The Tilt Formula for Generalized Simplices in Hyperbolic Space

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Abstract. For a simplex in Lorentzian space whose vertices are in the positive light cone, Weeks defined the “tilt” relative to each face. It gave us an efficient tool for deciding whether or not the dihedral angle between two simplices holding a face in common is convex. He also provided an efficient formula, called the “tilt formula,” to obtain tilts from the intrinsic hyperbolic structure of the simplex when its dimension is two or three. Sakuma and Weeks generalized it to general dimensions. In this paper we generalize the concept of the tilt and the tilt formula to the case where not all vertices are in the positive light cone. A key to our generalization is to give a correspondence between points and hyperplanes (or half-spaces) in Lorentzian space.

1. Introduction

In [EP] Epstein and Penner gave a method for decomposing any noncompact complete cusped hyperbolic manifold of finite volume with *weight* at each cusp into ideal polyhedra. This decomposition is called the *Euclidean decomposition*, and defined via a *convex hull construction* in Lorentzian space. Each vertex of the hull is in the positive light cone and corresponds to a lift of a cusp, and each face of the hull corresponds to an ideal polyhedron in the manifold. Especially, if all weights are equal, then the decomposition is invariant under the action of its fundamental group and is called the *canonical decomposition*.

By its definition, the canonical decomposition of a cusped hyperbolic manifold is determined by its intrinsic hyperbolic structure. So this decomposition becomes a useful tool for studying such a manifold. When the dimension is two, this gives us a canonical cellular decomposition of the Teichmüller space of hyperbolic surfaces (see, for example, [P]), and when the dimension is three, this is useful for the classification of hyperbolic manifolds, determination of their isometry group and so on (see, for example, [SW1]). So

an efficient tool was required to decide whether or not a given polyhedral decomposition is canonical.

One of the earliest suggestions to this requirement was given by Penner in Proposition 2.6(b) of [P], which gave us a criterion of convexity when simplices are (two-dimensional) ideal triangles. His idea was translated by Näätänen [N1, Lemma 3.3] into the case where simplices are triangles, and by Ushijima [U2, Proposition 3.5(2)] into the case where simplices are truncated triangles (i.e., orthogonal hexagons). These criteria are used for studying the Teichmüller space of such surfaces.

Independent of the above development, Weeks gave [W1, Proposition 3.1] a criterion of convexity when simplices are two- and three-dimensional ideal simplices. For a simplex in Lorentzian space whose vertices are in the positive light cone, he defined a real value called *tilt* on each of its faces. Using this value, we can easily decide whether or not the dihedral angle between two simplices holding a face in common is convex. Since the canonical decomposition is defined from a convex hull in Lorentzian space, it becomes a useful tool to determine whether or not a given polyhedral decomposition of the manifold is canonical in the previous sense. He also provided an efficient formula, called the *tilt formula*, to obtain tilts from the intrinsic hyperbolic structure of the simplex when its dimension is two (see Theorem 3.2 of [W1]) and three (see Theorem 5.1 of [W1]). He made a program, called “SnapPea,” for creating and studying three-dimensional hyperbolic manifolds [W2]. In this program the formula is used for determining the canonical decomposition of a given three-dimensional hyperbolic manifold. SnapPea is so useful that it is used in many researches, especially in topology and astrophysics (see, for example, [CDW] and [I]). Later Sakuma and Weeks [SW2] generalized the tilt formula to general dimensions.

On the other hand, Kojima [K1], [K2] gave a method for decomposing any complete hyperbolic manifold of finite volume *with nonempty totally geodesic boundary* into partially truncated polyhedra. We note here that this decomposition is not a complete generalization of Epstein–Penner’s one. However, this decomposition is also useful in studying such hyperbolic manifolds (see, for example, [U1] and [U2]), and convex hulls in Lorentzian space still play an important role. Therefore it is also meaningful to make an efficient tool for the decision of the canonical decomposition, namely, to generalize the tilt and the tilt formula. The purpose of the paper is to do so (see Definition 3.1 and Theorem 3.3).

