The Mean Number of Extreme Lines in a Convex Hull of Lines

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Abstract. The dual problem of the mean number of extreme lines among n chosen at random from a convex polygon of lines is defined, evaluations made for $n = 4, 6, 20$ and the asymptotic mean are shown to be the same as in the classic case of n points.

1. Introduction

The aim of this paper is to formulate the problem and approaches to finding the mean number of extreme lines amongst n chosen at random from within a convex polygon of lines. It is the dual of the classical problem of the mean number of vertices of the convex hull of n points. These mean values are obtained explicitly (with some numerical integration) for $n = 4, 6$ and 20 and found to be very close to the corresponding classical results. The asymptotic mean agrees with the same basic result found by Rényi and Sulanke (1963). The framework for convexity and convex hulls of line sets has been previously defined in Gates (1993), but a minimal description is given here.

The classical problem involves n points chosen uniformly randomly and independently inside a convex body, K ; let $e_n^c(K)$ denote the expected number of vertices of the convex hull and let $A_n(K)$ denote its expected area. Discussion of $e_n^c(K)$ in $d = 2$ and then in higher dimensions is one of the oldest problems of geometric probability. For the planar case, finding $e_4^c(K)$ is equivalent to Sylvester's problem of finding the probability that the convex hull of four points is a quadrilateral; this is also equivalent to finding $A_3(K)$. Results on the computation of $e_4(K)$ for various simple K (such as triangle, quadrilateral, regular polygon, ellipse) can be found in Santaló (1976) and Buchta (1985). If K is a quadrilateral of unit area with diagonals dividing K into triangles of areas $\Delta_1, 1 - \Delta_1$ and $\Delta_2, 1 - \Delta_2$, then an old result of Herglotz is that $A_3(K) = \frac{1}{12} - \frac{1}{9}\Delta_1(1 - \Delta_1)\Delta_2(1 - \Delta_2)$. Recently Buchta and Reitzner (1997b) found an expression for $A_n(K)$ when K is a planar convex polygon in terms of the areas of

polygons associated with K . In 1923 Blaschke showed that $e_4^c(K)$ is least for the case that K is a triangle. Groemer (1974) proved $e_{d+1}^c(K)$ is greatest for K an ellipsoid (only) in d dimensions. Recently Giannopoulos (1992) showed that $A_n(K) < A_n(\text{triangle})$ unless K is a triangle (this maximum value is given in a later section here). In the three-dimensional case the expected volume of four uniform points in a simplex has been computed by Mannion (1994) and by Butcha and Reitzner (1992); it is often surmised, but not proven, that this expected volume is greater than for any other convex body.

In connection with this paper, the main reference is Rényi and Sulanke (1963); in particular they obtained an expression for the asymptotic behaviour of $e_n^c(K)$, for K a planar polygon (also in the case of K with smooth boundary). In arbitrary dimension, the asymptotics of $e_n^c(K)$ (as $n \rightarrow \infty$) was explored by Dwyer (1988), Affentranger and Wieacker (1991) and then Bárány and Buchta (1993), for polytopes, and also by Bárány and Larman (1988) for arbitrary convex bodies. A quite different, more stochastic, approach was made by Groeneboom (1988) for planar polygons; by showing that the convex hull vertices, after rescaling, become essentially a Poisson point process near any of the parent vertices, he showed that the number of hull vertices is asymptotically Normal, obtaining the asymptotic variance as well as the mean. A similar approach in Cabo and Groeneboom (1994) obtained results for the boundary length and area of the convex hull. Summaries of these results and more references can be found in survey papers by Schneider (1988), by Weil and Wieacker (1993) and also by Schneider (1997); the latter contains outline references and also some open problems. A brief recent survey of extremal results can be found in Bauer and Schneider (1995). Buchta and Reitzner (1997a, b) use their exact expression for $A_n(K)$, for K a planar polygon, to obtain an asymptotic expansion for $e_n^c(K)$, up to a term n^{-2} .

2. Formulation and Some Finite Results

We work in the set Γ of directed lines in R^2 . Each $g \in \Gamma$ has a left-hand and right-hand closed half-space, $L(g)$ and $R(g)$, respectively. We deal with the lines selected uniformly (with respect to the invariant measure) at random from the compact convex hull, H_0 , of r lines $g_1, g_2, \dots, g_r$. If L_0 is the intersection of all the $L(g_i)$ and R_0 is the intersection of all the $R(g_i)$, then H_0 consists of all lines g such that $L_0 \subset L(g)$ and $R_0 \subset R(g)$; let H_0 have measure μ_0 . L_0 and R_0 are two infinite polygonal sets with edges formed by the g_i . If no two elements of $g_1, g_2, \dots, g_r$ are parallel, to check that $g \in H_0$ it suffices to check that the vertices of L_0 are within $L(g)$ and the vertices of R_0 are within $R(g)$.

If $n \geq 3$ independent uniform random lines $G_1, G_2, \dots, G_n$ are chosen from H_0 they have a convex hull $H(\subset H_0)$; let L and R denote the left-hand and right-hand sets of H and let N_n be the number of extreme lines amongst these n (where G_i is extreme if it is outside the convex hull of the other lines). L is the intersection of half-planes; suppose it has l vertices; suppose that R has r vertices. The extreme lines of H are the lines joining consecutive vertices of L ($l - 1$ of these) and the lines joining consecutive vertices of R ($r - 1$ of these) together with the two support lines from L to R (which form edges of L and R). The number of these extreme lines is $N_n = l + r$, the number of vertices of $L \cup R$, each point of which is of the form $G_i \cap G_j$. In the classic case of n points in a convex region a useful fact used by Rényi and Sulanke (and many others) is that two

points form an edge of the convex hull if and only if the other $n - 2$ points all fall on one side or all on the other side of the line joining them. The dual of this fact is that $G_i \cap G_j$ will form a vertex of $L \cup R$ if the other $n - 2$ lines are all positive (anticlockwise) or all negative (clockwise) about this point.

