***K*-Levels of Concave Surfaces**

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Abstract. We show that for any line l in space, there are at most $k(k+1)$ tangent planes through l to the k -level of an arrangement of concave surfaces. This is a generalization of Lovász's lemma, which is a key constituent in the analysis of the complexity of k -levels of planes. Our proof is constructive, and finds a family of concave surfaces covering the "laminated at-most- k -level." As a consequence, (1) we have an $O((n-k)^{2/3}n^2)$ upper bound for the complexity of the k -level of n triangles of space, and (2) we can extend the k -set result in space to the k -set of a system of subsets of n points.

1. Introduction

Investigation of the combinatorial properties of k -levels of arrangements of curves and surfaces is a central topic in combinatorial geometry. The k -level problem can be considered as the parametric version of the *selection* problem, which finds the k smallest elements among n weighted elements. If we replace the weight of each element by a function with $d-1$ parameters, we have an arrangement of n hypersurfaces in d -dimensional space, and its k -level shows the trajectory of the k th smallest element. Note that, in this paper, we count the levels in an arrangement from the bottom, unless explicitly specified. The union of i -levels for $i = 1, 2, \dots, k$ is called the *at-most- k -level*.

So far, only k -levels of arrangements of hyperplanes are investigated well in the literature except for some special arrangements of curves [4], [15], [16]. A point p on a surface (curve, if $d = 2$) is called a *concave point* if there is a hyperplane through p which is above or on the surface in a suitable open neighbor of p . Such a hyperplane is called a *tangent hyperplane* to the surface at p . A surface is called a *concave surface* if

every point on it is a concave point. A vertex of a k -level is called a *concave vertex* if it is a concave point on the k -level regarded as a surface.

Lovász [13] gave an $O(k^{1/2}n)$ upper bound for the complexity of the k -level of an arrangement of n lines in the plane by using the following lemma:¹

Lemma 1 (Two-Dimensional Lovász's Lemma). *From any point in the plane, there are at most $2k$ tangent lines to concave vertices of the k -level of an arrangement of lines in the plane.*

Here, a tangent line may cross the k -level at parts other than the tangent point; in other words, the tangent condition is determined locally at the (tangent) vertex. Agarwal et al. [1] pointed out that the lemma is immediately obtained from the fact that there exist a family of k concave chains in the arrangement such that all concave vertices of the k -level are concave vertices of the chains.

Utilizing the family of concave chains (which we often call the *concave chain decomposition* of the k -level), Dey [6] improved the upper bound of the k -level complexity to $O(k^{1/3}n)$. Eppstein [9] pointed out that the complexity of k concave chains is asymptotically equivalent to the number of transitions of a parametric matroid of rank k , and his theory can be further extended to parametric polymatroids [12].

This framework can be extended to an arrangement of concave curves. A simple observation is that the lower envelope of an arrangement of lines is concave, and it has at most two tangent lines from any point in the plane. This property holds not only arrangement of lines but also arrangement of concave curves in general, and we also have the following family of concave chains:

Lemma 2 (Concave Chain Decomposition for Concave Curves). *There exist a family of k concave chains of curves in an arrangement of concave curves such that their set union is the at-most- k -level.*

Since the k -level is the top level among the at-most- k -levels, all concave vertices of the k -level of the arrangement appear as vertices of the chains in the concave chain decomposition. Therefore, an analogue of Lemma 1 holds for the k -level of an arrangement of concave curves. Although the complexity of its k -level is difficult to analyze in general, the above framework is useful: for example, the k -level (counted from the top) of n segments can be transformed into the $(n - k)$ -level (counted from the bottom) of an arrangement of n concave curves each of which consists of three linear pieces (the original segment and two steep halflines), and hence an $O(n^{4/3})$ bound [6] can be easily derived from it.

Our target is the three-dimensional case. Lovász's lemma can be extended to the following three-dimensional version:

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¹ Lovász originally considered the dual problem, and this formulation was explicitly given by Agarwal et al. [1].

Lemma 3 (Three-Dimensional Lovász's Lemma). *From any line l in the space, there are at most $k(k + 1)$ tangent planes through l to concave vertices of the k -level of an arrangement of planes in the three-dimensional space.*

The lemma was first proved by Bárány et al. [3] in a weaker form, and Agarwal et al. [1] gave an elegant proof (for the general dimensional case) by using a projective transformation and a beautiful probabilistic argument of Clarkson [5]. An $O(k^{5/3}n)$ bound for the complexity of the k -level of a three-dimensional arrangement is obtained [7], [1] from this lemma.²

Agarwal et al. [1] generalized the three-dimensional Lovász's lemma for the k -level of an arrangement of triangles, but the bound was considerably weakened ($O(n^{2.5})$ tangents instead of $k(k + 1)$). This prevents application of the $O(n^{8/3})$ upper bound for the k -level (for $k = \Omega(n)$) of an arrangement of planes to the complexity of the k -level of more general objects. This was one of the motivations of our work.

