Approximation Results for Kinetic Variants of TSP

Mikael Hammar¹ and Bengt J. Nilsson²

¹Department of Computer Science, Lund University,
Box 118, S-221 00 Lund, Sweden
mikael@cs.lth.se

²School of Technology and Society, Malmö University College,
Citadellsvägen 7, 205 06 Malmö, Sweden
bengt.nilsson@ts.mah.se

Abstract. We study the approximation complexity of certain kinetic variants of the Traveling Salesman Problem (TSP) where we consider instances in which each point moves with a fixed constant speed in a fixed direction. We prove the following results:

1. If the points all move with the same velocity, then there is a polynomial time approximation scheme for the Kinetic TSP.
2. The Kinetic TSP cannot be approximated better than by a factor of 2 by a polynomial time algorithm unless $P = NP$, even if there are only two moving points in the instance.
3. The Kinetic TSP cannot be approximated better than by a factor of $2^{\Omega(\sqrt{n})}$ by a polynomial time algorithm unless $P = NP$, even if the maximum velocity is bounded. n denotes the size of the input instance.

The last result is especially surprising in the light of existing polynomial time approximation schemes for the static version of the problem.

1. Introduction

Consider a cat in a field with an ample supply of mice. The cat's objective is to catch all the mice while exerting the minimum amount of energy. The cat therefore wishes to use the shortest possible path to chase the mice. A major difficulty is the fact that the mice are moving. This problem is an instance of the *Kinetic Traveling Salesman Problem*.

The *Traveling Salesman Problem*, TSP for short, is probably the best known intractable problem. It asks for the shortest closed tour that visits the nodes in a given weighted complete graph exactly once. This deceptively simple problem lies at the heart of combinatorial optimization and has spawned a wealth of research in complexity theory and operations research; see [9].

TSP optimization has been shown to be NP-hard, even for instances that obey restriction to the weights in the complete graph. Specifically, if the graph is metric or even embeddable in the Euclidean plane, the problem is NP-hard [7], [11].

TSP is in fact extremely computationally difficult. It cannot be approximated to within *any* constant factor by a polynomial time algorithm, unless $P = NP$ [13]. For metric graphs the situation is better. Christofides [6] presents a polynomial time $\frac{3}{2}$ -approximation algorithm, but better than a constant factor approximation algorithm is not possible in polynomial time unless $P = NP$ [3].

In seminal work Arora [1] and independently Mitchell [10] showed that there exist *polynomial time approximation schemes* for TSP when the graph is embedded in the Euclidean plane. Polynomial time approximation schemes, PTAS for short, are polynomial time algorithms that, for any $\varepsilon > 0$, produce a $(1 + \varepsilon)$ -approximation to a given problem. The running time of a PTAS is polynomial in the input size, for any fixed ε . Recently, the running times of PTASs for d -dimensional Euclidean TSP has been significantly improved [2].

In the Kinetic TSP, we look at TSP for moving points in the Euclidean plane. We consider instances in which each point moves with a fixed velocity. This is a natural and both theoretically and practically important generalization of TSP (e.g. several scheduling problems can be reduced to solving variants of Kinetic TSP). The complexity status of the problem, especially with respect to approximation, is a paradigmatic question. Helvig et al. [8] give a polynomial time algorithm to solve the problem when the moving points are restricted to lie on the real line. They also give a $(2 + \varepsilon)$ -approximation algorithm for the Kinetic TSP if the number of points with nonzero speed is small. We prove the following results:

1. If the points all move with the same speed and in the same direction, then there is a PTAS for the Kinetic TSP. This generalizes the result of Chalasani et al. [4], [5].
2. The Kinetic TSP cannot be approximated better than by a factor of 2 by a polynomial time algorithm unless $P = NP$, even if there are only two moving points in the instance.
3. The Kinetic TSP cannot be approximated better than by a factor of $2^{\Omega(\sqrt{n})}$ by a polynomial time algorithm unless $P = NP$, even if the maximum speed is bounded. The n denotes the size of the input instance.

The last result is especially surprising in the light of existing PTASs [1], [2], [10], [12] for the static version of the problem.

In the next section we state definitions and give preliminary results concerning Kinetic TSP. Specifically, we give an overview of the original reduction of Garey et al. [7] proving the NP-hardness of the Euclidean TSP, since it plays an important role in our later reductions. In Section 3 we prove the existence of a PTAS for the case when all points move with the same speed in the same direction. In Sections 4 and 5 we prove the stated inapproximability results and we conclude the presentation with a discussion of open problems.

2. Preliminaries and Notation

In the *Euclidean Traveling Salesman Problem* we are given a set of points $S = \{s_1, s_2, \dots, s_n\}$ in the Euclidean plane. The objective is to compute the shortest tour that visits all

points. The Euclidean *Minimum Hamiltonian path Problem*, MHP, is the problem of finding a shortest path that visits all points of S , starting at s_1 and ending at s_n .