This paper is organized as follows: in Section 2 we recall some basic facts about Lorentzian space and hyperbolic geometry. One important task of this section is to give a correspondence between points and hyperplanes (or half-spaces) in Lorentzian space. This correspondence is a generalization of the well-known duality between points in the hyperboloid of one sheet and half-spaces in hyperbolic space. In Section 3 we first define a *generalized n -simplex* in the projective model of the n -dimensional hyperbolic space. Roughly speaking, this is a partially truncated n -dimensional simplex of finite volume. Next we define the *tilt* of a generalized n -simplex relative to an internal face, and obtain a relationship between tilts and the convexity of two adjoining generalized n -simplices (see Proposition 3.2). Finally we state Theorem 3.3, which shows how to obtain tilts from the intrinsic hyperbolic structure of a generalized n -simplex, by imitating the method in [SW2].

2. Lorentzian Space and Hyperbolic Geometry

The $(n+1)$ -dimensional Lorentzian space $\mathbb{E}^{1,n}$ is the real vector space $\mathbb{R}^{n+1}$ of dimension $n+1$ with the Lorentzian inner product $\langle \mathbf{x}, \mathbf{y} \rangle := -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n$, where $\mathbf{x} = (x_0, x_1, \dots, x_n)$ and $\mathbf{y} = (y_0, y_1, \dots, y_n)$. In this paper we assume $n \geq 2$. If $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is zero (resp. positive, imaginary), then $\mathbf{x}$ is said to be *light-like* (resp. *space-like*, *time-like*). The coordinate x_0 of $\mathbb{E}^{1,n}$ is called the *height*. We denote by L^+ the set of light-like vectors with positive height; $L^+ := \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0 \text{ and } x_0 > 0\}$.

Let $H_T^+ := \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1 \text{ and } x_0 > 0\}$ be the upper sheet of the (standard) hyperboloid of two sheets. The restriction of the quadratic form induced by $\langle \cdot, \cdot \rangle$ on $\mathbb{E}^{1,n}$ to the tangent space of H_T^+ is positive definite and gives a Riemannian metric on H_T^+ . The space obtained from H_T^+ equipped with the metric above is called the *hyperboloid model* of the n -dimensional hyperbolic space. If $\mathbf{x}$ and $\mathbf{y}$ are points in H_T^+ and d denotes the hyperbolic distance between $\mathbf{x}$ and $\mathbf{y}$, then the following relation holds (see, for example, p. 45 of [N2] or Proposition 2.4.5(a) of [T]): $\langle \mathbf{x}, \mathbf{y} \rangle = -\cosh d$. The (standard) *hyperboloid of one sheet* H_S is defined to be $H_S := \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$.

We denote by $\mathcal{P}$ the radial projection from $\mathbb{E}^{1,n} - \{\mathbf{x} \in \mathbb{E}^{1,n} \mid x_0 = 0\}$ to an affine hyperplane $\mathbf{P}_1^n := \{\mathbf{x} \in \mathbb{E}^{1,n} \mid x_0 = 1\}$ along the ray from the origin $\mathbf{o} = (0, 0, \dots, 0)$. The projection $\mathcal{P}$ is a homeomorphism on H_T^+ to the n -dimensional open unit ball $\mathbf{B}^n$ in $\mathbf{P}_1^n$ centered at $(1, 0, 0, \dots, 0)$. We call $\mathbf{B}^n$ with the metric induced by the projection the *projective model* of the n -dimensional hyperbolic space. This projection also induces the mapping from $\mathbb{E}^{1,n} - \{\mathbf{o}\}$ to the n -dimensional real projective space $\mathbf{P}^n = \mathbf{P}_1^n \cup \mathbf{P}_\infty^n$, where $\mathbf{P}_\infty^n$ is the set of lines in the affine hyperplane $\{\mathbf{x} \in \mathbb{E}^{1,n} \mid x_0 = 0\}$ through $\mathbf{o}$ and form the *sphere at infinity*. Now we use the notation $\mathcal{P}$ for the mapping obtained as above for brevity since there will be no confusion.