Indeed, it is true that the lower envelope of an arrangement of concave surfaces is also concave, and hence it has at most two tangent planes from any line in the space. Therefore, it is natural to try to extend Lovász's lemma to an arrangement of concave surfaces. Our main result is the following theorem:

Theorem 4 (Lovász's Lemma for the k -Level of Concave Surfaces). *For any line l in space, there are at most $k(k + 1)$ tangent planes through l to the k -level of an arrangement of concave surfaces.*

As a corollary, we have an $O((n - k)^{2/3}n^2)$ bound on the k -level of n triangles, which improves the known result [1] by a factor of $n^{1/9}$ (for $k = \Omega(n)$).

More generally, we have an $O(k^{2/3}n^2)$ upper bound for the complexity of the k -level of an arrangement of concave polyhedral surfaces created as a subset of an arrangement of n planes. Its dual problem is also meaningful: given a set $\mathcal{P}$ of subsets of a set S of n points in the space, the semispace (intersection with a halfspace) of $\mathcal{P}$ is called a k -set with respect to $\mathcal{P}$ if there are exactly k members of $\mathcal{P}$ intersecting it. We can show that the number of different k -sets with respect to $\mathcal{P}$ is $O(k^{2/3}n^2)$.

Our proof of Theorem 4 is constructive and intuitive in the following sense: It is plausible to conjecture that a three-dimensional analogue of the concave chain decomposition (Lemma 2) exists. If it were true, a recent result [2] by Aronov and Dey can be applied to investigate complexities. Unfortunately, even for an arrangement of planes, it is wrong that there always exists a family of $k(k + 1)/2$ concave surfaces whose set union is the at-most- k -level. Moreover, it does not always hold that there exists a family of $k(k + 1)/2$ concave surfaces covering the k -level. See the Appendix for a counterexample. Therefore, we consider a weaker analogue, and construct a family of concave surfaces covering the *laminated at-most- k -level* (defined in Section 2). Although we need to add artificial planes to the original arrangement, which increases the number of concave surfaces, the construction leads to Theorem 4.

² The $O(k^{5/3}n)$ bound has been improved to $O(k^{3/2}n)$ by Sharir et al. [14].

2. Laminated At-Most- k -Level and Its Concave Decomposition

2.1. Laminated At-Most- k -Level

If we prove Theorem 4 for an arrangement of concave polyhedral surfaces, it is easy to see (as explained in the paragraph below Proposition 5) that it also holds for an arrangement of any concave surfaces. Hence, we deal with polyhedral surfaces.

Consider a set $\mathcal{C}$ of concave polyhedral surfaces in the space. We gather all the planes supporting facets of concave polyhedral surfaces in $\mathcal{C}$ to obtain an arrangement $\mathcal{A}$ of the set $\mathcal{F} = \{F_1, \dots, F_n\}$ of these planes. Note that each element of $\mathcal{C}$ is obtained as the lower envelope $C(S)$ of the planes in a subset S of $\mathcal{F}$. A point p in a surface C in $\mathcal{C}$ is in the k -level if and only if there are at most $k - 1$ surfaces (in $\mathcal{C}$) below p and at most $n - k$ surfaces beyond p . The set L_k of points in the k -level forms a continuous monotone surface (i.e., its projection to the x - y plane is one-to-one). We want to prove the following proposition:

Proposition 5. *For any line l in the space, there are at most $k(k + 1)$ tangent planes through l to L_k .*

The number $k(k + 1)$ is independent of the number of faces in the arrangement and the statement has a “closed” property in topological sense; therefore, it also holds at a limit where the surfaces are not polyhedral (e.g., smooth).

For the i -levels L_i for $i = 1, 2, \dots, k$ of $\mathcal{C}$, their union $\bigcup_{i=1}^k L_i$ is called the at-most- k -level. We write mL_i to denote the multiset obtained as the union of m copies of L_i ; in other words, a structure obtained by giving the multiplicity m to each facet in L_i . We consider $\mathbf{L} = \bigcup_{i=1}^k (k - i + 1)L_i$ (union as a multiset), which we call the *laminated at-most- k -level*.

In order to prove Proposition 5, we apply a space-sweep method to generate a family of concave polyhedral surfaces covering $\mathbf{L}$ (as a multiset) so that all the concave vertices on L_k appear as vertices of these surfaces. Before considering the three-dimensional problem, we discuss two-dimensional problems that play key roles in the solution of the three-dimensional case.