As Chalasani et al. [5] we distinguish between *space-points* and *moving points*. A space-point is a point in a coordinate system, whereas a moving point is a point-object in space, the Euclidean plane in our case, that travels with a given velocity. The coordinates of a moving point s can be described by the function $s(t) = (x + tv \cos \alpha, y + tv \sin \alpha)$, where $v \geq 0$ is the point's speed and α is its direction. If $v = 0$ we say that the point is static. The *traveling salesman* is described by a special point that can move with variable speed and direction. The initial position s_0 of the salesman is assumed to be $(0, 0)$ and its maximal speed is assumed to be 1. The path taken by the salesman is denoted P and $P(t)$ is the position of the salesman at time t . If $P(t) = s(t)$, then we say that the salesman *visits* s at time t . P is called a *salesman path* of S if all points in S have been visited by the salesman. If the salesman also returns to its initial position, then we call the resulting tour a *salesman tour*.

We can now define the Kinetic TSP for moving points in the plane.

Definition 1. A set of moving points $S(t) = \{s_1(t), s_2(t), \dots, s_n(t)\}$ in the plane with the Euclidean metric is given. A point $s_i(t)$ in $S(t)$ moves with the speed $v_i < 1$. The objective of the *Kinetic Traveling Salesman Problem* (KTSP) is to compute a salesman tour, starting and ending at the initial point $s_0 = (0, 0)$, that minimizes the traveling time. The *Translational Traveling Salesman Problem* (TTSP) is a restricted version of KTSP, where all points of S have the same speed and direction.

The following lemma provides an important fact concerning the speed of the salesman. It is a direct consequence of the fact that the salesman can travel faster than any other moving point.

Lemma 1. *An optimal salesman moves with maximal speed.*

From now on we assume that the salesman travels with maximal speed and since the maximal speed is 1, there is no difference between the distance traveled and the traveling time.

Generally, we use *OPT* to denote a shortest tour or path, unless more specific notation is needed. Given a tour or a path P in the Euclidean plane, we let $\mathcal{C}(P)$ denote the length of P .

Garey et al. [7] prove that the Euclidean TSP is NP-hard by a reduction from the problem *exact cover by 3-sets*, X3C:

Given a family $\mathcal{F} = \{F_1, F_2, \dots, F_r\}$ of 3-element subsets of a set U of $3k$ elements, does there exist a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of pairwise disjoint sets such that $\bigcup_{F \in \mathcal{F}'} F = U$?

If there exists such a subfamily for $\mathcal{F}$, then we say that $\mathcal{F} \in \text{X3C}$, otherwise we say that $\mathcal{F} \notin \text{X3C}$.

Let S be an instance of TSP produced by Garey et al.'s reduction, such that $|S| = n$. The points of S lie on a unit grid G of size less than $n \times n$ and a naive tour visiting all points of S has length $l < 2n$. These results follow directly from Garey et al.'s

Fig. 1. Typical instances produced by the X3C reduction.

construction. Figure 1(a) is a schematic overview of a TSP-instance produced by the reduction. Each line segment in the figure represents a set of points on a line such that the distance between a point and its successor is one unit. The line segments build up a tube system and at tube intersection points constructions called crossovers are placed. The thickness of a tube is $\Theta(n^{2/3})$ and the distance between parallel tubes is at least four times the tube thickness. Thus, the optimal tour, OPT , is forced to visit the points along the line segments in sequence. At crossovers the tour can choose to jump to a new line or to continue in the same direction as before.

Garey et al. prove that the length of OPT is less than or equal to some specific value L^* if and only if an exact cover exists for the X3C-instance. Their construction also makes sure that an optimal tour has integral length. Thus, if there is no exact cover, then OPT has at least the length $L^* + 1$. The following lemma follows directly from this fact.

Lemma 2. *There is no polynomial time approximation algorithm for TSP producing a tour APX such that $C(APX) < C(OPT) + 1$ unless $P = NP$.*

The lemma also holds for MHP, since the reduction of Garey et al. also works for that problem. In this case the reduction from X3C results in instances of the type shown in Fig. 1(b), where we have opened the bottom tube. Observe that the length of the minimum Hamiltonian path in Fig. 1(b) is equal to the length of the corresponding minimum traveling salesman tour in Fig. 1(a). Instances produced by the X3C reduction will from now on be called *GGJ-instances*.

3. A PTAS for TTSP

We start this section by analyzing the translational MHP. A set $S(t) = \{s_1(t), \dots, s_n(t)\}$ of moving points is given together with a starting point $s_0 = (0, 0)$. All points move in the same direction α and with the same speed v . We can assume, without loss of

generality, that $\alpha = \pi/2$. Thus a point $s_i(t)$ is defined as

$$s_i(t) = \left(x + tv \cos \frac{\pi}{2}, y + tv \sin \frac{\pi}{2} \right) = (x, y + tv).$$

The traveling distance c_{ij} between two points $s_i(t)$ and $s_j(t)$ is the time needed by a salesman, moving with speed 1, to travel from s_i to s_j , i.e.

$$c_{ij} = \frac{v}{1-v^2}(y_j - y_i) + \sqrt{\frac{(x_j - x_i)^2}{1-v^2} + \frac{(y_j - y_i)^2}{(1-v^2)^2}}.$$

Note that the traveling distance is independent of time and that the cost function c_{ij} is asymmetric. Consider the function

$$d(s_i, s_j) = \frac{c_{ij} + c_{ji}}{2} = \sqrt{\frac{(x_j - x_i)^2}{1-v^2} + \frac{(y_j - y_i)^2}{(1-v^2)^2}}$$

and the bijective mapping $f_v(s)$ from a moving point $s = (x, y)$ to a static point in the Euclidean plane:

$$f_v(s) = \left(\frac{x}{\sqrt{1-v^2}}, \frac{y}{1-v^2} \right).$$

Let $P = (s_{i_1}, \dots, s_{i_n})$ be a path in the translational instance, then

$$\mathcal{C}(P) = \sum_{k=1}^{n-1} c_{i_k, i_{k+1}} = \frac{v(y_{i_n} - y_{i_1})}{1-v^2} + \sum_{k=1}^{n-1} d(s_{i_k}, s_{i_{k+1}}).$$

With this it is easy to prove the following result.