A vector subspace of $\mathbb{E}^{1,n}$ is said to be *time-like* if it has a time-like vector, *space-like* if every nonzero vector in it is space-like, or *light-like* otherwise. Suppose P is a time-like linear hyperplane, and let R be a half-space in $\mathbb{E}^{1,n}$ bounded by P . Then we can associate a unique vector $\mathbf{w} \in H_S$ so that $\langle \mathbf{w}, \mathbf{q} \rangle \leq 0$ for any $\mathbf{q} \in R$. This establishes a well-known duality between points on H_S and half-spaces in $\mathbb{E}^{1,n}$ bounded by time-like linear hyperplanes. Now we give a generalization of this duality. For an arbitrary vector $\mathbf{u}$ in $\mathbb{E}^{1,n}$, we define a half-space R_u and a hyperplane P_u in $\mathbb{E}^{1,n}$ as follows:

$$R_u := \left\{ \mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq \frac{\langle \mathbf{u}, \mathbf{u} \rangle - 1}{2} \right\},$$

$$P_u := \left\{ \mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{u} \rangle = \frac{\langle \mathbf{u}, \mathbf{u} \rangle - 1}{2} \right\} = \partial R_u.$$

We denote by Γ_u (resp. Π_u) the intersection of R_u (resp. P_u) and H_T^+ . We call $\mathbf{u}$ a *normal vector* to P_u (or Π_u). Then we can easily obtain the following proposition:

Proposition 2.1.

- (1) If $\mathbf{u} \in H_T^+$, then $\Pi_u = \{\mathbf{u}\}$.
- (2) If $\mathbf{u} \in L^+$, then Π_u is a horosphere whose center is a line directed by $\mathbf{u}$.
- (3) If $\mathbf{u} \in H_S$, then Π_u is a geodesic hyperplane.

Remark 2.2. More generally, if $\langle \mathbf{u}, \mathbf{u} \rangle < 0$ and $u_0 > 0$, then $\Pi_{\mathbf{u}}$ is a sphere centered at $n(\mathbf{u}) := \mathbf{x} / \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$, and if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, then $\Pi_{\mathbf{u}}$ is an equidistant hypersurface whose axial hyperplane is $\Pi_{n(\mathbf{u})}$.

Furthermore, if the other cases hold, i.e., if $\langle \mathbf{u}, \mathbf{u} \rangle \leq 0$ and $u_0 \leq 0$, then $\Pi_{\mathbf{u}} = P_{\mathbf{u}} \cap H_T^+ = \emptyset$. This means that almost every point in $\mathbb{E}^{1,n}$ can be given an appropriate geometric meaning as the correspondence to all fundamental geometric objects in the hyperbolic space.

Now we define the *generalized distance* between two geometric objects. Here the word “generalized” means that the distance defined below is a unification of distances (in the usual sense) and the angles.

Definition 2.3 (Generalized Distance). For a point $\mathbf{x}$ in H_S and $\mathbf{y}$ in $(H_T^+ \cup L^+ \cup H_S) \cap R_x$, the *generalized distance* d between Π_x and Π_y is defined as follows:

Case 1. If $\mathbf{y} \in H_T^+ \cap R_x$, then d is the distance (in the usual sense) between Π_x and $\Pi_y = \{\mathbf{y}\}$.

Case 2. If $\mathbf{y} \in L^+ \cap R_x$, then d is the signed distance (defined below) between Π_x and Π_y .

Case 3. If $\mathbf{y} \in H_S \cap R_x$, then $(\langle \mathbf{x}, \mathbf{y} \rangle \leq 0)$ and we have two different cases:

Case 3.1. If $-1 < \langle \mathbf{x}, \mathbf{y} \rangle (\leq 0)$, i.e., if Π_x and Π_y intersect each other, then $d := \sqrt{-1}\theta$, where θ is the dihedral angle between Π_x and Π_y measured in $\Gamma_x \cap \Gamma_y$.

Case 3.2. If $\langle \mathbf{x}, \mathbf{y} \rangle \leq -1$, i.e., if Π_x and Π_y are parallel or ultraparallel, then d is the distance (in the usual sense) between Π_x and Π_y .