2.2. Concave Chain Decomposition of a Two-Dimensional Laminated At-Most- k -Level

We consider a set of concave chains $\{C_1, \dots, C_m\}$ whose edges are among the edges of an arrangement of n lines $\{l_1, \dots, l_n\}$ in the plane. The k -level L_k is the trajectory of the k th lowest chain. Note that this is not the k -level of the arrangement of lines, but the k -level of the arrangement of concave chains. The union $\bigcup_{j=1}^k L_j$ is the at-most- k -level, and $\mathbf{L} = \bigcup_{i=1}^k (k - i + 1)L_i$ is the laminated at-most- k -level. Since it will not cause confusion, we use the same notation as in the three-dimensional case here.

Lemma 6. *There is a set of k concave chains in the arrangement covering the at-most- k -level $\bigcup_{j=1}^k L_j$ such that all concave vertices of L_k are vertices of these chains.*

Proof. This set of concave chains can be easily obtained by using a plane-sweep method from left to right. At the left end (i.e., $x = -\infty$), we start with the k lowest lines in the arrangement. Each chain goes straight until the highest one meets a concave vertex (it must be a vertex of L_k). Then the highest turns along another line that intersects it at the vertex, and the process continues. Since there are only concave turns, we have k concave chains. $\square$

Corollary 7. *There is a set of $k(k+1)/2$ concave chains in the arrangement satisfying that (1) the chains cover the laminated at-most- k -level (i.e., each edge in the i -level is contained in $k-i+1$ chains), and (2) all concave vertices of the at-most- k -level are vertices of these chains.*

Figure 1 illustrates such a decomposition for the case of an arrangement of lines. We can generate a set of concave chains satisfying the above corollary as the union of the sets of concave chains obtained in Lemma 6 for L_j ($j = 1, 2, \dots, k$). However, there are many other possible combinations of concave chains satisfying the corollary, and we utilize this fact to solve the three-dimensional problem. Note that the above results themselves can be shown in a more general framework using parametric polymatroids [12].

Fig. 1. Laminated at-most-3-level in the plane and decomposition into concave chains.

Fig. 2. Possible topologies of cross sections. The bold parts are their at-most-2-levels.

2.3. A Motivating Observation

To prove Proposition 5, we want to generate a family of concave surfaces analogous to the two-dimensional concave chain decomposition (Lemma 6). A naive idea using the space-sweep method is to keep track of the concave chain decomposition (given in Lemma 6) of each cross section of $\mathcal{C}$ with the sweep plane. Unfortunately, this does not work well.

Consider the 2-level of the arrangement of three planes F_i , F_s , and F_j . The topology of its cross section is shown by either the left-hand part of Fig. 2 (type **X**, since its at-most-2-level resembles the letter X) or the right-hand part (type **A**) of the same figure.

Each of them has a unique concave chain decomposition into two concave chains, as shown in Fig. 2. Suppose that the initial state is the left-hand part of the figure, and the intersection q comes above the line s (the cross section of F_s with the sweep plane) when we translate the sweep plane, and that the **X**-type topology changes to **A**-type. We can observe that there is no way to continue the pair of chains of type **X** to type **A**.

However, if we consider the laminated at-most- k -level (i.e., give multiplicity 2 to the lowest level as shown in Fig. 3), there are two different concave chain decompositions of the laminated at-most- k -level of the **A**-type configuration into three concave chains, one is $D(A(1))$, and the other is $D(A(2))$, as shown in Fig. 4. Here, we only describe the two top levels containing the corresponding vertex of the k -level on the sweep plane. On the other hand, the laminated at-most- k -level of the **X**-type configuration has a unique decomposition $D(X)$ into three concave chains, as shown in Fig. 5. It is indeed possible to continue the three chains of $D(X)$ into those of $D(A(1))$ to obtain three concave surfaces. This observation inspired us to use the laminated at-most- k -level.

2.4. Construction of a Concave Surface Decomposition for the Three-Dimensional Case

Now, we give the construction for the three-dimensional case. Without loss of generality, we can assume that no pair of concave surfaces in $\mathcal{C}$ share a supporting plane (i.e., a plane containing their facets). Indeed, the statement of Lovász's lemma is independent of the number of planes, and we can replace the plane supporting j facets by a set of j parallel copies (infinitesimally near to each other) of the plane. Recall that, in our definition, a facet contained in j concave surfaces is counted j times to determine the level-ranking of the surfaces. Therefore, the above assumption can be removed. We can also assume that the arrangement is simple; that is, no four planes intersect at a point. A symbolic perturbation can remove this assumption easily.

Fig. 3. Three-dimensional images of a type **A** cross section and a type **X** cross section.

Fig. 4. Concave chain decompositions of an **A**-type laminated at-most- k -level.