Lemma 3. *Let S_T be an instance of the translational MHP with specified starting and ending points, and let S_E be the corresponding Euclidean instance after the transformation using f_v . A salesman path in S_T is optimal if and only if the corresponding Euclidean Hamiltonian path in S_E is optimal.*

Proof. Let $P = (s_{i_1}, \dots, s_{i_n})$ denote a salesman path in S_T and let P' denote the corresponding Hamiltonian path in S_E . Observe that $\mathcal{C}(P') = \sum_{j=1}^{n-1} d(s_{i_j}, s_{i_{j+1}})$. Therefore, $\mathcal{C}(P) = \mathcal{C}(P') + v(y_{i_n} - y_{i_1})/(1-v^2)$. Note that the last term is fixed since both the starting point and the ending point were given. It follows that P is optimal if and only if P' is optimal. $\square$

Consider an arbitrary instance of the translational MHP with specified starting point s_0 and ending point s_n . We can transform this instance into an instance of the Euclidean MHP using the bijective mapping f_v . Given this new instance, we compute a Hamiltonian path APX_E such that $\mathcal{C}(APX_E) \leq (1 + \varepsilon)\mathcal{C}(OPT_E)$, using a PTAS for the Euclidean MHP problem. Such schemes can be constructed by modifying any PTAS for

the Euclidean TSP [1], [2], [10]. Here OPT_E denotes a minimal Hamiltonian path in the Euclidean instance. Let APX_T and OPT_T denote the corresponding paths in the translational MHP-instance. Observe that OPT_T is an optimal salesman path by Lemma 3. We have

$$\begin{aligned} \mathcal{C}(APX_T) &= \frac{v(y_n - y_1)}{1 - v^2} + \mathcal{C}(APX_E) \\ &\leq \frac{v(y_n - y_1)}{1 - v^2} + (1 + \varepsilon)\mathcal{C}(OPT_E) \\ &= \mathcal{C}(OPT_T) + \varepsilon\mathcal{C}(OPT_E) \\ &\leq \left(1 + \frac{\varepsilon}{1 - v}\right)\mathcal{C}(OPT_T). \end{aligned}$$

The last inequality holds since OPT_T is at least as long as the shortest path between s_1 and s_n , i.e. $\mathcal{C}(OPT_T) \geq |y_n - y_1|/(1 + v)$. Hence,

$$\begin{aligned} \mathcal{C}(OPT_E) &= \mathcal{C}(OPT_T) - \frac{v(y_n - y_1)}{1 - v^2} \leq \mathcal{C}(OPT_T) + \frac{v|y_n - y_1|}{(1 - v)(1 + v)} \\ &\leq \mathcal{C}(OPT_T) + \frac{v\mathcal{C}(OPT_T)}{1 - v} = \frac{\mathcal{C}(OPT_T)}{1 - v}. \end{aligned}$$

It follows that APX_T is a $(1 + \varepsilon/(1 - v))$ -approximation of the optimal path. Thus, we have a PTAS for the translational MHP.

With this approximation scheme we can now give a PTAS for the translational TSP. We use the same approach as Chalasani et al. [4], [5]. The difficulty of the translational TSP is that the initial point s_0 is not moving, which creates an asymmetry that we must be able to handle. Assume that $OPT = (s_0, s_1(t), \dots, s_n(t), s_0)$ is an optimal salesman tour. It follows easily that the optimal salesman path starting from s_0 and ending at $s_n(t)$ is a part of that tour. For each possible such ending point $s_i(t)$, $i \geq 1$, we compute a $(1 + \varepsilon/2(1 - v))$ -approximate salesman path. This gives us n paths that can easily be turned into salesman tours. The algorithm returns the shortest of these tours.

Theorem 1. *The algorithm described above is a PTAS for the translational TSP.*

Proof. Let OPT_H denote the length of the optimal salesman path, starting from s_0 and ending at $s_i(t)$. The length of the optimal salesman tour OPT is $\mathcal{C}(OPT) = OPT_H + l$, assuming s_i is the last moving point visited by the tour OPT and l is the length of the last segment in the tour (between the points $s_i(t)$ and s_0). There is a tour APX among the tours our algorithm computes, that also has $s_i(t)$ as the last unvisited point. The length of APX is

$$\mathcal{C}(APX) = \left(1 + \frac{\varepsilon}{2(1 - v)}\right) OPT_H + l',$$

where l' is the length between $s_i(t)$ and s_0 . A salesman that travels along APX visits $s_i(t)$ at time $t = (1 + \varepsilon/2(1 - v))OPT_H$, whereas the optimal salesman visits $s_i(t)$ at time $t = OPT_H$. The time difference is $(\varepsilon/2(1 - v))OPT_H$ and, in that time period, $s_i(t)$