Here the *signed distance* between a geodesic hyperplane Π_x and a horosphere Π_y is the signed distance between Π_x and $\mathbf{z}$, where $\mathbf{z}$ is the intersecting point of Π_y and the time-like linear plane spanned by $\mathbf{x}$ and $\mathbf{y}$ (if $\mathbf{y} \notin P_x$), and the sign is defined to be positive (resp. negative) if $\mathbf{z} \in \Gamma_x$ (resp. $\mathbf{z} \notin \Gamma_x$), that is, if $\langle \mathbf{x}, \mathbf{z} \rangle \leq 0$ (resp. $\langle \mathbf{x}, \mathbf{z} \rangle > 0$). If $\mathbf{y} \in P_x$, then we define the signed distance as $+\infty$.

Summarizing several basic facts on hyperbolic geometry (see, for example, Theorems 3.2.8, 6.2.10, and 3.2.12 of [R]), we can obtain the following proposition:

Proposition 2.4. Let $\mathbf{x}$ be a point in H_S . For an arbitrary point $\mathbf{y} \in (H_T^+ \cup L^+ \cup H_S) \cap R_x$, the following equality holds:

$$\langle \mathbf{x}, \mathbf{y} \rangle = -\frac{e^d + ve^{-d}}{2},$$

where $v := \langle \mathbf{y}, \mathbf{y} \rangle$ and d is the generalized distance between Π_x and Π_y .

3. Definitions of Generalized n -Simplices and Tilts

3.1. The Definition of Generalized n -Simplices

Let $U = \{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a set of independent points in $H_T^+ \cup L^+ \cup H_S$. We divide U into two disjoint subsets, say U_{in} and U_{ex} as follows: $U_{\text{in}} := \{\mathbf{u} \in U \mid \mathbf{u} \in H_T^+ \cup L^+\}$

Fig. 1. An example of a generalized 2-simplex in $\mathbf{B}^2$.

and $U_{\text{ex}} := \{\mathbf{u} \in U \mid \mathbf{u} \in H_S\} = U - U_{\text{in}}$. Without loss of generality, we may assume $U_{\text{ex}} = \{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_k\}$ and $U_{\text{in}} = \{\mathbf{u}_{k+1}, \mathbf{u}_{k+2}, \dots, \mathbf{u}_n\}$ for some k by changing indices if necessary. For any $i \in \{0, 1, \dots, n\}$, we denote by F_i the linear hyperplane spanned by $\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \dots, \mathbf{u}_n\}$. Let Δ_U be the n -simplex in $\mathbb{E}^{1,n}$ with the vertex set U .

Now we suppose U satisfies the following two conditions:

Condition 1. For any $i \in \{0, 1, \dots, k\}$, $R_{u_i} - P_{u_i} \supset \{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \dots, \mathbf{u}_n\}$.

Condition 2. For any $i \in \{0, 1, \dots, n\}$, $F_i \cap H_T^+ \neq \emptyset$.

Let V be $\mathcal{P}(U)$ and let $\mathbf{v}_i$ be $\mathcal{P}(\mathbf{u}_i)$. Under these conditions, we define a *generalized n -simplex Δ_V in $\mathbf{B}^n$ with vertex set V* as $\mathcal{P}(\Delta_U \cap \bigcap_{i=0}^k R_{u_i})$ (see Fig. 1).

We regard the $\mathbf{v}_i$'s as vertices of Δ_V . Each face of Δ_V is either contained in P_{u_i} for some $i \in \{0, 1, \dots, k\}$ or in F_j for some $j \in \{0, 1, \dots, n\}$. According to Kojima [K1], [K2], we call the former an *external face* of Δ_V , and the later an *internal face* of Δ_V (especially, this internal face is the *opposite face* of $\mathbf{v}_j$). Now Condition 1 means that Π_{u_i} cuts off nothing but $\mathbf{v}_i$ from V , and Condition 2 means that for each vertex $\mathbf{v}_i$ its opposite (internal) face exists.

