\delta - 4)(\delta - 6) + (\delta - 5) = \delta^2 - 9\delta + 19. \end{aligned}$$

Fig. 5.1

Let x, y, z be three consecutive vertices on C_n with y being a Type II vertex. Let u be the last neighbor of x (or, equivalently, the first neighbor of y) and let v be the last neighbor of y (or, equivalently, the first neighbor of z). The polygon bounded by the paths $P[xz]$ on C_n and $P[uv]$ on C_{n+1} , and the edges ux and vz is called an *M-section* of A_n , denoted by $\mathcal{M} = \mathcal{M}[xuvz]$. We have

$$|E(\mathcal{M} \cap C_n)| = 2, \quad |E(\mathcal{M} \cap C_{n+1})| = \delta - 5.$$

Let Q be a polygon of G and let S be a segment in $Q \cap A_n$. Let $P[x_1x_s]$ be the path of $S \cap C_n$ and let $P[y_1y_t]$ be the path of $S \cap C_{n+1}$. If y_1 is the last neighbor of x_1 and y_t is the first neighbor of x_s , and x_2, x_{s-1} are both Type II vertices, then S is called a *minimal segment* of $Q \cap A_n$. It is the union of some forward subdivisions and the M-section $\mathcal{M}[x_{s-2}y_{t-\delta+5}y_tx_s]$, where $y_{t-\delta+5}$ is the last neighbor of x_{s-2} . The minimal segment can also be viewed as the union of some backward subdivisions and the M-section $\mathcal{M}[x_1y_1y_{\delta-4}x_3]$, where $y_{\delta-4}$ is the first neighbor of x_3 .

Lemma 5.1. *Let Q be a polygon of G and let S be a minimal segment of $Q \cap A_n$. If S contains at most $\delta - 5$ subdivisions, then*

$$\frac{|E(S \cap C_{n+1})| - |E(S \cap C_n)| + 2}{|E(S \cap C_{n+1})| + |E(S \cap C_n)|} > \sqrt{\frac{\delta - 6}{\delta - 2}}. \quad (5.1)$$

Proof. Denote the left side of inequality (5.1) by $g(S)$. If $x_2 = x_{s-1}$, S is an M-section, so that $g(S) = (\delta - 5)/(\delta - 3)$. Then it is straightforward to verify (5.1). Now assume $x_2 \neq x_{s-1}$ and there are $k (\geq 3)$ Type II vertices $x_{i_1}, \dots, x_{i_k}$ in $P[x_1x_s]$ with $x_{i_1} = x_2$ and $x_{i_k} = x_{s-1}$. For each i_j ($j = 1, \dots, k - 1$), $x_{i_{j-1}}x_{i_j}$ is the supporting edge of a forward Type II section, and each of these forward Type II sections is the leading section of a Type I or a Type II subdivision. Since S contains at most $\delta - 5$ subdivisions, we have $k - 1 \leq \delta - 5$. Furthermore, since there are either $\delta - 5$ or $\delta - 6$ Type I subdivisions between two closest Type II subdivisions, S contains at most one Type II subdivision.

Case 1: $\mathcal{S}$ is the union of $k - 1$ forward Type I subdivisions and the M -section $\mathcal{M}[x_{s-2}y_{t-\delta+5}y_tx_s]$. Then

$$\begin{aligned} g(\mathcal{S}) &= \frac{[(k-1)(\delta-3)(\delta-5) + (\delta-5)] - [(k-1)(\delta-4) + 2] + 2}{[(k-1)(\delta-3)(\delta-5) + (\delta-5)] + [(k-1)(\delta-4) + 2]} \\ &= \frac{(k-1)(\delta^2 - 9\delta + 19) + \delta - 5}{(k-1)(\delta^2 - 7\delta + 11) + \delta - 3}. \end{aligned}$$

Denote this quantity by $g_1(\delta, k)$. Since

$$\frac{\partial g_1(\delta, k)}{\partial k} = \frac{-2}{[(k-1)(\delta^2 - 7\delta + 11) + \delta - 3]^2} < 0,$$

$g_1(\delta, k)$ is a decreasing function of k . Therefore to prove the lemma for Case 1, it suffices to show the inequality holds when $k = \delta - 5$; that is, $g_1(\delta, \delta - 5) > \sqrt{(\delta - 6)(\delta - 2)}$, which is a straightforward verification. Thus the lemma is true in this case.