Fig. 5. Concave decomposition of an X-type laminated at-most- k -level.

It seems to be a natural extension of the two-dimensional case that we can construct a family of $k(k+1)/2$ concave surfaces containing all concave vertices of L_k . Unfortunately, there are counterexamples showing that this is impossible (see the Appendix).

In our construction, some vertical hyperplanes parallel to l are newly added to the arrangement, which introduces some new artificial vertices to the arrangement. Therefore, the construction depends on the line l . We say that a tangent plane from l to the arrangement is *active* if the tangent point is an original concave vertex of the arrangement; otherwise (i.e., if the tangent point is an artificial vertex) it is called *inactive*. The tangent points are also called active and inactive accordingly. We prove the following theorem that implies Proposition 5:

Theorem 8. *Given a line l in space, there exists a set Y of concave surfaces covering the laminated at-most- k -level $\mathbf{L}$ as a multiset satisfying the conditions that (1) all concave vertices of L_k are contained as concave vertices of the concave surfaces in Y , and (2) there are at most $k(k+1)$ active tangent planes to these surfaces from the line l .*

3. Proof of the Theorem

3.1. Space-Sweep Method

Let H_0 be the y - z plane defined by $x = 0$. Without loss of generality, we can assume that l is contained in H_0 and not parallel to the z -axis. We also assume that no pair of vertices in $\mathcal{A}$ is located on a plane parallel to H_0 . This can be attained by rotating the z -axis infinitesimally keeping the k -level structure unchanged.

We add to the arrangement two artificial planes $y = \tau$ and $y = -\tau$ vertical to H_0 for a sufficiently large τ so that all vertices of the arrangement are located in the slab bounded by these two parallel hyperplanes. We restrict the surfaces in $\mathcal{C}$ within this slab, so that each surface is cut at these planes and continues on these planes outside the slab. These two artificial planes are added in order to simplify the proof; later, we will add more artificial planes that will play essential roles.

We start from H_0 and execute the space-sweep procedure using a sweep plane parallel to H_0 in two directions. We consider only one direction (say, toward $x = \infty$), since the other can be treated similarly. For any sweep plane H , the cross section of the arrangement is an arrangement $\mathcal{A}(H)$ of lines. Moreover, the cross section $\mathbf{L}(H)$ of the laminated at-most- k -level is the laminated at-most- k -level in $\mathcal{A}(H)$ of the k -level of the concave chains obtained as the cross sections of the surfaces of $\mathcal{C}$. Hence, from Corollary 7, $\mathbf{L}(H)$

can be decomposed into $k(k+1)/2$ concave chains. We select a suitable decomposition of $\mathbf{L}(H)$ for each H and trace the topological change of these concave chains while sweeping H from H_0 to $x = \infty$ to construct a family of concave surfaces.

The topological change naturally occurs when H passes through a vertex q of $\mathcal{A}$. This sweep plane is denoted by $H(q)$. Between the sweep planes $H(p)$ and $H(q)$ of two consecutive vertices p and q in the sweep process, we can keep the topology of $\mathbf{L}(H(p))$ to have a family $\mathbf{S}(H(p))$ of concave surfaces $S_1(H(p)), \dots, S_{k(k+1)/2}(H(p))$ between the slab bounded by $H(p)$ and $H(q)$.

Let U be the arrangement of three planes, say F_s, F_i, F_j , meeting at q , and let H' and H'' be the hyperplanes just before and after $H(q)$, respectively. Let $\mathbf{L}_U$ be the part of $\mathbf{L}$ restricted to the planes of U .

We classify the vertex q into three cases: The first is the triple intersection, where F_s, F_i, F_j belong to three different input concave chains of $\mathcal{C}$; the second is the double intersection, where two of them are in the same input concave chain. The last case is the case where q itself is a concave vertex in an input concave chain. (Recall that we have assumed that no pair of input concave surfaces shares a supporting plane.) The last case is trivial, and therefore we consider the other two cases.

Topology Change at Triple Intersection. We first consider the case in which q is a triple intersection. The vertex q is further classified into the following three cases:

Case 1: q is a concave vertex on the k -level: i.e., the lower envelope of U is on the k -level in the neighborhood of q . Note that this is the only case in which the vertex is a concave vertex of L_k .

Case 2: q is a saddle-type vertex on the k -level: i.e., two lower levels of U are in the union of the $(k-1)$ -, k -, and $(k+1)$ -levels in the neighborhood of q .

Case 3: q is a vertex below the k -level: i.e., the whole of U is on or below the k -level in the neighborhood of q .