For a generalized n -simplex Δ_V with weight at all ideal vertices (here a “weight” is a real number), there exists a unique n -simplex Δ_U in $\mathbb{E}^{1,n}$ whose projection corresponds to Δ_V . We note here that the correspondence between an ideal vertex $\mathbf{v}$ of Δ_V with weight w and its preimage $\mathbf{u}$ is defined as follows: $\mathcal{P}(\mathbf{u}) = \mathbf{v}$ and $w = -\log(u_0\sqrt{2})$, where u_0 is a height of $\mathbf{u}$. From now on we use the term “generalized n -simplex” as the simplex itself with weights at all ideal vertices. We call Δ_U the *lift* of a generalized n -simplex Δ_V in $\mathbf{B}^n$, U the *lift* of the vertex set V , and $\mathbf{u}$ the lift of the vertex $\mathcal{P}(\mathbf{u}) \in V$.

3.2. The Definition of Tilt and the Tilt Proposition

Fix a generalized n -simplex Δ_V in $\mathbf{B}^n$, and take one of its internal faces Φ_i . Then there is a unique point $\mathbf{m}_i$ in H_S such that $\Phi_i \subset P_{\mathbf{m}_i}$ and $\Delta_V \subset R_{\mathbf{m}_i}$. We define the *normal vector $\mathbf{p}$ to the lift Δ_U of Δ_V* by the condition that $\langle \mathbf{p}, \mathbf{x} \rangle = -1$ for all $\mathbf{x} \in \Delta_U$.

Definition 3.1. The tilt t_i of Δ_V relative to Φ_i is defined as follows:

$$t_i := \langle \mathbf{m}_i, \mathbf{p} \rangle.$$

Proposition 3.2 following, corresponding to Proposition 3.1 of [W1], states how to use the tilt for deciding the convexity.

Proposition 3.2 (Tilt Proposition). *Let Δ_{U_0} and Δ_{U_1} be two adjacent n -simplices in $\mathbb{E}^{1,n}$ that project to generalized n -simplices in $\mathbf{B}^n$. We denote by t_0 (resp. t_1) the tilt of Δ_{U_0} (resp. Δ_{U_1}) relative to the joint face. Then the dihedral angle formed by Δ_{U_0} and Δ_{U_1} is convex (flat, concave, respectively) if and only if $t_0 + t_1 < 0$ ($= 0$, > 0 , respectively).*

Proof. Let $U_0 = \{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $U_1 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n+1}\}$. We denote by $\mathbf{m}$ the normal vector to the hyperplane containing the joint face satisfying that $\Delta_{U_0} \subset R_{\mathbf{m}}$. Then, for an arbitrary $i \in \{1, 2, \dots, n\}$, we have $\langle \mathbf{u}_i, \mathbf{m} \rangle = 0$. Furthermore, we have $\langle \mathbf{u}_0, \mathbf{m} \rangle < 0$ and $\langle \mathbf{u}_{n+1}, \mathbf{m} \rangle > 0$.

Since vectors $\mathbf{m}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ form a basis of $\mathbb{E}^{1,n}$, there exist unique real numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ such that

$$\mathbf{u}_0 = \alpha_0 \mathbf{m} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_n \mathbf{u}_n.$$

Similarly we have

$$\mathbf{u}_{n+1} = \alpha'_0 \mathbf{m} + \alpha'_1 \mathbf{u}_1 + \alpha'_2 \mathbf{u}_2 + \dots + \alpha'_n \mathbf{u}_n$$

for some $\alpha'_0, \alpha'_1, \dots, \alpha'_n \in \mathbb{R}$. We note that $\alpha'_0 > 0$, since $\langle \mathbf{u}_{n+1}, \mathbf{m} \rangle > 0$.