Case 2: $\mathcal{S}$ is the union of the following: $k - 2$ forward Type I subdivisions, one forward Type II subdivision, and the M -section $\mathcal{M}[x_{s-2}y_{t-\delta+5}y_tx_s]$. Then

$$\begin{aligned} g(\mathcal{S}) &= \frac{[(k-2)(\delta-3)(\delta-5) + (\delta^2 - 9\delta + 19) + (\delta-5)] - [(k-2)(\delta-4) + (\delta-5) + 2] + 2}{[(k-2)(\delta-3)(\delta-5) + (\delta^2 - 9\delta + 19) + (\delta-5)] + [(k-2)(\delta-4) + (\delta-5) + 2]} \\ &= \frac{\delta^2 - 9\delta + 19}{\delta^2 - 7\delta + 11}. \end{aligned}$$

Then the inequality $g(\mathcal{S}) > \sqrt{(\delta - 6)(\delta - 2)}$ is easily verified. This finishes the proof. $\square$

Theorem 5.2. *Let G be a δ -regular triangulation of the plane with $\delta \geq 7$. Then*

$$h(G) > \frac{(\delta - 6)(\delta^2 - 8\delta + 15)}{(\delta - 4)(\delta^2 - 8\delta + 13)}. \quad (5.2)$$

Proof. Denote the right side of (5.2) by $f(\delta)$. By Lemma 4.2, we only need to show $l(\partial Q)/a(Q) > f(\delta)$ for any polygon Q in G .

Claim 1. *If $\delta \geq 7$, $f(\delta) < \sqrt{(\delta - 6)(\delta - 2)} < 1$.*

Proof of Claim 1. The proof is a routine verification. $\square$

The proof of the theorem proceeds by induction on the number of triangles in Q . First we assume that Q is a single triangle. Therefore $l(\partial Q)/a(Q) = 3 > 1 > f(\delta)$ and so the theorem holds in this case. Now assume that Q is a polygon containing more than one triangle. If Q contains no interior vertex, then $a(Q) = |E(\partial Q)| - 2$. Since $|E(\partial Q)|/(|E(\partial Q)| - 2) > 1 > f(\delta)$, the theorem also holds. Therefore we may assume

that Q contains an interior vertex, v_0 . Let $\{L_1, \dots, L_n, \dots\}$ be the lens sequence centered at v_0 . Let n be the largest index such that $Q \cap A_n \neq \emptyset$. Let $\mathcal{S}$ be a segment of $Q \cap A_n$ with $\mathcal{S} \cap C_n = P[x_1 x_s]$ and $\mathcal{S} \cap C_{n+1} = P[y_1 y_t]$.

Case 1: y_t is not the first neighbor of x_s . Let y'_t be the first neighbor of x_s . Let Q^- be the polygon obtained from Q by deleting the triangles incident to x_s which have an edge on C_{n+1} . Suppose k triangles are deleted. Then $l(\partial Q)/a(Q) = (l(\partial Q^-) + k)/(a(Q^-) + k)$. By the induction hypothesis, $l(\partial Q^-)/a(Q^-) > f(\delta)$. Since $k/k = 1 > f(\delta)$, then $l(\partial Q)/a(Q) > f(\delta)$, by Lemma 4.1. Therefore the theorem holds in this case.