For Case 1, $\mathbf{L}_U$ itself is a concave surface with a multiplicity of one. Figure 6 illustrates the topological change on the cross section. Here, s, i, j correspond to F_s, F_i , and F_j , respectively. Therefore, this is a topological change on a unique surface in $\mathbf{S}(H(p))$ into a surface in $\mathbf{S}(H(q))$, and is a continuous and concave change; hence we can connect these two surfaces into one concave surface. The vertex q is realized as a concave vertex of the surface.

For Case 2, the cross section $\mathbf{L}_U(H')$ is of one of two types, **X** and **A** shown in Figs. 5 and 4, while the cross section $\mathbf{L}_U(H'')$ should be of the other type. As we saw in Section 2.3 the **X**-type configuration has a unique decomposition $D(X)$ of $\mathbf{L}_U(H')$ into three concave chains, while the **A**-type configuration has two possible decompositions $D(A(1))$ and $D(A(2))$, shown in Fig. 4. If $\mathbf{L}_U(H')$ is of type **X**, we decompose $\mathbf{L}_U(H'')$

Fig. 6. Topology change for Case 1.

Fig. 7. Topology change from $D(A(2))$ to $D(A(1))$.

(which must be of type **A**) into $D(A(1))$, so that the concave chains naturally continue as we saw in Section 2.3. This process creates no vertex or face. Conversely, if $\mathbf{L}_U(H')$ is of type **A** and is decomposed into $D(A(1))$, it continues to the unique decomposition of $\mathbf{L}_U(H'')$ (which must be of type **X**).

The difficult case is when (1) $\mathbf{L}_U(H')$ is an **A**-type configuration, and (2) its decomposition is given as $D(A(2))$ (note that the decomposition is inherited from the previous sweep steps). Of course, $\mathbf{L}_U(H'')$ must be of type **X**. In this case we first convert the $D(A(2))$ decomposition into $D(A(1))$ for $\mathbf{L}_U(H')$ by introducing new concave surfaces. More precisely, we terminate two concave surfaces S_1 and S_2 associated with the broken-line chain and the dashed-and-dotted-line chain in the left part of Fig. 7 by adding the (vertical) faces on H' and creating two new concave surfaces S'_1 S'_2 associated with broken-line chain and dashed-and-dotted-line chain in the right part of Fig. 7.

Then the $D(A(1))$ decomposition obtained can be continued into the $D(X)$ decomposition of $\mathbf{L}_U(H'')$. The key observation is as follows:

Lemma 9. *Among four surfaces S_1 , S_2 , S'_1 , and S'_2 , two surfaces have inactive tangent points from l . Moreover, the inactive tangent points are on the plane H' , and they are different from each other.*

Proof. All of S_1 , S_2 , S'_1 , and S'_2 have a face on the hyperplane F_s in Fig. 7. Consider the vertical sections $H' \cap S_1$ and $H' \cap S_2$, and the associated tangent points r_1 and r_2 to these vertical (planar) concave chains from l . Note that the tangent points must exist, since we added artificial planes $y = \tau$ and $y = -\tau$ to the arrangement. Thus, we can easily observe that r_1 must be a tangent point on a tangent plane through l to either S_1 or S'_1 . Similarly, r_2 must be a tangent point on either S_2 or S'_2 . These tangent points are inactive, since they are not vertices in the original arrangement. It is clear that they are different from each other. $\square$

For Case 3, suppose that p is the intersection of three hyperplanes with multiplicities m , $m+1$, and $m+2$ for $m \geq 1$ in the laminated at-most- k -level. Then we can see that any concave chain decomposition of **X** must contain m copies for each of three lines, since each edge in the upper envelope of U must go straight in order to have concavity. The same fact is observed for an **A**-type configuration (left part of Fig. 8): the decomposition must have m copies of each of lines i and j , since they appear in the upper envelope of U , and if we remove these lines, the remaining configuration with multiplicities is shown in the right figure of Fig. 8, for which it is easy to observe that any concave decomposition must contain m copies of the line s .

Fig. 8. A-type configuration at a deep level and removal of m copies of lines i and j .

Removing these straight lines (which are naturally continued through q), we have the situation of Case 2, and the previous argument holds.

Topological Change at a Double Intersection. Now we consider a double intersection q . We consider the same subcases—Cases 1–3—as the triple intersection case. Case 1 can be handled similarly, and hence we consider Cases 2 and 3. Note that no concave vertex on L_k appears for these cases. Assume that the segments F_i , F_s , and F_j on the cross section $\mathbf{L}_U(H(q))$ meeting at q are ordered in increasing order of the slopes of the lines.

If F_i and F_s are on the same input concave surface and F_j is on another, the concave chain decompositions of $\mathbf{L}_U(H')$ and $\mathbf{L}_U(H'')$ are uniquely determined, and we can observe that no topological change occurs in our concave surface decomposition. Similarly, if F_j and F_s are on the same input concave surface, no topological change occurs.