Let $P = G_1 \cap G_2$ and $p_+(P)$ denote the probability that a random line of H_0 is positive about P and $p_-(P) = 1 - p_+(P)$. Then the probability that P forms a vertex of $L \cup R$ is $(p_+(P))^m + (p_-(P))^m$, where m is used throughout to denote $n - 2$ (so $m \geq 1$). We calculate $p_+(P)$ from the measure of lines with $L_0 \cup \{P\} \subset L(g)$ and with $R_0 \subset R(g)$; this measure can be expressed in terms of interpoint distances. As $N_n = \sum_{1 \leq i < j \leq n} I(i, j)$, where $I(i, j) = 1$ if $G_i \cap G_j$ is a vertex $L \cup R$ and 0 otherwise, the expected number of extreme lines is

$$\begin{aligned} e_n &= E(N_n) = \binom{n}{2} E(I(1, 2)) \\ &= \binom{n}{2} \int_{R^2 - L_0 \cup R_0} ((p_+(P))^m + (p_-(P))^m) f(P) dv, \end{aligned} \quad (1)$$

where dv denotes the Lebesgue measure in the plane and $f(P)$ is the probability density function of the intersection point of two random lines from H_0 . In Gates (1993) the density, for P in $R^2 - (L_0 \cup R_0)$, was shown to be

$$f(P) = 2\mu_0^{-2}(\omega - \sin \omega), \quad (2)$$

where ω is the angular range of the pencil of lines of H_0 through P .

Some elementary algebraic identities are relevant here. If we define polynomials

$$u_m(x) = x^m + (1 - x)^m,$$

then, for odd m , u_m is a linear combination of u_r for $r < m$. For example,

$$\begin{aligned} u_3 &= 1/2(3u_2 - 1); & u_5 &= 1/2(5u_4 - 5u_2 + 2); \\ u_7 &= 1/4(14u_6 - 35u_4 + 42u_2 - 17); \end{aligned}$$

which imply

$$\begin{aligned} e_5 &= 5/2(e_4 - 2), & e_7 &= 7/4(2e_6 - 5e_4 + 12), \\ e_9 &= 3/2(3e_8 - 14e_6 + 42e_4 - 102). \end{aligned} \quad (3)$$

Each point P of $R^2 - (L_0 \cup R_0)$ has two support points on the boundary of $L_0 \cup R_0$, the two vertices at the extremes of the pencil of lines through P . Thus $R^2 - (L_0 \cup R_0)$ is partitioned into compartments, each corresponding to a pair of vertices. Some of these compartments are split into two components. Points within a compartment will be parametrized by a pair of angles from the corresponding vertex pair.

Consider the case when the parent polygon is a triangle ($r = 3$). Let g_1, g_2, g_3 be the three generating lines (no two parallel) labelled at increasing orientations and assume that g_2 is positive about $g_1 \cap g_3$. To generate a triangle the three lines have

Fig. 1

to be non-concurrent and with orientations strictly within a semicircle. Let A, B, C denote both the points of intersection $g_2 \cap g_3, g_1 \cap g_3, g_1 \cap g_2$, respectively, and also the internal angles of the triangle formed (see Fig. 1). Figure 1 shows the three compartments $[AB], [AC]$ and $[BC]$, with $[AB]$ split into $[AB]_1$ and $[AB]_2$ and $[BC]$ into $[BC]_1$ and $[BC]_2$. To avoid cumbersome notation we use an adaptive system. For $P \in [AB]$ let α be the angle from g_3 back to AP and let β be the angle from g_3 back to BP ; with $0 \leq \beta \leq \pi - B$ and $P \in [AB]_1$ if $\alpha \leq \beta$. For $P \in [AC]$ let α be the angle from g_2 forward to AP and let γ be the angle from g_2 back to PC ; with $0 \leq \alpha \leq A$ and $0 \leq \gamma \leq C$. For $P \in [BC]$ let β be the angle from g_1 forward to PB and let γ be the angle from g_1 forward to PC , with $0 \leq \beta \leq \pi - B$ and $0 \leq \gamma \leq C$ and $p \in [BC]_1$ if $\gamma \leq \beta$.

For $P \in [AB]$ let p_* be the probability that a random line has the same orientation about P as g_3 , so that p_* is $p_-(P)$ on $[AB]_1$ and $p_* = p_+(P)$ on $[AB]_2$. On $[AC]$ and $[BC]$ p_* is defined as $p_+(P)$ or $p_-(P)$ by the orientation of g_2 and g_1 , respectively.

