Packings of Unequal Circles in a Convex Set

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Abstract. In the Euclidean plane let T be a convex set, and let $K_1, \dots, K_n$ be a family of $n \geq 2$ circles packed into T . We show that the density of each such packing is smaller than $\pi/\sqrt{12}$, the density of the densest packing of equal circles in the plane, provided the radii of the circles are not too different. This extends a result of G. Fejes Tóth, where T was a polygon with at most six sides.

1. Introduction

For circle packings of the Euclidean plane, L. Fejes Tóth [7, p. 79] noticed that the densest packing of equal circles is so good that its density $\pi/\sqrt{12}$ cannot be improved by using circles with slightly different radii. Florian [9] showed that the radii may lie in the interval $[1, 0.906 \dots]$, and G. Blind [1], [2] extended this interval to $[1, 0.742 \dots]$. In other words, in this relatively large interval you cannot do better than to use equal circles. It is not known whether the interval can be further extended. However, L. Fejes Tóth [6, p. 175] found a packing with density greater than $\pi/\sqrt{12}$ for each interval larger than $[1, 0.645 \dots]$, and Danzer [4] could do so for each interval even larger than $[1, 0.654 \dots]$.

Now, packings of the whole Euclidean plane are closely related to finite packings of a convex hexagon, where a hexagon is a polygon with at most six sides. In fact, G. Fejes Tóth [5] (see also [8]) showed that any finite circle packing of a convex hexagon with radii in the interval $[1, 0.742 \dots]$ also has density at most $\pi/\sqrt{12}$.

In the present paper we show that this phenomenon even occurs for finite circle packings of any convex set T , though in this case circles and the shape of T could fit together much better.

Fig. 1. A packing with three circles.

Theorem 1. *In the Euclidean plane let T be a convex set, and let $K_1, \dots, K_n$ be a family of $n \geq 2$ circles packed into T , with radii $r_1, \dots, r_n$, respectively. Let δ be the density of this packing. Then, denoting the area of T also by T , we have*

$$\delta := \frac{\pi(r_1^2 + \dots + r_n^2)}{T} < \frac{\pi}{\sqrt{12}} = 0.906899 \dots \quad (1.1)$$

if

$$\frac{\min r_i}{\max r_i} \geq 0.742990 \dots =: a_0. \quad (1.2)$$

The bound a_0 in (1.2) has the following meaning. Recall that the area of a regular p -gon circumscribed to a circle of radius 1 is $a(p) = p \tan(\pi/p)$. Then a_0 is defined by

$$a_0^2 := \frac{a(6) - a(7)}{a(5) - a(6)}.$$

Since $\pi/\sqrt{12}$ is the density of the densest packing of equal circles in the plane, the bound given in (1.1) is best possible. However, for small n better bounds exist. For each n , let $\delta(n)$ be the maximum of the densities in (1.1) for all convex T and all packings with n circles satisfying (1.2). Our proof of Theorem 1 simultaneously yields an upper bound for $\delta(3)$, and a lower bound for $\delta(3)$ is given by the density of the packing in Fig. 1 consisting of one circle of radius 1 and two circles of radius a_0 . Moreover, $\delta(2)$ can be determined explicitly. We obtain

Theorem 2. $\delta(2) = 0.885066 \dots$ and $0.8712 \dots \leq \delta(3) < 0.8866$.

2. Proof of the Theorems

Because of (1.2) we may assume

$$a_0^2 \leq r_i^2 \leq 1 \quad \text{for } 1 \leq i \leq n. \quad (2.1)$$

First, we prove that $\delta(2) = 0.885066 \dots$ as claimed in Theorem 2. Consider two touching circles, one with radius 1 and one with radius $r \leq 1$, and let T be their convex hull. Then

clearly the maximum density $\delta(2)$ is $\delta(2) = \max_r (\pi(1+r^2)/T)$. This function of r can be explicitly stated, and the usual technique of considering its first two derivatives shows that it decreases in r , so it attains its maximum for $r = a_0$, which yields $\delta(2) = 0.855066 \dots$, as claimed.

Next, we prove Theorem 1, where we may now assume $n \geq 3$. We may assume that T is compact. Using the power lines defined by the circles $K_1, \dots, K_n$, we decompose T into convex domains $T_1, \dots, T_n$ with $K_i \subset T_i$ for $1 \leq i \leq n$. Denoting the area of T_i also by T_i , we have $T = T_1 + \dots + T_n$.

Instead of (1.1) we prove the stricter inequality

$$\delta + \frac{b_0}{T} \leq \frac{\pi}{\sqrt{12}} \quad (2.2)$$

for some fixed constant $b_0 > 0$ (note that T is bounded from below by $3\pi a_0^2$), which is equivalent to

$$[T_1 - r_1^2 \sqrt{12}] + \dots + [T_n - r_n^2 \sqrt{12}] \geq \frac{b_0 \sqrt{12}}{\pi}. \quad (2.3)$$

This enables us to assume without loss of generality that T is smooth, strictly convex, and that the vertices of the decomposition are all 3-valent. The number of such vertices is $2n - 2$ by Euler's theorem.