Hence, consider the case in which F_i and F_j are on the same input concave surface C and F_s is on C' . Let C_U and C'_U be the restrictions of the input surfaces at the local neighbor of q . $\mathbf{L}_U(H')$ has two layers: the upper one has multiplicity m and the lower one has multiplicity $m + 1$. However, we can observe that any chain decomposition of $\mathbf{L}_U(H')$ (resp. $\mathbf{L}_U(H'')$) has at least $m - 1$ copies of each of $C_U \cap H'$ and $C'_U \cap H'$ (resp. $C_U \cap H''$ and $C'_U \cap H''$). Therefore, it suffices to consider the case in which $m = 1$.

The possible configuration is either **A**-type or *separate-type* as in Fig. 9. The *separate-type* configuration has a unique decomposition shown in the right part of Fig. 9. If $\mathbf{L}_U(H')$ is *separate-type*, it can be extended to the $D(A(2))$ decomposition of $\mathbf{L}_U(H'')$ as shown in Fig. 10. Conversely, if $\mathbf{L}_U(H'')$ is of type **A** and decomposed into $D(A(2))$, it can be naturally extended to the *separate-type* configuration of $\mathbf{L}_U(H')$.

The only case in which we need to introduce a new concave surface is that in which the concave chain decomposition of $\mathbf{L}_U(H')$ is $D(A(1))$. In this case we first transform the $D(A(1))$ decomposition of $\mathbf{L}_U(H')$ into a $D(A(2))$ decomposition by terminating

Fig. 9. Separate-type configuration.

Fig. 10. Transition from separate-type to A-type ($D(A(2))$ decomposition).

two surfaces and creating two new surfaces. It is easy to see that the following analogue of Lemma 9 holds:

Lemma 10. *Consider the two terminated and two created surfaces when we transform the $D(A(1))$ decomposition to the $D(A(2))$ decomposition. Among these four surfaces, two surfaces have inactive tangent points from l on the hyperplane H' . Moreover, these inactive tangent points are different from each other.*

Therefore, this space-sweep method (sweeping to the right from H_0) constructs a family of concave surfaces starting with $k(k+1)/2$ initial surfaces, and if N surfaces are newly introduced during the sweep procedure, we have at least N inactive tangent points from l . Without loss of generality, we can assume a nondegeneracy condition that no pair of vertices is on the same sweep plane. Thus, it does not occur that an inactive tangent point is counted twice in the sweeping process, since an inactive tangent point is on a sweep plane infinitesimally near to a vertex of the arrangement, and two inactive tangent points on a sweep plane are different from each other.

Therefore, we have at most $k(k+1)/2$ active tangents. Combined with the sweep process, which sweeps toward the left, there are at most $k(k+1)$ active tangent points. This completes the proof of Theorem 8.

3.2. Applications

We consider applications of our generalized Lovász's lemma to geometric problems.

Proposition 11. *There are $k(k+1)/2$ local maxima in the k -level of an arrangement of concave surfaces.*

Proof. A special case of Theorem 4 where l is “the line at infinity parallel to the x -axis” in the projective geometric sense. Here, the set of planes through l is the set of planes parallel to the x - y plane. $\square$

Theorem 12. *The number of concave vertices on the k -level of a family of concave polyhedral surfaces in an arrangement of n planes in the space is $O(n^2 k^{2/3})$.*

Proof. The proof is exactly the same as that for the k -level [7] of planes. Suppose there are Y concave vertices (including original vertices of the input polyhedral surfaces). If we consider the dual, we have a triangle associated with the three meeting hyperplanes for each concave vertex. Hence, we have Y triangles whose vertices are among the set of dual points of the n planes. Two dual triangles intersect each other if and only if in the primal space we can draw a tangent plane from one suitable ridge of one of the concave vertices to another concave vertex. Hence, there are $O(k^2 Y)$ intersections, but it is known from the triangle intersection bound [7] that there must be $\Omega(Y^4/n^6)$ intersections. Hence, $Y = O(n^2 k^{2/3})$. $\square$

Corollary 13. *The complexity of the k -level of a family of concave polyhedral surfaces on an arrangement of n planes in the space is $O(n^2 k^{2/3})$.*

Proof. Each vertex of the k -level is represented as a concave transition point of either the k -, $(k-1)$ -, or $(k-2)$ -level. Hence the corollary follows from Theorem 12. $\square$

By using the theorem, we obtain the following result on the k -level of triangles by using the theorem:

Theorem 14. *The complexity of the k -level of n triangles in space is $O((n-k)^{2/3} n^2)$.*

Proof. Consider a set T of n triangles, and let $\mathcal{A}$ be its arrangement. We can assume that no triangle is vertical to the x - y plane. Instead of considering the k -level counted from the bottom, we consider the k -level counted from the top, since we can apply reflection with respect to the x - y plane.