Case 2: y_t is the first neighbor of x_s , and x_{s-1} is a Type I vertex. Then the edge $x_{s-1}x_s$ is the supporting edge of a backward Type I section U . Let $Q^- = Q \setminus U$. Then

$$\frac{l(\partial Q)}{a(Q)} = \frac{l(\partial Q^-) + |E(U \cap C_{n+1})| - |E(U \cap C_n)|}{a(Q^-) + |E(U \cap C_{n+1})| + |E(U \cap C_n)|} = \frac{l(\partial Q^-) + (\delta - 4) - 1}{a(Q^-) + (\delta - 4) + 1}.$$

By the induction hypothesis, $l(\partial Q^-)/a(Q^-) > f(\delta)$. Since $(\delta - 5)/(\delta - 3) > f(\delta)$, then $l(\partial Q)/a(Q) > f(\delta)$, by Lemma 4.1.

Case 3: y_t is the first neighbor of x_s , and x_{s-1} is a Type II vertex.

Subcase 3.1: $\mathcal{S}$ contains at least $\delta - 5$ subdivisions (Type I or Type I and Type II combined). Then the edge $x_{s-1}x_s$ is the supporting edge of a backward Type II section. Denote this section by W_1 . If to the left of W_1 there are $\delta - 5$ Type I sections, then W_1 and these $\delta - 5$ Type I sections together form a Type I subdivision. Denote this subdivision by $\mathcal{U}$. Let $Q^- = Q \setminus \mathcal{U}$. Recalling the formulas for $|E(\mathcal{U} \cap C_n)|$ and $|E(\mathcal{U} \cap C_{n+1})|$, we obtain

$$\begin{aligned} \frac{l(\partial Q)}{a(Q)} &= \frac{l(\partial Q^-) + |E(\mathcal{U} \cap C_{n+1})| - |E(\mathcal{U} \cap C_n)|}{a(Q^-) + |E(\mathcal{U} \cap C_{n+1})| + |E(\mathcal{U} \cap C_n)|} \\ &= \frac{l(\partial Q^-) + (\delta - 5)(\delta - 3) - (\delta - 4)}{a(Q^-) + (\delta - 5)(\delta - 3) + (\delta - 4)}. \end{aligned}$$

By the induction hypothesis, $l(\partial Q^-)/a(Q^-) > f(\delta)$. Since

$$\frac{(\delta - 5)(\delta - 3) - (\delta - 4)}{(\delta - 5)(\delta - 3) + (\delta - 4)} > f(\delta),$$

then $l(\partial Q)/a(Q) > f(\delta)$, by Lemma 4.1. If to the left of W_1 , there are only $\delta - 6$ consecutive Type I sections, then W_1 and these $\delta - 6$ Type I sections together form a Type II subdivision which we denote by $\mathcal{W}$. Since between two closest Type II subdivisions there are either $\delta - 5$ or $\delta - 6$ Type I subdivisions, then to the left of $\mathcal{W}$ there are at least $\delta - 6$ consecutive Type I subdivisions $\mathcal{U}_1, \dots, \mathcal{U}_{\delta-6}$. Let $Q' = \mathcal{W} \cup \mathcal{U}_1 \cup \dots \cup \mathcal{U}_{\delta-6}$. As above, $|E(Q' \cap C_n)| = (\delta - 6)(\delta - 4) + (\delta - 5)$ and $|E(Q' \cap C_{n+1})| = (\delta - 6)(\delta - 3)(\delta - 5) + (\delta^2 - 9\delta + 19)$. Let $Q^- = Q \setminus Q'$. Then

$$\frac{l(\partial Q)}{a(Q)} = \frac{l(\partial Q^-) + |E(Q' \cap C_{n+1})| - |E(Q' \cap C_n)|}{a(Q^-) + |E(Q' \cap C_{n+1})| + |E(Q' \cap C_n)|}.$$

By the induction hypothesis, $l(\partial Q^-)/a(Q^-) > f(\delta)$. Since

$$\begin{aligned} & \frac{|E(Q' \cap C_{n+1})| - |E(Q' \cap C_n)|}{|E(Q' \cap C_{n+1})| + |E(Q' \cap C_n)|} \\ &= \frac{((\delta - 6)(\delta - 3)(\delta - 5) + (\delta^2 - 9\delta + 19)) - ((\delta - 6)(\delta - 4) + (\delta - 5))}{((\delta - 6)(\delta - 3)(\delta - 5) + (\delta^2 - 9\delta + 19)) + ((\delta - 6)(\delta - 4) + (\delta - 5))} \\ &= f(\delta), \end{aligned}$$

then $l(\partial Q)/a(Q) > f(\delta)$, by Lemma 4.1.