Suppose the triangle has been sized to give the total line measure

$$\mu_0 = |AB| + |BC| - |AC| = 1, \quad \text{so that} \quad |AB| = \sin(C)(\sin(A) + \sin(C) - \sin(B))^{-1}.$$

For $P \in [AB]$, $p_* = |AB| + |BP| - |AP| = |AB|U_{AB}$, where $U_{AB} = 1 - (\sin(\beta) - \sin(\alpha)) / \sin(\beta - \alpha)$. Then the probability of m lines passing all on one side of P is

$$p_*^m + (1 - p_*)^m = (|AB|U_{AB})^m + (1 - |AB|U_{AB})^m. \quad (4)$$

If the Cartesian co-ordinates of A and B are (a_1, a_2) and (b_2, b_2) , then those of P are

$$(x, y) = (a_1, a_2) + \frac{\sin \beta}{\sin(\beta - \alpha)} ((b_1 - a_1) \cos \alpha + (b_2 - a_2) \sin \alpha, \\ (b_2 - a_2) \cos \alpha - (b_1 - a_1) \sin \alpha)$$

for $0 \leq \alpha \leq A, 0 \leq \beta \leq \pi - B$.

We can transform the probability element, from (2), to $\alpha - \beta$ co-ordinates to obtain $p_{AB}(\alpha, \beta) d\alpha d\beta$, where

$$p_{AB}(\alpha, \beta) = 2|AB|^2 q(\beta - \alpha) \sin(\alpha) \sin(\beta), \quad (5)$$

with the function q defined on $[0, \pi)$ by

$$q(x) = \begin{cases} (x - \sin(x)) / \sin^3(x), & 0 < x < \pi, \\ \frac{1}{6}, & x = 0. \end{cases} \quad (6)$$

For $P \in [BC]$ the same expressions apply, with AB changed to BC and α replaced by γ to form U_{BC} and $p_{BC}(\beta, \gamma)$.

For $P \in [AC]$ we have $p_* = |AC|U_{AC}$, where $U_{AC} = (\sin(\alpha) + \sin(\gamma)) / \sin(\alpha + \gamma) - 1$ and the corresponding probability density is

$$p_{AC}(\alpha, \gamma) = 2|AC|^2 q(\alpha + \gamma) \sin(\alpha) \sin(\gamma). \quad (7)$$

Determining e_n requires the evaluation of

$$\iint ((|AB|U_{AB})^m + (1 - |AB|U_{AB})^m) p_{AB}(\alpha, \beta) d\alpha d\beta \\ + \iint ((|BC|U_{BC})^m + (1 - |BC|U_{BC})^m) p_{BC}(\beta, \gamma) d\beta d\gamma \\ + \iint ((|AC|U_{AC})^m + (1 - |AC|U_{AC})^m) p_{AC}(\alpha, \gamma) d\alpha d\gamma. \quad (8)$$

It is possible to perform one stage of integration analytically by making the substitutions $\omega = \beta - \alpha$ and $t = \beta + \alpha$, but the second stage has to be carried out numerically. Evaluations of (8) were carried out for two triangles; Triangle 1 had $A = B = C = \pi/3$ and Triangle 2 had $A = C = \pi/4$ and $B = \pi/2$; values of $n = m + 2 = 4, 6$ and 20 were used and the results shown in Table 1. Corresponding values for $n = 5$ and 7 can be deduced using (3).

In the dual case there is also a special case that does not occur classically, this is when two of the parent lines generating H_0 are parallel. Suppose that g_1 and g_2 are parallel, distance h apart, with $g_2 \subset L(g_1)$, and that g_3 makes an angle A with them and crosses from $R(g_1)$ to $L(g_1)$. Denote $g_2 \cap g_3$ by A and $g_1 \cap g_3$ by B and regard the point at infinity $g_1 \cap g_2$ as “ C ”. Again we can decompose $R^2 - (L_0 \cup R_0)$ into three compartments with $[AB]$ having two components. The measure of lines in this “vertical” triangle can be easily shown to be $h \tan(A/2)$, so that $h = \cot(A/2)$ if we assumed this measure has been sized to 1. For $P \in [AB]$ the expressions for p_* and the density for P are as before.

Table 1

Parent polygon	$n = 4$	$n = 6$	$n = 20$
Triangle 1	3.66706	4.56833	7.10502
Triangle 2	3.66745	4.56995	7.11226
Vertical triangle	3.66744	4.56991	7.11267
Classical triangle	3.66667	4.56667	7.09548

For $P \in [AC] = L(g_1) \cap R(g_2) \cap L(g_3)$ we parametrize by the angle α of AP with g_2 and the distance y of P from g_2 . For such a P , $p_* = p_-(P)$ is calculated as $y \tan(\alpha/2)$ and the probability element is

$$dP = 2(\alpha - \sin(\alpha))y(\sin(\alpha))^{-2}d\alpha dy \quad (9)$$

for $0 \leq \alpha \leq A$ and $0 \leq y \leq h$. The integration of $p_*^m + (1 - p_*)^m$ is then

$$\int_0^A 2(\alpha - \sin(\alpha))(\sin(\alpha))^{-2} \int_0^h y[(y \tan(\alpha/2))^m + (1 - y \tan(\alpha/2))^m] dy d\alpha. \quad (10)$$

The integral for the BC case is the same. Numerical evaluations were made for $A = \pi/2$ (and so $h = 1$) for $n = 4, 6$ and 20 and these are included in Table 1.

For comparison with the classical case we give the values of e_n^c for n points in a (any) triangle. From the original Rényi and Sulanke (1963) paper, where we can assume an equilateral triangle (and put back the neglected term), we easily find

$$\begin{aligned} e_n^c(\text{triangle}) &= 4 \binom{n}{2} \int_0^1 \int_0^1 [(1 - xy)^m + (xy)^m] xy dx dy \\ &= 2 \sum_{k=1}^{n-1} \frac{1}{k}, \end{aligned} \quad (11)$$

which can also be found in Buchta (1985).