If a domain T_i contains boundary points of T , then T_i is an *exterior domain*; otherwise, T_i is an *interior domain*. Let e be the number of exterior domains. As each exterior domain has at least two vertices on the boundary of T , the number of vertices on the boundary of T is $e + \varepsilon$ for some integer $\varepsilon \geq 0$. Let p_i be the vertex number of T_i . Then $p_1 + \dots + p_n + e + \varepsilon = 3(2n - 2)$. Thus

$$p_1 + \dots + p_n = 6n - 6 - e - \varepsilon \quad \text{for some integer } \varepsilon \geq 0. \quad (2.4)$$

Denote the exterior angles of the vertices of T_i by $\beta_1^i, \dots, \beta_{p_i}^i$, see Fig. 2. Put $y_i := \sum_{k=1}^{p_i} \frac{1}{2} \beta_k^i$. Then $y_i = \pi$ for interior domains T_i , and $y_i < \pi$ for exterior domains, since T is strictly convex. In order to determine the deficit $\pi - y_i$ for an exterior domain T_i , consider the supporting line S of T in a point P of the boundary ∂T of T . If P moves along the arc $\partial T \cap T_i$, then S turns around the angle $2(\pi - y_i)$, so that $\sum (\pi - y_i) = \pi$ where the summation extends over all exterior domains. Thus

$$y_1 + \dots + y_n = n\pi - \pi. \quad (2.5)$$

Note that $p_i \geq 2$ and that $p_i = 2$ is only possible for an exterior domain, that is, for $y_i < \pi$. Thus $y_i/p_i < \pi/2$.

Now clearly

$$T_i \geq r_i^2 \left(\pi - y_i + \tan \frac{\beta_1^i}{2} + \dots + \tan \frac{\beta_{p_i}^i}{2} \right)$$

and so, by Jensen's inequality,

$$T_i \geq r_i^2 \left(\pi - y_i + p_i \tan \frac{y_i}{p_i} \right). \quad (2.6)$$

Fig. 2. The exterior angles of the vertices of T_i .

With the function

$$E(p, y) := \pi - y + p \tan \frac{y}{p}$$

defined on the convex domain D given by $p > 0$, $0 \leq y \leq \pi$, and $y/p < \pi/2$, we may rewrite (2.6) as

$$T_i \geq r_i^2 E(p_i, y_i). \quad (2.7)$$

We now distinguish three cases, namely the case that at least four exterior domains occur, the case that three exterior and at least one interior domains occur, and the case of three domains, that is, $n = 3$. To complete the proof we need some results of [3], which we present as lemmas, one corresponding to each of these cases.

The first lemma is shown in Section 8 of [3], it corresponds to inequality (8.2) there.

Lemma 1. *There is a constant $B_1 > 0$ such that*

$$r_1^2[E(p_1, y_1) - \sqrt{12}] + \cdots + r_n^2[E(p_n, y_n) - \sqrt{12}] \geq B_1,$$

if r_i satisfies (2.1), if (p_i, y_i) are points of D satisfying (2.4) and (2.5), and if at least four exterior domains exist.

The second lemma summarizes the content of p. 269 of [3].

Lemma 2. *If three exterior and at least one interior domains exist, then there is an interior domain T_4 and a function $\Delta(p_4)$ such that*

$$T_4 \geq r_4^2[E(p_4, y_4) + \Delta(p_4)]$$

with $\Delta(p_4 = 3) = 0$ and $\Delta(p_4 \geq 4) \geq 0.24925$. (2.8)

Moreover, there is a constant $B_2 > 0$ such that

$$r_1^2[E(p_1, y_1) - \sqrt{12}] + \cdots + r_n^2[E(p_n, y_n) - \sqrt{12}] + a_0^2\Delta(p_4) \geq B_2,$$

if r_i satisfies (2.1), if (p_i, y_i) are points of D satisfying (2.4) and (2.5), and if three exterior and at least one interior domains exist.

In the case $n = 3$, there are just two ways to decompose a convex domain into three convex domains so that the vertices of the decomposition are all 3-valent. Thus for $n = 3$ we have either $p_1 = p_2 = p_3 = 3$ or $p_1 = p_2 = 2, p_3 = 4$. Moreover, (2.5) becomes $y_1 + y_2 + y_3 = 2\pi$. The third lemma corresponds to the very end of Section 8 in [3]

Lemma 3. *There is a constant $B_3 > 0$ such that*

$$r_1^2[E(p_1, y_1) - 3.5436] + \cdots + r_n^2[E(p_3, y_3) - 3.5436] \geq B_3, \quad (2.9)$$

if r_i satisfies (2.1), and if (p_1, y_1) , (p_2, y_2) , and (p_3, y_3) are points as described above.

Note that the value 3.5436 in (2.9) is greater than $\sqrt{12} = 3.4641 \dots$.

Returning to our proof of Theorem 1 for $n \geq 3$, recall that we must show the validity of (2.3) taking into account (2.7) and, in the second case, (2.8). So the above lemmas immediately complete the proof of Theorem 1.

In order to complete the proof of Theorem 2, we must show $\delta(3) < 0.8866$, which is equivalent to

$$[T_1 - r_1^2 3.5434 \dots] + \cdots + [T_3 - r_3^2 3.5434 \dots] > 0, \quad (2.10)$$

where $3.5434 \dots = \pi/0.8866$. The validity of (2.10) follows from (2.7) and Lemma 3, so that Theorem 2 is also true.

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