moves the length $(v\varepsilon/2(1-v))OPT_H$. The triangle inequality assures us that l' does not exceed $l + (v\varepsilon/2(1-v))OPT_H$. The cost of our approximate tour is thus

$$\begin{aligned} \mathcal{C}(APX) &\leq \left(1 + \frac{\varepsilon}{2(1-v)}\right) OPT_H + l + \frac{v\varepsilon}{2(1-v)} OPT_H \\ &= \left(1 + \frac{(1+v)\varepsilon}{2(1-v)}\right) OPT_H + l \\ &< \left(1 + \frac{\varepsilon}{1-v}\right) OPT_H + l. \end{aligned}$$

The last inequality holds since the speed does not exceed 1. The ratio between the costs of APX and OPT becomes

$$\frac{\mathcal{C}(APX)}{\mathcal{C}(OPT)} < \frac{(1 + \varepsilon/(1-v))OPT_H + l}{OPT_H + l} \leq \left(1 + \frac{\varepsilon}{1-v}\right).$$

The proof is completed, since the cost of the returned salesman tour does not exceed that of APX . $\square$

Observe that our technique differs from that of Chalasani et al. [5] only in the bijective mapping f_v with which the transformation was performed. Thanks to this mapping we were able to generalize their result. If the speeds are bounded by a constant $c < 1$, then the algorithm is a true PTAS.

4. Two Reductions for KTSP

KTSP is in general not as easily approximated as the translational TSP. Helvig et al. [8] give a $(2 + \varepsilon)$ -approximation algorithm for KTSP-instances with n points of which $O(\log n / \log \log n)$ are moving. The simple construction described in Fig. 2 shows that their algorithm is close to optimal.

The figure contains a GGJ-instance, chosen as in Fig. 1(a), at distance D from the origin and two moving points, p_1 and p_2 , both with speed $v < 1$. We assume that the salesman starts at the origin at time $t = 0$. Point p_1 moves horizontally and lies at the same altitude as the bottom line of the GGJ-instance and intersects line l at time D .

Fig. 2. KTSP-instance with two moving points.

Point p_2 moves parallel to p_1 and reaches the origin at time $2D + L^*$. Recall that a GGJ-instance is constructed so that if $\mathcal{F} \in \text{X3C}$, then an optimal salesman tour is at most L^* long. Otherwise the length is at least $L^* + 1$.

Because of the large speed of the moving points, it follows that the first point to be visited is p_1 . Furthermore, point p_2 must be visited before it reaches the origin. Otherwise, the tour has unbounded length.

If $\mathcal{F} \in \text{X3C}$, then an optimal salesman can catch p_1 and continue along the GGJ-instance's optimal salesman tour. After all points in the GGJ-instance have been visited, the salesman still has time to catch p_2 before it reaches the origin.

If $\mathcal{F} \notin \text{X3C}$, then an optimal salesman does not have time to catch all the points in the GGJ-instance before he has to turn back and catch p_2 . This means that he must traverse the distance D at least four times. From this we get:

Theorem 2. *It is NP-Hard to get an approximation ratio less than 2 for KTSP, even if there are only two moving points in the instance.*

In Theorem 2 we assume that the speed v can be arbitrarily close to 1. With more moving points we can restrict the speed to $v = \frac{1}{2}$ and still get a lower bound of 2 on the approximation ratio.

Theorem 3. *It is NP-Hard to get an approximation ratio less than 2 for KTSP, even if the speed of the moving points is restricted to $v = \frac{1}{2}$.*

Proof. To prove the theorem we use two pairs of sets with $k = m$ points in each set evenly distributed on a line and a GGJ-instance as in Fig. 1(b) with m static points. Each pair contains one set with static points and one with moving points. The static points lie on a vertical line of length $D = m^2$, whereas the moving points lie on a line with the same height but with the length $(\sqrt{5}/2)D$. The slope of the second line is -2 in pair 1 and 2 in pair 2. The moving points perform a horizontal motion from left to right with the speed $v = \frac{1}{2}$.

Given these building blocks, we construct an instance of KTSP as described in Fig. 3. The salesman begins at the origin at time $t = 0$. At the same time the lowest moving point of pair 1 intersects the origin and, at time $D + L^*$, the topmost moving point in pair 2 collides with the corresponding static point in that pair. The pairs are constructed so that the salesman can move (with speed 1) along the vertical line of the static points,

Fig. 3. KTSP-instance with moving points of speed $\frac{1}{2}$.

visiting pairs of points from the two sets at their collision point if and only if the salesman visits the first two colliding points at the time of impact. It thus takes time D to visit all points in a pair, assuming the conditions above, since the length of the vertical line is D .

If there exists an exact cover for the X3C-instance, then an optimal salesman has time to visit all points in both pair 1 and the GGJ-instance before the first two points in pair 2 collide. The time that an optimal salesman needs in order to visit all points in the whole instance is thus at most $2D + L^* + W$, where $W \leq L^*$ is the length of the bottom line segment of the GGJ-instance.