3. Asymptotic Behaviour of e_n for a Triangle

In the classic case of n points at random in a convex polygon with r vertices Rényi and Sulanke (1963) established that $e_n^c \sim \frac{2}{3}r \ln(n)$, as $n \rightarrow \infty$. We denote by $\varphi(m)$ the function $\ln(m)/m^2$; terms which are $o(\varphi(m))$ will be described as negligible. We consider

$$\iint (p_*^m + (1 - p_*)^m) f(x, y) dx dy \quad (12)$$

for large m . The method of analysis is the classical Laplace method (see Wong, 1989) which says that (12) will be dominated by the region where p_* is close to 1 or 0, this is where P is near the boundary of $R^2 - (L_0 \cup R_0)$. We make approximations to (12) which then behave like (11). The details of the error bounds of the approximation are lengthy and are shown in one case and then omitted. The following lemmas allow bounds for this approximation to be established.

Lemma 1.

(a) With q defined in (6) the function

$$h(t) = \begin{cases} t^{-2}(q(t) - \frac{1}{6}), & t > 0, \\ \frac{3}{40}, & t = 0, \end{cases}$$

is continuous and increasing on $[0, \pi)$ and $h(1) < \frac{1}{10}$.

(b) $(1 - mx^2)e^{-mx} \leq (1 - x)^m \leq e^{-mx}$ for $0 \leq x \leq 1, m \geq 1$.

Proof. For (a), for t near 0,

$$\begin{aligned} h(t) &= (t^3/6 - t^5/120 \dots) - (t^3 - t^5/120 \dots)/6 t^{-2} (t - t^3 \dots)^{-3} \\ &= \frac{3}{40} + O(t), \end{aligned}$$

and the continuity at $t = 0$ is clear. To prove that h is increasing we show that the derivative is positive on $(0, \pi)$:

$$\begin{aligned} r(t) &= h'(t)(t^3 \sin^4(t)) \\ &= \frac{9}{8} + t \sin(2t) - t \sin(t) - 3t^2 \cos(t) - \frac{7}{6} \cos(2t) + \cos(4t)/24 \\ &\geq \frac{9}{8} + t(2t - 4/3t^3 + 4/15t^5 - 8/315t^7) \\ &\quad - t(t - t^3/6 + t^5/120 - t^7/5040 + t^9/9!) \\ &\quad - 3t^2(1 - t^2/2 + t^4/24 - t^6/6! - t^8/8!) \\ &\quad - \frac{7}{6}(1 - 2t^2 + 2t^4/3 - 4t^6/45 + 2t^8/315) \\ &\quad + (1 - 8t^2 + 32/3t^4 - 256/45t^6 + 512t^8/315 - 4096t^{10}/14175) \\ &= 33t^8/840 - 16489t^{10}/1360800. \end{aligned}$$

This lower bound is positive on $(0, \pi/2)$ and so $h'(t)$ is positive on this interval. For a bound on $[\pi/2, \pi)$ write t as $\pi/2 + y$, then

$$\begin{aligned} r(t) &= \frac{9}{8} - (y + \pi/2) \sin(2y) - (y + \pi/2) \cos(y) + 3(y + \pi/2)^3 \sin(y) \\ &\quad + \frac{7}{6} \cos(2y) + \cos(4y)/24 \\ &\geq \frac{9}{8} - (y + \pi/2)(2y) - (y + \pi/2)(1 - y^2/2 + y^4/24) + 3(y + \pi/2)^2(y - y^3/6) \\ &\quad + \frac{7}{6}(1 - 2y^2) + (1 - 8y^2)/24 \\ &= (\frac{7}{3} - \pi/2) + y(3\pi^2/4 - \pi - 1) + y^2(13\pi/4 - \frac{14}{3}) + y^3(\frac{7}{2} - \pi^2/8) \\ &\quad - 25\pi y^4/48 - 13y^5/24 \\ &> 0.5 + 3y + 5y^2 + 2y^3 - 2y^4 - y^5 \\ &\geq 0.5 + (12/\pi^2 + 10/\pi + 2)y^3 - 2y^4 - y^5 \\ &> 0.5 + y^3(7 - (y + 1)^2) \geq 0.5, \end{aligned}$$

for $y < \pi/2$ and the increasing nature of $h(t)$ is established.

For (b) we have, as standard,

$$(1 - x)^m \leq e^{-mx} \leq (1 + x)^{-m},$$

which gives the right-hand side of (b). Also $(1 - mx^2) \leq (1 - x^2)^m$ so

$$(1 - mx^2)e^{-mx} \leq (1 - x^2)^m(1 + x)^{-m} = (1 - x)^m. \quad \square$$

A Corollary to Lemma 1 is that (from (5)) $p_{AB}(\alpha, \beta) \leq K\alpha\beta$ and (from (7)) $p_{AC}(\alpha, \gamma) \leq K\alpha\gamma$, where K is some constant that depends on A, B, C .

Lemma 2. *If*

$$U(x, y) = \begin{cases} 1 - (\sin(x) - \sin(y))/\sin(x - y) & \text{for } x \neq y, \\ 1 - \cos(x) & \text{for } x = y, \end{cases}$$

then

- (a) $(2/\pi^2)xy \leq U(x, y)$ for $0 \leq x < \pi, 0 \leq y < \pi$;
- (b) $xy(\frac{1}{2} - x^2/24) \leq U(x, y) \leq xy(\frac{1}{2} + x^2/2)$ for $0 \leq y \leq x \leq 1$.