First, we claim that we can assume without loss of generality that the projected image of a vertex v of a triangle of T to the x - y plane does not lie on the projected image of an edge e of $\mathcal{A}$ unless v is an endpoint of e . Indeed, suppose that we have a vertex v of a triangle Δ such that the projected image of v is on the projected image of an edge e of $\mathcal{A}$. For any real number ε , we can suitably move v by a distance less than ε on the plane containing Δ so that the new triangle contains Δ and the degeneracy is removed if v is not an endpoint of e . Although this reformation may change the structure of the k -level, setting ε small enough, we can assure that a nonempty segment of each edge in the k -level remains in the k -level of the reformed arrangement, and hence the combinatorial complexity is not decreased. By applying a general theory of perturbation such as the SOS system [8], we can remove all the degeneracy simultaneously.

Then, for each triangle Δ , we generate four planes: one is the plane containing Δ , and the other three are almost vertical planes (to the x - y plane) satisfying that (1) each contains an edge of Δ , and (2) whose common intersection is above Δ . The lower envelope $C(\Delta)$ of these three planes and the plane containing Δ is a truncated trihedral cone, which is a concave polyhedral surface containing Δ and three very steep facets called side facets.

We consider the arrangement of the n truncated trihedral cones $C(\Delta)$ for $\Delta \in T$ generated as above. Let W be the set of all side facets of these trihedral cones. We can make the side facets steep enough so that there exists a nonempty part $\tilde{e} \subset e$ for each

edge e of $\mathcal{A}$ satisfying that $\tilde{e}$ is not below any side facet in W . Here, we say that $\tilde{e}$ is not below a side facet F if no vertical halfline emanated from a point on $\tilde{e}$ upward intersects F .

If e is on the k -level (counted from the top) of $\mathcal{A}$, $\tilde{e}$ must be a part of the $(n+1-k)$ -level (counted from the bottom) of the arrangement of $C(\Delta)$. Moreover, it can be observed that $\tilde{e}$ and $\tilde{f}$ are on different edges of the arrangement of $C(\Delta)$ if $e \neq f$. Thus, the complexity of the k -level of $\mathcal{A}$ is bounded by the complexity of the $(n+1-k)$ -level of n concave polyhedral surfaces containing $3n$ linear pieces in total. Hence, the theorem follows from Corollary 13. $\square$

We consider the dual of the k -level of concave surfaces (resp. chains) in the space (resp. plane). A set S of n points in the space (resp. plane) and a family $\mathcal{P}$ of subsets of S are given. We say a subset of S is a *semispace* of S if it is represented as an intersection of S and a halfspace (resp. halfplane). A semispace is called a k -set with respect to $\mathcal{P}$ if there are exactly k members of $\mathcal{P}$ intersecting it. If $\mathcal{P}$ is the set of all singletons, it coincides with the well-known k -set of S . We consider two k -sets V and V' are equivalent if the plane (resp. line) defining V can be continuously moved to that of V' without changing the members of $\mathcal{P}$ intersecting the halfspace (resp. halfplane). Figure 11 shows a halfplane (shaded region) defining a 5-set, where S has 20 points and $\mathcal{P}$ is a set of 10 triplets each of which is indicated by the spanning triangle. The triangles intersecting the halfspace are shown with bold lines.

The following is the dual of Theorem 12:

Proposition 15. *The number of different k -sets with respect to $\mathcal{P}$ (under the equivalence) is $O(nk^{1/3})$ for the plane and $O(n^2k^{2/3})$ for the space.*

Proof. We only consider the three-dimensional case. For a member P of $\mathcal{P}$ and a halfspace H^+ , the dual of the condition that $H^+ \cap P \neq \emptyset$ is that the dual point of H

Fig. 11. A halfspace defining a 5-set for a set of triangles.

is above the lower envelope of the arrangement of the dual planes of P . The rest of the proof is routine. $\square$

4. Concluding Remarks

We remark that it might be possible to modify Clarkson's probabilistic argument [5] to prove Proposition 5, although we currently do not know how to modify his definition of the *base* exactly.