Subcase 3.2: $\mathcal{S}$ contains less than $\delta - 5$ subdivisions (Type I or Type I and Type II combined). By the discussion in Cases 1 and 2, and the symmetry, we may assume that x_2 is a Type II vertex. Therefore S is a minimal segment. Let $Q^- = Q \setminus \mathcal{S}$. Then

$$\frac{l(\partial Q)}{a(Q)} = \frac{l(\partial Q^-) + |E(\mathcal{S} \cap C_{n+1})| - |E(\mathcal{S} \cap C_n)| + 2}{a(Q^-) + |E(\mathcal{S} \cap C_{n+1})| + |E(\mathcal{S} \cap C_n)|}.$$

By the induction hypothesis, $l(\partial Q^-)/a(Q^-) > f(\delta)$. By Lemma 5.1 and Claim 1,

$$\frac{l(\partial Q)}{a(Q)} = \frac{|E(\mathcal{S} \cap C_{n+1})| - |E(\mathcal{S} \cap C_n)| + 2}{|E(\mathcal{S} \cap C_{n+1})| + |E(\mathcal{S} \cap C_n)|} > \sqrt{\frac{\delta - 6}{\delta - 2}} > f(\delta).$$

Then, by Lemma 4.1, $l(\partial Q)/a(Q) > f(\delta)$. This finishes the proof. $\square$

6. A Lens Structure Theorem

In this section we prove that infinite locally finite plane graphs with one end and with minimum degree $\delta \geq 6$ have a lens structure.

Lemma 6.1 [PZ]. *Suppose finite plane graph G has its infinite face bounded by a cycle C and that u and v are two vertices on C . If the vertices on C , except possibly one or two vertices, have degree at least 4, then there is an interior vertex with degree less than 6.* $\square$

Lemma 6.2. *Any infinite locally finite plane graph G with one end and with minimum degree $\delta \geq 6$ is 3-connected.*

Proof. Suppose to the contrary that G is not 3-connected. Then there exists a 1-vertex cut or a 2-vertex cut. On one side of the cut there is a finite subgraph which satisfies the conditions of Lemma 6.1, and a contradiction is immediate. $\square$

The definition of a lens sequence for a general plane graph G is obtained from the one given in Section 2 for triangulations by replacing “triangles” with “faces.”

Theorem 6.3 (Lens Structure Theorem). *Suppose G is an infinite locally finite plane graph with one end and with minimum degree at least 6. Let $\{L_n\}$ be the lens sequence*

centered at a vertex v_0 . Then, for $n \geq 1$:

- (i) L_n is a 2-disk, ∂L_n is a cycle C_n , and C_n is vertex-disjoint from C_{n-1} . Hence $C_1, C_2, \dots$ together form a collection of concentric cycles.
- (ii) Each edge in $L_n \setminus L_{n-1}$ is either contained in the cycle C_n , or has one endvertex on C_{n-1} and the other on C_n .
- (iii) Each vertex on C_n is incident to at most two vertices on C_{n-1} .

Proof. This proof proceeds by induction on n , $n \geq 1$. By Lemma 6.2, G is 3-connected, and so L_1 is a 2-disk. Now assume the theorem is true for all $k \leq n$. Therefore L_n is a 2-disk bounded by the cycle C_n and contains n nested cycles $C_1, \dots, C_n$ centered at v_0 . Furthermore, the edges of L_n either lie in these cycles or join a vertex in C_j to a vertex in C_{j+1} , for some $0 \leq j \leq n-1$. We proceed to show that the theorem is also true for $k = n+1$. Let $x_1, \dots, x_m$ denote the vertices on C_n in clockwise order. Since $x_1 \in \partial L_n$, some faces of $\text{st}(x_1)$ are in L_n . By the lens structure of L_n , the faces of $\text{st}(x_1)$ contained in L_n together form a “half star.” Let $\text{st}(x_1)^- = \text{st}(x_1) \setminus L_n$. Then $\text{st}(x_1)^-$ is the half star of the faces which are not in L_n . Clearly, the 2-edge path $x_m x_1 x_2$ is a component of $\partial \text{st}(x_1)^- \cap \partial L_n$.