Proof. Let $R(x, y) = U(x, y)/xy$, as this is symmetrical in x and y consider only $0 \leq y \leq x$. Using basic trigonometric identities this can be written

$$R(x, y) = \frac{2 \sin(x/2) \sin(y/2)}{xy \cos((x - y)/2)}.$$

As a function of y on $[0, x]$, with x fixed, this is decreasing (as $y^{-1} \sin(y/2)$ and $(\cos((x - y)/2))^{-1}$ are decreasing). Thus

$$2(\sin(x/2)/x)^2 \leq R(x, y) \leq \tan(x/2)/x.$$

The lower bound is a decreasing function of x on $[0, \pi]$, hence showing that $R(x, y) \geq 2/\pi^2$, establishing (a).

For (b) we note that $2(\sin(x/2)/x)^2 \geq \frac{1}{2} - x^2/24$ for all x in $[0, \pi]$, giving the lower bound. Also

$$\tan(x/2)/x = \sin(x/2)/(x \cos(x/2)) \leq 1/(2 \cos(x/2)) \leq 1/(2(1 - x^2/8)) \leq (1 + x^2)/2,$$

with the last inequality valid for $x^2 \leq 7$. Hence the upper bound of (b) will be valid for $0 \leq y \leq x \leq 1$. $\square$

Consider the general triangle and $P \in [AB]$. The function $p_* = |AB|U(\alpha, \beta)$ is continuous on $[0, A] \times [0, \pi - B]$, is always non-negative on this domain, attains its maximum of 1 only at $\alpha = A$ and $\beta = \pi - B$ and has partial derivatives

$$D_1 p_*(\alpha, \beta) = |AB|(1 - \sin \beta \cos(\alpha - \beta))(\sin^2(\alpha - \beta))^{-1},$$

$$D_2 p_*(\alpha, \beta) = |AB|(1 - \sin \alpha \cos(\alpha - \beta))(\sin^2(\alpha - \beta))^{-1}.$$

On the sub-domain $A - C/2 \leq \alpha \leq A$, $\pi - B - C/2 \leq \beta \leq \pi - B$ we have $C/2 \leq \beta - \alpha$ and so both partial derivatives are at least $|AB|(2 \cos^2(C/4))^{-1}$. Using the Mean Value Theorem it follows that, on this sub-domain, $1 - p_* \geq |AB|(2 \cos^2(C/4))^{-1}(A - \alpha + \pi - B - \beta)$. Away from this corner, on $[0, A] \times [0, \pi - B - C/2] \cup [0, A - C/2] \times [0, \pi - B]$, the function $(A - \alpha + \pi - B - \beta)(1 - p_*)^{-1}$ is positive and continuous and so bounded above by $1/m_2$.

Hence if k_1 is the smaller of $|AB|(2 \cos^2(C/4))^{-1}$ and m_2 we have, on all of $[0, A] \times [0, \pi - B]$, that

$$p_* \leq 1 - k_1(A - \alpha + \pi - B - \beta) \quad (13)$$

with the constant $k_1 > 0$. Consequently, integrating over this domain,

$$\begin{aligned} \iint p_*^m p_{AB}(\alpha, \beta) d\alpha d\beta &\leq \iint \exp(-mk_1(A - \alpha + \pi - B - \beta)) K \pi^2 d\alpha d\beta \\ &\leq K \pi^2 / (m^2 k_1^2) = o(\varphi(m)). \end{aligned} \quad (14)$$

Thus the integral of p_*^m makes a negligible contribution to (8) for large m .

To examine the integral of $(1 - p_*)^m$ we will find that the dominant part comes where α and β are small. Let $c(m)$ denote $(\log(m))^{-1}$ and suppose that m is large enough to ensure that $0 < c(m) < \min(1, A, \pi - B)$. On $D_1 = [c(m), A] \times [0, \pi - B]$, from Lemma 1,

$$\iint (1 - p_*)^m p_{AB}(\alpha, \beta) d\alpha d\beta \leq \iint e^{-mp_*} K \alpha \beta d\alpha d\beta \leq \iint K e^{-mk_2 \alpha \beta} \alpha \beta d\alpha d\beta,$$

where $k_2|AB|/\pi^2$. Using the substitutions $p = \alpha\beta$ and $u = \beta/\alpha$ the last integral becomes

$$\begin{aligned} \int_0^{A(\pi-B)} K e^{-mpk_2} p \left\{ \int_{p/A}^{\min\{p/c(m)^2, (\pi-B)^2/p\}} (2u)^{-1} du \right\} dp \\ < K \ln(A/c(m)) \int_0^\infty p e^{-mpk_2} dp = K \ln(A/c(m)) / (mk_2)^2, \end{aligned}$$

which is $o(\varphi(m))$. As the initial integral is positive this also implies that the same integral over the domain $[c(m), A] \times [c(m), \pi - B]$ and, by reversing the variables, over $[0, c(m)] \times [c(m), \pi - B]$ are negligible. This leaves us to consider the integral over the domain $[0, c(m)] \times [0, c(m)]$; by symmetry this is twice the integral over the domain $D_2 = \{(\alpha, \beta): 0 \leq \beta \leq \alpha \leq c(m)\}$.

The following results can be established by making the substitution $y = mk(m)p$ for p .