If there is no exact cover for the X3C-instance, then an optimal salesman does not have enough time to visit all points in the GGJ-instance before the first two points in pair 2 collide. Thus, he is left with two options. Either he visits all points in the GGJ-instance before visiting the points of pair 2 or he saves some points in the GGJ-instance in order to fetch the other points optimally.

If the salesman chooses to save some points in the static instance, then he must go back again after visiting the moving points. The total traveling time is thus at least $4D + L^*$.

If he decides to visit all points in the static GGJ-instance at once, then he misses the first collision in pair 2 by at least one time unit. The salesman ends up chasing each point separately. If he visits the points setwise, he ends up close to the GGJ-instance and at least a distance D from the origin. Such an approach thus takes at least time $4D + L^*$. To avoid this he might zigzag between the two sets in pair 2. For each pair of points that the salesman visits, the gain is at most $3D/k$. However, since the points in the second set move with speed $v = \frac{1}{2}$, the cost of moving between the two sets back and forth doubles each time. The cost of this approach is thus $4D + L^* - 3Dz/k + 2^{z-1}$, where z is the number of zigzag movements. Minimizing this expression, using the knowledge that $L^* \geq m$, $D = m^2$ and $k = m$, gives us that the length of the tour is at least $4m^2 - cm \log m$, for some constant c .

Since the salesman moves with speed 1, length equals time. The length of an optimal salesman tour is thus at least $4m^2 - cm \log m$ if $\mathcal{F} \notin \text{X3C}$ and at most $2(m^2 + L^*)$ if $\mathcal{F} \in \text{X3C}$. The limit of the ratio gives us a lower bound on the approximation ratio and concludes the proof of Theorem 3:

$$\lim_{m \rightarrow \infty} \frac{4m^2 - cm \log m}{2(m^2 + L^*)} = 2. \quad \square$$

5. Exponential Inapproximation of KTSP

In this section we present a gap producing reduction from X3C to KTSP. First, we give a description of the KTSP-instance that we use in the reduction. The KTSP-instance is a uniformly expanding instance. A uniformly expanding instance contains moving points of the form $s_i(t) = (v_i t \cos \alpha, v_i t \sin \alpha)$. This implies that at time $t = 0$ all points are located at the origin and that the relative distances within the instance do not change over time. To make the construction work, we assume that the salesman begins his pursuit at time $t_0 > 0$ (if he starts at time $t = 0$, he visits all points at once, without moving). Observe that salesman tours in uniformly expanding TSP-instances

Fig. 4. One circle of the inapproximable KTSP-instance.

are in a sense symmetric. Consider namely a salesman tour P . If we reverse the order in which the points are visited, the length of the new tour is equal to the length of P .

The KTSP-instance consists of l circles $C_1, \dots, C_l$, with radii $r_1, r_2, \dots, r_l$. Each circle consists of $k = \lceil 9\pi l \rceil$ small expanding identical GJJ-instances produced by the X3C-reduction of Garey et al. The GJJ-instances are placed on the circles as shown in Fig. 4. A sequence of evenly distributed points is placed along the line segment between the rightmost point and the leftmost point of the previous GJJ-instance. The spacing of these points equals the spacing of the points along the bottom line segment of the GJJ-instance. At the topmost GJJ-instance's rightmost point we let the sequence of points point toward the origin. Let σ_i denote this special sequence on circle C_i ; see Fig. 4. Each GJJ-instance initially contains m points but we have extended the bottom line of the GJJ-instances with $5m$ points on each side; see Fig. 5. Thus, each GJJ-instance consists of $11m$ moving points. We let $l = (11m)^a$, for some $a > 1$. The number of points between consecutive GJJ-instances is less than m if $k > 35$. On the special sequences σ_i we place $5m$ points for technical reasons. The total number of points in the instance, denoted n , is therefore between $12mkl$ and $14mkl$, for large k and l . The rightmost point of a GJJ-instance is placed on the circle and the bottom line of the GJJ-instance follows the circle's tangent line at that point. Each circle C_i is expanding with a speed v_i , i.e. the rightmost point of each GJJ-instance on C_i has a speed v_i , directed away from the center and the radius of C_i is $r_i = v_i t$.

Fig. 5. A small expanding GJJ-instance.

We let $v_1 = 1/2e$ and $v_i = (1 + 9\pi/k)v_{i-1}$. This implies that the largest circle's speed, v_l , is bounded by

$$v_l = \left(1 + \frac{9\pi}{k}\right)^l v_1 = \left(1 + \frac{1}{l}\right)^l v_1 < ev_1 = \frac{1}{2}.$$

Note also that $r_1 \leq r_2 \leq \dots \leq r_l$. Observe that the GGJ-instances are uniformly expanding, i.e. the distance between two points s_i and s_j is $d(s_i, s_j) = d_{ij}t$, where d_{ij} is constant. The expansion rate is such that the distance between the leftmost and the rightmost point is $lv_i \tan(2\pi/k)$. Therefore, we let $u_i = v_i \tan(2\pi/k)$ define the expansion rate; see Fig. 5. The fastest point of a GGJ-instance is the leftmost point and this point has the speed

$$\sqrt{v_i^2 + \left(v_i \tan \frac{2\pi}{k}\right)^2} \leq v_i \left(1 + \frac{2\pi}{k}\right).$$

Thus, no point belonging to circle C_i has a speed exceeding $v_i(1 + 2\pi/k)$, which implies that the maximal speed is bounded. From now on we simply denote our newly constructed instance the *KTSP-instance*.