Claim 2. *This 2-edge path is the only component of $\partial \text{st}(x_1)^- \cap \partial L_n$.*

Proof of Claim 2. Suppose $\partial \text{st}(x_1)^- \cap \partial L_n$ has a second component; in other words, suppose one of the faces incident to x_1 and contained in $\text{st}(x_1)^-$ has a vertex x_k on its boundary, for some $x_k \in C_n$, $k > 2$. If $\partial \text{st}(x_1)^- \cap \partial L_n$ has more than two components, choose k as small as possible. Denote this face by f . (See Fig. 6.1.) Let H be the subgraph of G bounded by the path $P[x_1 x_k]$ (or $P[x_2 x_k]$ if x_2 is also on the boundary of f) and the section of ∂f from x_1 (x_2) to x_k in counterclockwise order. By the induction hypothesis, any vertex on $P[x_1 x_k]$ is adjacent to at most two vertices on C_{n-1} , and therefore the degree of each vertex on $P[x_1 x_k]$ (not including x_1 and x_k , or x_2 and x_k) in H is at least 4, and the remaining vertices other than x_1 (x_2) and x_k have degree at least 6 in H .

Fig. 6.1

However, by Lemma 6.1, such a graph H cannot exist. Hence, $\partial f \cap \partial L_n$ has no second component, and Claim 2 follows. $\square$

Now let $L_n[x_1] = \text{st}(x_1)^- \cup L_n$. By Claim 2, $L_n[x_1]$ is a 2-disk. $\partial L_n[x_1]$ is the union of the path of C_n from x_2 to x_m and the path of $\partial \text{st}(x_1)^-$ from x_m to x_2 (clockwise).

Let $\text{st}(x_2)^- = \text{st}(x_2) \setminus L_n[x_1]$ and $L_n[x_1, x_2] = L_n[x_1] \cup \text{st}(x_2)^-$.

Claim 3. $\partial \text{st}(x_2)^- \cap \partial L_n[x_1]$ consists of a single component.

Proof of Claim 3. This proof is similar to that of Claim 2. Clearly, $\partial \text{st}(x_2)^- \cap \partial L_n[x_1]$ has a component which is the union of the edge x_2x_3 and the path of $\partial \text{st}(x_1)^-$ from a vertex w to x_2 (clockwise about x_1). Suppose there is a face f in $\text{st}(x_2)^-$ which has a second attachment with $\partial L_n[x_1]$. If this second attachment is on the section of C_n from x_3 to some x_l , then, by an argument similar to that used for Claim 2, such an attachment does not exist. If the second attachment is on the section of $\text{st}(x_1)^-$ from x_m to w (clockwise about x_1), then let y be the vertex of this attachment closest to w clockwise on $\partial \text{st}(x_1)^-$. Let H be the subgraph bounded by the section of $\partial \text{st}(x_1)^-$ from y to x_2 (clockwise) and the section of ∂f from y to x_2 (counterclockwise with respect to f). All the vertices of H on the section of $\partial \text{st}(x_1)^-$ from y to x_2 (not including y and x_2) have degree at least 5 in H , and thus satisfy the condition of Lemma 6.1. Therefore, by Lemma 6.1, H does not exist, and Claim 3 follows. $\square$