Lemma 3. Suppose that $\{k(m)\}$ is a sequence with limit k as $m \rightarrow \infty$:

- (a) If $I(m) = \int_0^{c(m)^2} e^{-mk(m)p} p \log(c(m)^2/p) dp$, then $I(m)/\varphi(m) \rightarrow 1/k^2$.
- (b) If $J(m) = \int_0^{c(m)^2} e^{-mk(m)p} p^3 \log(c(m)^2/p) dp$, then $I(m)/\varphi(m) \rightarrow 0$, as $m \rightarrow \infty$.

Now consider

$$A(m) = \iint_{D_2} (1 - p_*)^m p_{AB}(\alpha, \beta) d\alpha d\beta.$$

From Lemma 1, as $\alpha - \beta \leq c(m) \leq 1$, we have

$$\begin{aligned} A(m) &\leq 2|AB|^2 \iint_{D_2} (\tfrac{1}{6} + c(m)^2/10) e^{-mp_*} \alpha \beta d\alpha d\beta \\ &\leq 2|AB|^2 (\tfrac{1}{6} + c(m)^2/10) \iint_{D_2} e^{-mk(m)\alpha\beta} \alpha \beta d\alpha d\beta, \end{aligned}$$

with $k(m) = |AB|(\frac{1}{2} - c(m)^2/24)$. Again making the substitutions $p = \alpha\beta$ and $u = \beta/\alpha$ and performing the integration with respect to u we obtain

$$A(m) \leq 2|AB|^2 (\tfrac{1}{6} + c(m)^2/10) \int_0^{c(m)^2} e^{-mk(m)p} p \ln(c(m)^2/p) / 2 dp,$$

by Lemma 3 we conclude that $\limsup(A(m)/\varphi(m)) \leq \frac{2}{3}$.

Also by Lemmas 1 and 2 we have that

$$A(m) \geq 2|AB|^2 \iint_{D_2} (\tfrac{1}{6} - ml(m)^2 p^2) e^{-ml(m)p} \alpha \beta (1 - c(m)^2/3) d\alpha d\beta,$$

where $l(m) = |AB|(1 + c(m)^2)/2$. The same substitution leads to

$$A(m) \geq 2|AB|^2 (\tfrac{1}{6} - c(m)^2/18) \int_0^{c(m)^2} (p - ml(m)^2 p^3) e^{-ml(m)p} (\log(c(m)^2/p)/2) dp.$$

By Lemma 3 we obtain $\liminf(A(m)/\varphi(m)) \geq \frac{2}{3}$. The two inequalities imply that $\lim(A(m)/\varphi(m)) = \frac{2}{3}$ and hence, combining domains, that

$$\lim \left[1/(\varphi(m)) \left(\iint ((1 - p_*)^m + p_*^m) p_{AB}(\alpha, \beta) d\alpha d\beta \right) \right] = \frac{4}{3}.$$

For the BC term the derivation is the same, also giving the limit of $\frac{4}{3}$.

For $P \in AC$, p_* is the probability of a random line being positive about P so

$$p_* = |AC|(\sin(\alpha) + \sin(\gamma) - \sin(\alpha + \gamma)) / \sin(\alpha + \gamma),$$

which is near 0 if P is near g_2 (small α or small γ) and is near 1 only if P is near B . As in the reasoning of (13) and (14) this implies that the contribution of the integral of p_*^m is asymptotically negligible. For P near g_2 ,

$$(1 - p_*)^m \cong (1 - |AC|\alpha\gamma/2)^m \cong e^{-k_3\alpha\gamma},$$

where $k_3 = |AC|/2$. Hence,

$$\begin{aligned} \int_0^C \int_0^A (1 - p_*)^m p_{AC}(\alpha, \gamma) d\alpha d\gamma &\cong |AC|^2/3 \int_0^{A.C} e^{-k_3 m p} p \int_{p/c(m)^2}^{A^2/p} (1/2u) du dp \\ &\cong 4/3 \varphi(m). \end{aligned}$$

Consequently we see that each of the integrals of $p_*^m + (1 - p_*)^m$ over AB , AC and BC gives an asymptotic value of $\frac{4}{3}\varphi(m)$, independent of the actual angles A , B , C .

In the special case of a vertical triangle the “ AB ” analysis is as before. For the AC case $p_* = (1 - y \tan(\alpha/2))$ is only near 1 if P is near B and the corresponding integral is negligible for large m . Also $(1 - p_*)^m \cong (1 - xy/2)^m \cong e^{-mxy/2}$, and, from (10),

$$\begin{aligned} \int (1 - p_*)^m dP &\cong \iint 2e^{-m\alpha y/2} (\alpha - \sin(\alpha)) y (\sin(\alpha))^{-2} dy d\alpha \\ &\cong \iint e^{m\alpha y/2} (\alpha y/3) dy d\alpha \cong \frac{4}{3} \varphi(m), \end{aligned}$$

again. Consequently, using (1), we have established the following theorem.