The construction above guarantees that an optimal salesman visits the GGJ-instances circlewise, one at a time. Furthermore, the time it takes to go from any moving point s_i to any other moving point s_j in the KTSP-instance depends linearly on the starting time. Assuming that a salesman starts at s_i at time t it would take him time $\kappa_{ij}t$ to go to s_j , where we consider κ_{ij} to be constant since it does not depend on t . The implication of this is that the time needed by the salesman to visit all points in the tour is multiplicative in the time it takes to visit one GGJ-instance. That is, if the salesman begins his pursuit at time t_0 and finishes the first GGJ-instance at time Kt_0 , then it will take him at least time $CK^{kl}t_0$ to visit all points in the KTSP-instance, where C denotes a constant and K is independent of t . Note that a nonoptimal salesman inevitably is going to do worse on each GGJ-instance and since the error is also multiplicative the inapproximability ratio is going to become large. The rest of the presentation is devoted to proving the following theorem.

Theorem 4. *For any $\gamma > 0$, there exists no polynomial time algorithm that achieves an approximation ratio of $2^{\Omega(n^{1/2-\gamma})}$ for KTSP, unless $P = NP$.*

In order to prove Theorem 4 we need the following background material. Consider a salesman moving between the points in the KTSP-instance. Let P be the path taken by the salesman. We define $\mathcal{C}_P(t)$ to be the length of this path (which is synonymous to the time it takes for the salesman to traverse path P). We let $\mathcal{T}_P(t)$ denote the time when the salesman arrives at the end of the path P , given the starting time t , i.e. $\mathcal{T}_P(t) = t + \mathcal{C}_P(t)$. A simple proof of induction yields that

$$\mathcal{T}_P(t) = t \prod_{[s_i, s_j] \in P} (1 + \kappa_{ij}) = tK_P$$

and, therefore,

$$\mathcal{C}_P(t) = t \prod_{[s_i, s_j] \in P} (1 + \kappa_{ij}) - t,$$

where κ_{ij} is the time it takes for the salesman to travel between s_i and s_j , starting from s_i at time t . We examine a small expanding GGJ-instance produced by an X3C-instance. We assume that the instance is located somewhere on circle C_i . The optimal salesman path for the GGJ-instance, starting at the rightmost point and ending at the leftmost point, is denoted *opt* if $\mathcal{F} \in \text{X3C}$ and *nopt* if $\mathcal{F} \notin \text{X3C}$. We would like to find an upper bound on *opt* and a lower bound on *nopt*. To do this, we need some definitions. First, if we were to stop the expansion of the instance at time t , then all points would share the speed and direction of the rightmost point. The result would be a translational GGJ-instance having the same size as the small expanding GGJ-instance at time t . We define $D(t)$ as the length of this new translational GGJ-instance's optimal path, starting at the rightmost point and ending at the leftmost point. Clearly,

$$D(t) \leq \mathcal{C}_{opt}(t) \leq D(t + \mathcal{C}_{opt}(t)). \quad (1)$$

Furthermore, let $f(t)$ denote the distance between the two closest points in the expanding instance at time t and let t_i be the point in time such that $f(t_i) = 1$, i.e. at that moment, the instance's size is identical to the size of the static instance described in Section 2. Observe that t_i depends on which circle the instance is located. $D(t)$ and $f(t)$ are both linear mappings, which implies that

$$\frac{D(t)}{D(t')} = \frac{f(t)}{f(t')} = \frac{t}{t'}.$$

With these definitions it is straightforward to prove the following.

Lemma 4. *The following bounds hold:*

$$3km \leq t_i \leq 10km, \quad (2)$$

$$D(t_i) \leq \frac{L^*}{1 - v_i} + \frac{10m}{\sqrt{1 - v_i^2}} < 4m + \frac{10m}{\sqrt{1 - v_i^2}} \leq 16m, \quad (3)$$

$$\mathcal{C}_{opt}(t) \leq \frac{D(t_i)}{t_i - D(t_i)} t, \quad (4)$$

$$\mathcal{C}_{nopt}(t) \geq \frac{D(t_i) + 1}{t_i} t. \quad (5)$$

Proof. (2) Let v_f denote the expansion rate of the distance between the two closest points in the instance. Then $f(t) = v_f t$ and, by Definition, $1 = f(t_i) = v_f t_i$. The distance between consecutive points on the bottom line is $f(t)$ at time t and the length of the bottom line is between $10m$ and $11m$ at time t_i . Furthermore, the expansion rate of the bottom line is u_i . Given this, we can bound v_f :

$$\frac{u_i}{10m} \geq v_f \geq \frac{u_i}{11m}.$$

Now, $t_i = 1/v_f$, so

$$\frac{10m}{u_i} \leq t_i \leq \frac{11m}{u_i}.$$

By the Taylor expansion, we get that

$$\frac{2\pi + \frac{1}{3}}{k} \geq \tan \frac{2\pi}{k} \geq \frac{2\pi}{k}$$

and, thus,

$$\frac{10mk}{(2\pi + \frac{1}{3})v_i} \leq t_i \leq \frac{11m}{2\pi v_i},$$

since $u_i = v_i \tan(2\pi/k)$. Inserting the bounds of v_i yields that $3km \leq t_i \leq 10km$.