Let $\mathbf{p}_0$ (resp. $\mathbf{p}_1$) be the normal vector to Δ_{U_0} (resp. Δ_{U_1}). Now $\mathbf{p}_0$ is also expressed as a linear combination of $\mathbf{m}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ as follows:

$$\mathbf{p}_0 = \beta_0 \mathbf{m} + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_n \mathbf{u}_n,$$

where $\beta_0, \beta_1, \dots, \beta_n \in \mathbb{R}$. Then we have the following equation:

$$\begin{pmatrix} \langle \mathbf{p}_0, \mathbf{u}_1 \rangle \\ \langle \mathbf{p}_0, \mathbf{u}_2 \rangle \\ \vdots \\ \langle \mathbf{p}_0, \mathbf{u}_n \rangle \end{pmatrix} = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle & \langle \mathbf{u}_1, \mathbf{u}_2 \rangle & \dots & \langle \mathbf{u}_1, \mathbf{u}_n \rangle \\ \langle \mathbf{u}_2, \mathbf{u}_1 \rangle & \langle \mathbf{u}_2, \mathbf{u}_2 \rangle & \dots & \langle \mathbf{u}_2, \mathbf{u}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{u}_n, \mathbf{u}_1 \rangle & \langle \mathbf{u}_n, \mathbf{u}_2 \rangle & \dots & \langle \mathbf{u}_n, \mathbf{u}_n \rangle \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix},$$

that is,

$$\sum_{i=1}^n \beta_i \langle \mathbf{u}_j, \mathbf{u}_i \rangle = -1 \quad \text{for } j \in \{1, 2, \dots, n\}. \quad (3.1)$$

Similarly we have

$$\mathbf{p}_1 = \beta'_0 \mathbf{m} + \beta'_1 \mathbf{u}_1 + \beta'_2 \mathbf{u}_2 + \dots + \beta'_n \mathbf{u}_n$$

for some $\beta'_0, \beta'_1, \dots, \beta'_n \in \mathbb{R}$. Then, in the same fashion as (3.1), we have

$$\sum_{i=1}^n \beta'_i \langle \mathbf{u}_j, \mathbf{u}_i \rangle = -1 \quad \text{for } j \in \{1, 2, \dots, n\}.$$

Using the relations above, we can compute $\langle \mathbf{p}_1, \mathbf{u}_{n+1} \rangle$ as follows:

$$\begin{aligned} -1 = \langle \mathbf{p}_1, \mathbf{u}_{n+1} \rangle &= \alpha'_0 \beta'_0 + \alpha'_1 (\beta'_1 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + \beta'_2 \langle \mathbf{u}_1, \mathbf{u}_2 \rangle + \dots + \beta'_n \langle \mathbf{u}_1, \mathbf{u}_n \rangle) \\ &\quad + \alpha'_2 (\beta'_1 \langle \mathbf{u}_2, \mathbf{u}_1 \rangle + \beta'_2 \langle \mathbf{u}_2, \mathbf{u}_2 \rangle + \dots + \beta'_n \langle \mathbf{u}_2, \mathbf{u}_n \rangle) \\ &\quad + \dots \\ &\quad + \alpha'_n (\beta'_1 \langle \mathbf{u}_n, \mathbf{u}_1 \rangle + \beta'_2 \langle \mathbf{u}_n, \mathbf{u}_2 \rangle + \dots + \beta'_n \langle \mathbf{u}_n, \mathbf{u}_n \rangle) \\ &= \alpha'_0 \beta'_0 - \sum_{i=1}^n \alpha'_i. \end{aligned}$$

Thus we have the following relation:

$$\sum_{i=1}^n \alpha'_i = \alpha'_0 \beta'_0 + 1. \quad (3.2)$$

The hyperplane P_i , where $i = 1$ or 2 , is defined to be $P_i := \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{p}_i \rangle = -1\}$. Then P_i contains Δ_{U_i} . Now the dihedral angle formed by Δ_{U_0} and Δ_{U_1} is convex if and only if P_0 separates $\mathbf{u}_{n+1}$ from the origin $\mathbf{o}$, or, equivalently, if and only if P_1 separates $\mathbf{u}_0$ from $\mathbf{o}$. This condition is equivalent to the one when $\langle \mathbf{u}_i, \mathbf{p}_j \rangle < -1$, where $i, j = (n+1, 0)$ or $(0, 1)$. So using (3.1) and (3.2), we can compute $\langle \mathbf{u}_{n+1}, \mathbf{p}_0 \rangle$ as follows:

$$\begin{aligned} -1 > \langle \mathbf{u}_{n+1}, \mathbf{p}_0 \rangle &= \alpha'_0 \beta_0 + \alpha'_1 (\beta_1 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + \beta_2 \langle \mathbf{u}_1, \mathbf{u}_2 \rangle + \dots + \beta_n \langle \mathbf{u}_1, \mathbf{u}_n \rangle) \\ &\quad + \alpha'_2 (\beta_1 \langle \mathbf{u}_2, \mathbf{u}_1 \rangle + \beta_2 \langle \mathbf{u}_2, \mathbf{u}_2 \rangle + \dots + \beta_n \langle \mathbf{u}_2, \mathbf{u}_n \rangle) \\ &\quad + \dots \\ &\quad + \alpha'_n (\beta_1 \langle \mathbf{u}_n, \mathbf{u}_1 \rangle + \beta_2 \langle \mathbf{u}_n, \mathbf{u}_2 \rangle + \dots + \beta_n \langle \mathbf{u}_n, \mathbf{u}_n \rangle) \\ &= \alpha'_0 \beta_0 - \sum_{i=1}^n \alpha'_i \\ &= \alpha'_0 \beta_0 - (\alpha'_0 \beta'_0 + 1). \end{aligned}$$

Thus we obtain $\beta_0 - \beta'_0 < 0$, since $\alpha'_0 > 0$. Furthermore, $t_0 = \langle \mathbf{m}, \mathbf{p}_0 \rangle = \beta_0$ and $t_1 = \langle -\mathbf{m}, \mathbf{p}_1 \rangle = -\beta'_0$, and we have $t_0 + t_1 < 0$ if and only if the dihedral angle formed by Δ_{U_0} and Δ_{U_1} is convex. The proofs of the other cases are analogous. We have thus proved Proposition 3.2. $\square$

3.3. The Tilt Formula

The definition of the canonical decomposition requires that a tilt is obtained from the intrinsic hyperbolic structure of a generalized n -simplex, and the following *tilt formula* shows how to do it.

Fig. 2. Values for getting tilts.

Fix a generalized n -simplex Δ_V in $\mathbf{B}^n$, and denote by $U = \{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n\}$ the lift of V . Let $v_i := \langle \mathbf{u}_i, \mathbf{u}_i \rangle$ and let d_i be the generalized distance between $\Pi_{\mathbf{u}_i}$ and $\Pi_{\mathbf{m}_i}$, where $\mathbf{m}_i$ is a point in H_S with $\Phi_i \subset P_{\mathbf{m}_i}$ and $\Delta_V \subset R_{\mathbf{m}_i}$. Then we define $Q_i := 2/(e^{d_i} + v_i e^{-d_i})$. We denote by ϱ_{ij} the generalized distance between $\Pi_{\mathbf{m}_i}$ and $\Pi_{\mathbf{m}_j}$. We note here that we can obtain v_i , d_i and ϱ_{ij} only from the intrinsic hyperbolic structure of Δ_V (see Fig. 2). Then we have Theorem 3.3 following, a generalization of Theorem 2.1 of [SW2]:

Theorem 3.3 (Tilt Formula). *The tilt of a generalized n -simplex relative to each of its (codimension one) internal faces may be computed as follows:*

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} 1 & -\cosh \varrho_{01} & -\cosh \varrho_{02} & \cdots & -\cosh \varrho_{0n} \\ -\cosh \varrho_{10} & 1 & -\cosh \varrho_{12} & \cdots & -\cosh \varrho_{1n} \\ -\cosh \varrho_{20} & -\cosh \varrho_{21} & 1 & \cdots & -\cosh \varrho_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cosh \varrho_{n0} & -\cosh \varrho_{n1} & -\cosh \varrho_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{pmatrix}.$$

We call the $(n+1) \times (n+1)$ matrix on the right side of the above formula the *Gram matrix of the generalized n -simplex Δ_V* (see [V]). The proof of this theorem is nearly a word-by-word translation of that of Theorem 2.1 of [SW2] together with replacing Lemma 2.4 of [SW2] by Proposition 2.4 in this paper. Thus we omit the proof.

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