The k -level of n concave surfaces can be considered as the locus of the minimum weight base of a special parametric matroid with two parameters [17]. We conjecture that Lovász's lemma holds for the locus of a parametric matroid in general, although our proof method does not work even for the transversal matroids (note that Proposition 15 is similar to but slightly different from the transversal matroid problem). It remains a curious open problem to obtain a nontrivial upper bound for the number of different minimum weight bases of a parametric matroid, for which it is not difficult to construct an $\Omega(n^2 r^{2/3})$ lower bound by overlaying two lower bound examples, where r is the rank of the matroid. Note that the complexity of the locus of the transition of a parametric matroid with two parameters does not always gives an asymptotic upper bound for the number of different minimum weight bases, since the latter is only bounded by the complexity of the projection of the 1-skeleton of the locus of the transition, which may create new crossings to increase the complexity for a general matroid (in contrast to the k -level problem and the one-parametric matroid problems). Indeed, the above mentioned $\Omega(n^2 r^{2/3})$ bound does not hold for the complexity of the locus in the construction.

Finally, we believe that the following generalization of Theorem 4 holds:

For any affine subspace l of codimension 2 in the d -dimensional space, there are at most $O(k^{d-1})$ tangent hyperplanes through l to the k -level of an arrangement of concave hypersurfaces.

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Appendix

As an analogue of the concave chain decomposition for the two-dimensional case, we may conjecture that there always exists a set of $k(k+1)/2$ concave surfaces which forms a concave surface decomposition of the at-most- k -level. A little weaker conjecture is that there always exists a set of $k(k+1)/2$ concave surfaces covering the k -level. However, neither is true, since we have a counterexample for $k=2$ as shown below. We remark, however, that this does not deny the existence of $O(k^2)$ concave surfaces which form a concave surface decomposition of the at-most- k -level.

We construct an arrangement of four planes such that the union (ignoring multiplicity) of any three concave surfaces in the arrangement cannot be the at-most-2-level. We further

Fig. 12. The slice at $y = 0$.

show that it is also impossible to find three concave surfaces in the arrangement whose union contains the 2-level.

The following is the set of our planes: $H_1: z = x$, $H_2: z = -x$, $H_3: z = x/2 + ay - 3$, and $H_4: z = -x/2 + y - 1$. Here, a can be any number larger than 3, and we choose $a = 7$ to explain the structure.

We examine the slice of the at-most-2-level by a plane parallel to the x - z plane. We trace the change of the topology of the slice from $y = 0$ to $y = \infty$. Figure 12 is the slice at $y = 0$ (bold segments show the 2-level). In the slice, H_2 , H_3 , and H_4 meet at the point $p = (2, -2)$, where the first coordinate represents the x -value and the second represents the z -value. Thus, at $y = 0$, the topology of the at-most-2-level changes from **X** to **A**. The local decomposition of the type **A** configuration at $0 < y < \varepsilon$ for a small ε must be $D(A(1))$ (suspicious readers can try all other possibilities). Hence, the slice of three concave surfaces around p at $0 < y < \varepsilon$ is as shown in Fig. 13.

The next topological change occurs at $y = 0.4$, where H_1 , H_3 , and H_4 meet at a point $q = (-0.4, 0.4)$. Here, locally, an **A** configuration changes to an **X** configuration. The change of the local decomposition must be from $D(A(1))$ to $D(X)$. Now we know the complete topology of the slice of three concave surfaces. The structures for $0 < y < 0.4$ and $y > 0.4$ are illustrated in Fig. 14. The third topological change occurs at $y = \frac{3}{7}$, and the fourth occurs at $y = 1$. At $y = 1$, H_1 , H_2 , and H_4 meet at $(0, 0)$, and the current topology around $(0, 0)$ is $D(A(2))$ as seen in Fig. 15. We must change it to an **X**-type configuration without breaking the convexity of surfaces; this is impossible.

Fig. 13. Topology change from $y < 0$ (left drawing) to $y > 0$ (right drawing).

Fig. 14. Topology for $0.4 < y < \frac{3}{7}$.

Fig. 15. Topology for $\frac{3}{7} < y < 1$.

Fig. 16. Possible topology at $\frac{3}{7} < y < 1$ of surfaces covering the 2-level for $y < 1$ (other than that of Fig. 15).

Thus, we have shown that no triple of concave surfaces can cover the at-most-2-level of this arrangement. We further show that no triple of concave surfaces can cover the 2-level of this arrangement.

Locally, when we change an **X** configuration into an **A** configuration (or vice versa) at a vertex, if we do not restrict the three surfaces to be subsets of the at-most-2-level, there is another possibility than the transition from $D(X)$ to $D(A(1))$. That is the case where three surfaces are three straight planes locally near the transition vertex. Indeed, the transition is just changing the ordering of the planes.

Suppose that one of the transitions at $y = 0$ and $y = 0.4$ is this type of transition. Then we can easily see that both transitions must be of this type of transition. Thus, the only possible configuration at $\frac{3}{7} < y < 1$ is the one shown in Fig. 16, and we cannot continue the convex surfaces at $y = 1$ so that they cover the 2-level.

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