(3) The value of $D(t_i)$ is less than $L^*/(1 - v_i)$ plus the total extra time needed to traverse the extended parts of the bottom line, by straightforward calculations. The salesman needs $2 \cdot 5m/\sqrt{1 - v_i^2}$ time units to traverse the extended parts, so

$$D(t_i) \leq \frac{L^*}{1 - v_i} + \frac{10m}{\sqrt{1 - v_i^2}}.$$

The speed of the rightmost point is at most $\frac{1}{2}$ and $L^* < 2m$ so

$$D(t_i) \leq 4m + \frac{10m}{\sqrt{1 - v_i^2}} < 16m.$$

(4) We start with the inequality

$$\mathcal{C}_{opt}(t) \leq D(t + \mathcal{C}_{opt}(t))$$

taken from (1). Using the linearity of $D(t)$ we get that

$$D(t + \mathcal{C}_{opt}(t)) = \frac{D(t_i)(t + \mathcal{C}_{opt}(t))}{t_i}.$$

This implies that

$$\mathcal{C}_{opt}(t) \leq \frac{D(t_i)}{t_i - D(t_i)} t,$$

since $t_i \geq 3km > 16m > D(t_i)$.

(5) Since the GGJ-instance is expanding it follows that

$$\mathcal{C}_{nopt}(t) \geq \mathcal{C}_{opt}(t) + f(t).$$

By the linearity of $D(t)$ and $f(t)$ we have that

$$C_{\text{nopt}}(t) \geq D(t) + f(t) = \frac{D(t_i) + 1}{t_i} t,$$

since $C_{\text{opt}}(t) \geq D(t)$. $\square$

With these bounds we can prove a lower bound on the approximation ratio for the small expanding GGJ-instances. To simplify the analysis later on, we compute the approximation ratio in terms of the arrival time. We assume that the salesman visits his first point of the instance at time t .

Lemma 5. *It is NP-hard to find a salesman path apx , for an expanding GGJ-instance, with*

$$\mathcal{T}_{\text{apx}}(t) \geq \left(1 + \frac{1}{11km}\right) \mathcal{T}_{\text{opt}}(t).$$

Proof. The following ratio measures the gap between the arrival times of apx and opt . According to Lemma 4

$$\begin{aligned} \frac{\mathcal{T}_{\text{apx}}(t)}{\mathcal{T}_{\text{opt}}(t)} &\geq \frac{\mathcal{T}_{\text{nopt}}(t)}{\mathcal{T}_{\text{opt}}(t)} = \frac{C_{\text{nopt}}(t) + t}{C_{\text{opt}}(t) + t} \\ &\geq \frac{((D(t_i) + 1)/t_i)t + t}{(D(t_i)/(t_i - D(t_i)))t + t} = 1 + \frac{1}{t_i} - \frac{D(t_i)^2 + D(t_i)}{t_i^2}. \end{aligned}$$

$D(t_i) < 16m$ and $3km \leq t_i \leq 10km$ by Lemma 4. The quotient, expressed in terms of k and m , is thus

$$\frac{\mathcal{T}_{\text{apx}}(t)}{\mathcal{T}_{\text{opt}}(t)} > 1 + \frac{1}{10km} - \frac{256m^2 + 16m}{9m^2k^2}$$

and since $k = \lceil 9\pi(11m)^a \rceil$ we get that

$$\frac{\mathcal{T}_{\text{apx}}(t)}{\mathcal{T}_{\text{opt}}(t)} > 1 + \frac{1}{10km} - \frac{1}{km} \frac{(256m^2 + 16m)}{9 \cdot 9\pi m(11m)^a},$$

which is larger than $1 + 1/11km$ if $a \geq 2$ and $m \geq 1$. $\square$

We return to the KTSP-instance that contains the small expanding GGJ-instances. We assume that the salesman starts at the origin at time $t_0 > 0$. We will prove that the KTSP-instance is inapproximable by using the inapproximability result in Lemma 5 for the small expanding GGJ-instances.

Consider a salesman tour P for the KTSP-instance. A subpath of P starting at a point s_i and ending at the point s_j is called an edge of P if no other points are visited along the subpath. Edges that connect two different GGJ-instances are called leaps. Leaps that connect the leftmost point of the topmost GGJ-instance on circle C_i with the rightmost point of the corresponding GGJ-instance on circle C_{i+1} will be denoted J_i and we have

that $\mathcal{T}_{J_i}(t) = K_{J_i}t$, for some K_{J_i} . A subpath of P between consecutive GGJ-instances on circle C_i , following the sequence of points between the leftmost and the rightmost point of the instances, is denoted x_i . Using these definitions we give a tight bound on the cost of an optimal salesman tour.

Lemma 6. *An optimal salesman tour, OPT , starting at the origin at time t_0 is finished at time*

$$\mathcal{T}_{OPT}(t_0) = K_{J_0} \prod_{j=1}^l \left((K_{opt} K_{x_i})^k \frac{K_{J_i}}{K_{x_i}} \right) t_0,$$

where J_0 is the leap between the origin and the first circle.