Theorem 1. *For n lines uniformly and randomly chosen from the convex hull of three non-concurrent lines, with directions confined to a semicircle, then the expected number of extreme lines, e_n , is asymptotically $2 \ln(n)$ as $n \rightarrow \infty$.*

4. More Than Three Parent Lines

If the parent polygon is defined by more than three lines the previous analysis will apply at each “corner”. Suppose the lines are labelled as follows. Let g_1 denote the line of minimum orientation (in the range $[0, \pi)$) and let $g_2, g_3, \dots, g_s$ denote the lines of increasing orientation that form the edges of the polygon R_0 and let $g_s, g_{s+1}, \dots, g_r, g_1$ be the lines that form the edges of L_0 with decreasing orientation. A vertex of R_0 is of the form $g_i \cap g_{i+1}$ ($1 \leq i \leq s-1$); a vertex of L_0 is of the form $g_i \cap g_{i+1}$ ($s \leq i \leq r$, with $r+1$ identified with 1). Two vertices will be adjacent if they are of the form $g_i \cap g_{i+1}$ and $g_{i+1} \cap g_{i+2}$ or are $g_r \cap g_1$ and $g_1 \cap g_2$.

For a point P of $R^2 - (L_0 \cup R_0)$ with adjacent support vertices and with p_* the probability of a random line of H_0 having the same orientation about P as g_{i+1} the analysis of the integrals of p_*^m and $(1 - p_*)^m$ is the same as in the triangular case, with the former giving negligible contributions and the latter a contribution of $\frac{4}{3} \ln(n)$ to e_n . For a point with two support vertices which are not adjacent we get negligible contributions. If the two vertices are of the form $g_i \cap g_{i+1}$ and $g_{i+2} \cap g_{i+3}$ (indices modulo r) it is possible to have $p_* = 0$ but only in the case that the point is $g_{i+1} \cap g_{i+2}$, the argument then is as given in (13) and (14) to conclude that there can only be a negligible contribution. If the two vertices are more than one vertex apart, then p_* and $1 - p_*$ will be both bounded away from 1 (by the measure of an intervening triangle) and so p_*^m and $(1 - p_*)^m$ will shrink to zero exponentially fast and be negligible relative to $\varphi(m)$. These considerations—which are essentially the same as in the classic case—allow us to conclude that if the parent polygon H_0 is generated (non-redundantly) by r

lines with r vertices to $L_0 \cup R_0$, then, asymptotically, the expected number of extreme lines among n is $\frac{2}{3}r \ln(n)$, that is, the same result as in the classic case.

5. Discussion of Further Possibilities

The main technique in this paper dualizes that of Rényi and Sulanke. There is scope for exploring the duals of other classical techniques. As an illustration consider the upper and lower bounds and their asymptotic nature determined by Dwyer (1988) in the simplest case of a 2-gon of lines in the plane; this will consist of the set Γ_2 of lines having the point $A = (1, 0)$ on the right and the point $B = (-1, 0)$ on the left. Thus Γ_2 consists of lines passing upwards through the interval $[-1, 1]$ of the x -axis. If we parametrize a line by its orientation θ and the distance, p , that the origin is to the left, then Γ_2 corresponds to the parameter set $\{(p, \theta): -\sin \theta \leq p \leq \sin \theta, 0 \leq \theta \leq \pi\}$, which has total measure 4. We now look for upper and lower bounds for the probability that the first of n random lines in Γ_2 is extreme.

First, any g_1 can divide Γ_2 into four regions, with g_1 a vertex of each. We do this by defining two points: the point X is where g_1 crosses the segment AB and the point S is where g_1 crosses the vertical line, $s = (-1, \pi/2)$ through B . Let g'_1 denote the line from S to A . Then Γ_2 is partitioned into four triangles of lines

$$\begin{aligned} \langle \{e, g_1, g'_1\} \rangle &= \{g: X \in L(g); A, S \in R(g)\}, \\ \langle \{w, g_1, g'_1\} \rangle &= \{g: X, S \in L(g); A \in R(g)\}, \\ \langle \{e, g_1, s\} \rangle &= \{g: B \in L(g); S, X \in R(g)\}, \\ \langle \{w, g_1, s\} \rangle &= \{g: B, S \in L(g); X \in R(g)\}, \end{aligned}$$

where e is the line from A to B and w is the line from B to A . If g_1 is in the “first quadrant” ($p \geq 0, 0 \leq \theta \leq \pi/2$), then the first of these has the smallest measure of

$$SX + AX - AS = \frac{\sin \theta + p}{\sin \theta \cos \theta} + \frac{\sin \theta - p}{\sin \theta} - \sqrt{4 + \left(\frac{p + \sin \theta}{\cos \theta}\right)^2} \geq t,$$

where t is defined as $t = (\sin^2 \theta - p^2)/4$ (and so $0 \leq t \leq \frac{1}{4}$). If there are $n - 1$ other lines in Γ_2 , then g_1 cannot be extreme if all the four triangles above are occupied. Thus for n random lines $G_1, G_2, \dots, G_n$

$$\begin{aligned} \Pr(G_1 \text{ is extreme}) &\leq \int_{0 \leq \theta \leq \sin \theta, 0 \leq \theta \leq \pi/2} 4(1 - t/4)^{n-1} dp d\theta \\ &= \int_{-\sin \theta \leq p \leq \sin \theta, 0 \leq \theta \leq \pi/2} 2(1 - t/4)^{n-1} dp d\theta. \end{aligned}$$

In the integral we now change to variables t and $u = (\sin \theta - p)/2$, so that

$$dp d\theta = 2[u^2 - (u^2 + t)^2]^{-1/2} du dt.$$

The above inequality becomes

$$\Pr(G_1 \text{ is extreme}) \leq 4 \int_0^{1/4} (1 - t/4)^{n-1} h(t) dt,$$

where

$$h(t) = \int_{\alpha_t}^{1-\alpha_t} 4[u^2 - (u^2 + t)^2]^{-1/2} du,$$

with $\alpha_t = (1 - \sqrt{1 - 4t})/2$. In fact h is the probability density function of the statistic t ; t itself seems to play the role of the “product of uniforms” statistic that frequently occurs in classical proofs.