Let $\text{st}(x_l)^- = \text{st}(x_l) \setminus L_n[x_1, \dots, x_{l-1}]$ and $L_n[x_1, \dots, x_l] = L_n[x_1, \dots, x_{l-1}] \cup \text{st}(x_l)^-$, for $2 \leq l < m$. Assume $L_n[x_1, \dots, x_t] = L_n[x_1, \dots, x_{t-1}] \cup \text{st}(x_t)^-$ is a 2-disk, for $t = 2, \dots, l$. We now show that $L_n[x_1, \dots, x_{l+1}] = L_n[x_1, \dots, x_l] \cup \text{st}(x_{l+1})^-$ is a 2-disk, where $\text{st}(x_{l+1})^- = \text{st}(x_{l+1}) \setminus L_n[x_1, \dots, x_l]$. It suffices to show that $\partial L_n[x_1, \dots, x_l] \cap \partial \text{st}(x_{l+1})^-$ consists of a single component. Let w be the vertex on $\partial L_n[x_1, \dots, x_{l+1}]$ preceding x_l (clockwise about x_{l+1}). Then the 2-edge path $wx_{l+1}x_{l+2}$ is a component of $\partial L_n[x_1, \dots, x_l] \cap \partial \text{st}(x_{l+1})^-$. Suppose there is a face f in $\text{st}(x_{l+1})^-$ which has a second attachment with $\partial L_n[x_1, \dots, x_l]$. If this second attachment is on the section of C_n from x_{l+2} to x_m , then by an argument similar to that used in Claim 2, such an attachment does not exist. If the second attachment is on $\partial L_n[x_1, \dots, x_l]$, then by an argument similar to that used in Claim 3, such an attachment does not exist. Therefore $\partial L_n[x_1, \dots, x_l] \cap \partial \text{st}(x_{l+1})^-$ consists of a single component which is a 2-edge path. Hence, $L_n[x_1, \dots, x_{l+1}] = L_n[x_1, \dots, x_l] \cup \text{st}(x_{l+1})^-$ is a 2-disk. This inductive process constructively proves that $L_{n+1} = L_n[x_1, \dots, x_m]$ is a 2-disk; denote by C_{n+1} its boundary cycle. Since C_{n+1} is the symmetric difference of C_n and the boundaries of the faces incident to C_n , it follows that $C_n \cap C_{n+1} = \emptyset$. This proves (i).

Since $L_{n+1} = L_n[x_1, \dots, x_m]$, each edge of $L_{n+1} \setminus L_n$ is either a rim edge or a spoke in some $\text{st}(x_j)^-$. Furthermore, the rim edges are in $\partial L_{n+1} = C_{n+1}$ while the spokes join a vertex in C_{n+1} to a vertex in C_n . Hence (ii) is true.

Suppose that there is a vertex on C_{n+1} adjacent to more than two vertices on C_n . Let v_a, v_b , and v_c be three consecutive such vertices on C_n in clockwise order on C_n . By the induction hypothesis, there are at most two edges joining v_b to the vertices on C_{n-1} . This implies that v_b has degree at most 5, a contradiction. Hence, (iii) is also true. $\square$

Corollary 6.4. *The sets $V(G)$, $E(G)$, and $F(G)$ of any infinite locally finite plane graph G with one end and minimum degree at least 6 give rise to a plane cell 2-complex.*

Theorem 6.3 does not hold if we relax the restriction on δ to read $\delta \geq 5$. For, choose several faces and insert a subgraph with minimum degree 5 into each. Then the lens L_n is not necessarily a 2-disk and ∂L_n may be a union of two or more cycles.

7. The Lower Bound in a General Setting

In this section we show that the lower bound in Theorem 5.2 holds in the more general case in which the plane graphs considered are not necessarily regular and are not even required to be triangulations.

Theorem 7.1. *Inequality (5.2) holds for any plane cell 2-complex G with minimum degree at least 7.*

Proof. First recall that, by Lemma 4.2, we may assume that G is a plane triangulation and we only need to show that (5.2) holds for any polygon Q of G . If Q consists of a single triangle, $l(\partial Q)/a(Q) = \frac{3}{1} > f(\delta)$, and the theorem is true. If Q contains more than one triangle but no interior vertex, $a(Q) = |E(\partial Q)| - 2$, so that $l(\partial Q)/a(Q) > 1 > f(\delta)$, and so the theorem holds. Now we assume that Q has an interior vertex, which we denote by v_0 . By Theorem 6.3, G has a lens structure, centered at v_0 . Let n be the largest index such that $Q \cap A_n \neq \emptyset$ and let S be a segment of $Q \cap A_n$. Let $S \cap C_n = P[x_1 x_s]$ and let $S \cap C_{n+1} = P[y_1 y_t]$.