Proof. First, notice that if the GGJ-instances are visited counterclockwise, one circle at a time, using an optimal salesman path, then this bound is achieved. Consider therefore a salesman tour P that diverts from the supposedly optimal tour. In order to do better, we must gain some time by performing leaps. There must exist at least one leap exiting each circle, because of the salesman tour property. We examine P restricted to the interior of a GGJ-instance on circle C_i and compare this restriction to an optimal salesman path for that GGJ-instance. We turn our attention to the original part of the GGJ-instance and save the extended parts for later. Let opt' denote an optimal salesman path restricted to the original part of the instance. We look at the GGJ-instance at time t_i . Since we only consider points in the original part, it follows from the proof of Lemma 4 that $D(t_i) < 4m$. Since $D(t_i) < t_i$, we can still use the third result of Lemma 4 and since $k > 16m$, we have that

$$\mathcal{C}_{opt'}(t_i) \leq \frac{D(t_i)t_i}{t_i - D(t_i)} = D(t_i) + \frac{D(t_i)^2}{t_i - D(t_i)} < 4m + \frac{16m^2}{km} < 4m + 1.$$

However, the length of any leap J , connecting circle C_i with another circle at time t_i is at least the minimum of

$$\min_{1 \leq j < i} \left\{ \frac{v_i - (1 + 2\pi/k)v_j}{1 + v_j} t_i \right\}$$

and

$$\min_{i < j \leq n} \left\{ \frac{v_j - (1 + 2\pi/k)v_i}{1 - v_j} t_i \right\},$$

which is at least $10m$. Furthermore, the distance to the closest GGJ-instance on the same circle is at least $5m \sin(\pi(k-4)/2k)$, as shown in Fig. 6. This is also a lower bound on the shortest leap connecting the original part with another GGJ-instance, since all points on the same circle are moving away from each other. Now,

$$\mathcal{C}_J(t_i) > 5m \sin \frac{\pi(k-4)}{2k} > 4m + 1 > \mathcal{C}_{opt'}(t_i)$$

for sufficiently large k and m , which implies that no gain is possible by performing leaps from the original part of a GGJ-instance. It remains to show that leaps performed

Fig. 6. Shortest distance between two instances at time t_i .

from the extended parts of a GGJ-instance lead to no gain. We observe that such leaps must be performed between the extended parts of two GGJ-instances, since the length of the leap cannot exceed $4m + 1$. At these parts no gain is possible since the distance between consecutive points along the optimal salesman path is the shortest possible. Thus, we have proved that there is exactly one leap exiting each circle. For each circle C_i , $1 \leq i < l$, the leap that is performed is J_i , due to the special sequence σ_{i+1} . This implies that the GGJ-instances are taken counterclockwise one circle at a time, so we have proved our claim. $\square$

Corollary 1. *There exists no polynomial-time algorithm producing a tour APX for our KTSP-instance such that, unless $P = NP$,*

$$\mathcal{T}_{APX}(t_0) < K_{J_0} \prod_{j=1}^l \left(\left(\left(1 + \frac{1}{11km} \right) K_{opt} K_{x_i} \right)^k \frac{K_{J_i}}{K_{x_i}} \right) t_0.$$

Proof. It follows by Lemma 5 and the proof of Lemma 6. $\square$

We can now calculate a lower bound on the approximation ratio, assuming that $P \neq NP$.

Proof of Theorem 4. We are actually interested in the ratio $\mathcal{C}_{APX}(t_0)/\mathcal{C}_{OPT}(t_0)$ but since $\mathcal{C}_{APX}(t_0)/\mathcal{C}_{OPT}(t_0) > \mathcal{T}_{APX}(t_0)/\mathcal{T}_{OPT}(t_0)$ we might as well use the latter ratio. From Lemma 6 and the resulting corollary we have that

$$\begin{aligned} \frac{\mathcal{T}_{APX}(t_0)}{\mathcal{T}_{OPT}(t_0)} &\geq \frac{K_{J_0} \prod_{j=1}^l (((1 + 1/11km) K_{opt} K_{x_i})^k (K_{J_i}/K_{x_i})) t_0}{K_{J_0} \prod_{j=1}^l ((K_{opt} K_{x_i})^k (K_{J_i}/K_{x_i})) t_0} \\ &= \left(1 + \frac{1}{11km} \right)^{kl} \in 2^{\Omega(n^{1/2-\gamma})}, \end{aligned}$$

for $\gamma = 1/2a$, where a can be chosen arbitrarily large. Remember that n lies between $12mkl$ and $14mkl$, $l = (11m)^a$ and $k = 9\pi l$. $\square$

6. Conclusions

We have investigated kinetic variants of TSP. Our major result proves an exponential lower bound on the approximation factor for such problems unless $P = NP$ even when the velocities are bounded. Even so, we feel that the bound is coarse and can probably be improved. Also, the question of good upper bounds on the approximation ratio comes to mind.

The use of the mapping f_v described in Section 3 is actually a generic method that can be used to solve a large class of tour problems when instances perform constant translational movement in time.

The translational TSP PTAS that we present can be used in an $O(1)$ -approximation algorithm for KTSP, assuming that the number of different velocities of the instances is bounded by a constant and that the maximum velocity of the instances considered is bounded.

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