For a lower bound for the probability that G_1 is extreme we now dualize the technique of associating a hyperplane through a point in a convex region. In our case for a line g_1 in the first quadrant of Γ_2 we look for a point X^* on g_1 so that the triangle of lines $\{g: B, X^* \in L(g), A \in R(g)\}$ has minimum measure. This triangle of lines has measure $2 + AX^* - BX^*$; this is minimized when

$$X^* = (p^{-1} \sin \theta (\cos^2 \theta + p^2), p^{-1} \cos \theta (\sin^2 \theta - p)),$$

and the minimum measure becomes simply $2 - 2\sqrt{1 - 4t}$. Essentially we have determined the “economic cap” (Bárány and Larman, 1988) through g_1 . If there are $n - 1$ other lines in Γ_2 , then a sufficient condition for g_1 to be extreme is that this “cap” is empty. Thus

$$\Pr(G_1 \text{ is extreme}) \geq \int_0^{1/4} \left(\frac{1 + \sqrt{1 - 4t}}{2} \right)^{n-1} h(t) dt \geq \int_0^{1/4} (1 - 2t)^{n-1} h(t) dt.$$

Consequently we can argue that the expected number, e_n , of extreme lines amongst n in Γ_2 is bounded by $L_n \leq e_n \leq U_n$, where

$$L_n = n \int_0^{1/4} (1 - 2t)^{n-1} h(t) dt \quad \text{and} \quad U_n = 4n \int_0^{1/4} (1 - t/4)^{n-1} h(t) dt.$$

To provide simple estimates of $h(t)$, write $Q_1 = u - u^2 - t$ and $Q_2 = u + u^2 + t$, so that $Q_1 Q_2$ is the quartic in the denominator of the $h(t)$ integral. For $\alpha_t \leq u \leq 1$,

$$Q_1 \leq u - \alpha_t, \quad Q_2 \leq 2u + t,$$

thus

$$\begin{aligned} h(t) &\geq \int_{\alpha_t}^{1/2} 4[(u - \alpha_t)(2u + t)]^{-1/2} du \\ &= 2\sqrt{2} \{ \ln(2\sqrt{2(1 - 2\alpha_t)(1 + t)} + 2 + t - 2\alpha_t) - \ln(t + 2\alpha_t) \} \geq 2\sqrt{2} \ln(1/4t). \end{aligned}$$

Also

$$Q_1 \geq \frac{1 - 4t}{2(1 - 2\alpha_t)} \min\{(u - \alpha_t), (1 - \alpha_t - u)\}, \quad Q_2 \geq u + t,$$

so

$$\begin{aligned}
 \int_{\alpha_t}^{1-\alpha_t} \frac{1}{\sqrt{Q_1 Q_2}} du &\leq \sqrt{\frac{2(1-2\alpha_t)}{1-4t}} \\
 &\quad \times \left(\int_{\alpha_t}^{1/2} \frac{1}{\sqrt{(u-\alpha_t)(u+t)}} du + \int_{1/2}^{1-\alpha_t} \frac{1}{\sqrt{(1-u-\alpha_t)}} du \right) \\
 &\leq 2\sqrt{\frac{2(1-2\alpha_t)}{1-4t}} \int_{\alpha_t}^{1/2} \frac{1}{\sqrt{(u-\alpha_t)(u+t)}} du \\
 &= 2\sqrt{\frac{2(1-2\alpha_t)}{1-4t}} \{ \ln[\sqrt{(1-2\alpha_t)(1+2t)} + 1 + t - \alpha_t] \\
 &\quad - \ln[\alpha_t + t] \}.
 \end{aligned}$$

The last expression can be shown to be no greater than $2\sqrt{10}\ln(9/8t)$ and so a fairly inefficient upper bound for the $h(t)$ is $8\sqrt{10}\ln(9/8t)$.

Thus

$$L(n) \geq n \int_0^{1/4} 2\sqrt{2} \ln(1/4t) (1-2t)^{n-1} dt \approx \sqrt{2} \ln(n),$$

the last approximation is “standard” and follows the same lines as in the classical case (see, for example, Dwyer, 1988). Also

$$U(n) \leq 32\sqrt{10}n \int_0^{1/4} (1-t/4)^{n-1} \ln(9/8t) dt \approx 128\sqrt{10}\ln(n).$$

It follows, by the arguments of this section, that $e_n = \Omega(\ln(n))$, as in the classical case (here only for the 2-gon).

Other classical methods may be adaptable; the approach of Groeneboom uses the boundary process indexed by support lines of a point set. The dual approach would use a boundary process of lines indexed by the points of $\partial(L_0 \cup R_0)$.

A Poisson process line is familiar, but re-scaling of sets of lines may present a problem as may the (probably) more complex expressions for Poisson void probabilities, which Groeneboom found amenable.

The ideas here extend to hyperplanes in higher dimensions. In Gates (to appear) the three- and higher-dimensional Sylvester–Klee problem is discussed with some numerical, but no asymptotic results.

Finally we note that the definition of convex hulls of (directed) lines used here is *different* to that introduced by Goodman and Pollack (1995); with their definition a general triad of lines is already convex (and so the expected measure of its hull is zero).

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