Even if G is not regular, each Type I (Type II, respectively) vertex v on C_n is still adjacent to one vertex (two vertices) on C_{n-1} . However, v may be adjacent to more than $\delta - 3$ ($\delta - 4$, respectively) vertices on C_{n+1} , and so most of the counting formulas in Section 5 for the numbers of edges no longer hold.

Claim 4. *We may assume that S is a segment S' of $Q' \cap A'_n$, where Q' is a polygon in some triangulation G' , centered at v_0 , which is δ -regular, and $A'_n = L'_{n+1} \setminus L'_n$.*

Proof of Claim 4. For each vertex v on $C_n \cap S$ with degree greater than δ , contract $\deg_G(v) - \delta$ consecutive rim edges on $\text{st}(v) \cap C_{n+1}$ into a single vertex with collapsing the incident faces into a single edge. This process transforms S to S' . Since

$$\frac{l(\partial Q)}{a(Q)} = \frac{l(\partial Q') + \deg_G(v) - \delta}{a(Q') + \deg_G(v) - \delta} \quad \text{and} \quad \frac{\deg_G(v) - \delta}{\deg_G(v) - \delta} = 1 > f(\delta),$$

it suffices to show that inequality (5.2) holds for Q' in G' , by Lemma 4.1. In other words, we may suppose that all the vertices on C_n have degree equal to δ .

Next, suppose x_a and x_b are two closest Type II vertices on $P[x_1 x_s]$. Then $P[x_a x_b]$ is the supporting path of a subdivision $\mathcal{T}$. Note that if G is δ -regular, between two closest Type II vertices on C_n there are either $\delta - 5$ Type I vertices (corresponding to a Type I subdivision) or $\delta - 6$ Type I vertices (corresponding to a Type II subdivision). Let u

be the apex vertex of $\mathcal{T}$. If u is a Type I vertex (hence $\mathcal{U}$ is a Type I subdivision) and $P[x_a x_b]$ contains $k + (\delta - 5)$ Type I vertices, $k \geq 1$, then let $x_{i_1}, \dots, x_{i_k}$ be the first k Type I vertices and let $U_1, \dots, U_k$ be the corresponding Type I sections supported by the edges $x_a x_{i_1}, \dots, x_{i_{k-1}} x_{i_k}$, respectively. Let $\mathcal{T}' = U_1 \cup \dots \cup U_k$. Contract the edges of $\mathcal{T}' \cap (C_n \cup C_{n+1})$ to shrink $\mathcal{T}'$ to a single edge joining C_n to C_{n+1} . Suppose this process transforms Q to Q' . Then

$$\frac{l(\partial Q)}{a(Q)} = \frac{l(\partial Q') + k(\delta - 4)}{a(Q') + k(\delta - 4) + k}.$$

Since $k(\delta - 4)/(k(\delta - 4) + k) = (\delta - 4)/(\delta - 3) > f(\delta)$, it suffices to show that $l(\partial Q')/a(Q') > f(\delta)$. Therefore we may assume that if u is a Type I vertex, then the subdivision $\mathcal{T}$ consists of exactly one Type II section and $\delta - 5$ Type I sections, which is the same as a Type I subdivision in a δ -regular triangulation. Similarly, if u is a Type II vertex, we may assume that the subdivision $\mathcal{T}$ consists of exactly one Type II section and $\delta - 6$ Type I sections, just as a Type II subdivision in a δ -regular triangulation does. This finishes the proof of Claim 4. $\square$

Now use Claim 4 and proceed as in the proof of Theorem 5.2 to finish the proof. $\square$

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