About 4-Isosceles Planar Sets*

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Abstract. A finite planar set is k -isosceles for $k \geq 3$, if every k -point subset of the set contains a point equidistant from the other two. This paper gives affirmative answers to a few open problems about 4-isosceles sets.

1. Introduction

A finite planar set is k -isosceles for $k \geq 3$, if every k -point subset of the set includes a 3-set which forms an isosceles triangle, i.e., contains a point equidistant from the other two. In [1] Fishburn discussed 3-isosceles planar sets and 4-isosceles planar sets. At the end of his paper, he put forward several open questions about 4-isosceles planar sets. We solve the first two questions and provide a remedy for Theorem 1 in [1]. Let $1, 2, 3, 4, \dots$ be points on a horizontal line from left to right. When the successive interpoint distances are $a, b, c, \dots$, we say that $\{1, 2, 3, 4, \dots\}$ has the linear successive distance pattern $a : b : c : \dots$. In this paper we use the same terminology as in [1].

2. 4-Isosceles Planar Set with Linear Restrictions

The conclusion about a 4-isosceles planar 5-set on a line is essential in proving the other conclusions in Theorem 1 in [1]. We give a remedy for it with one more distance pattern.

Theorem 1. *A 5-set on a line is 4-isosceles if and only if its linear successive distance pattern is $2 : 1 : 1 : 2$ or $1 : 1 : 1 : 3$ or $1 : 1 : 2 : 3$ or $2 : 2 : 1 : 3$.*

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Proof. It is easy to show that a 4-set on a line is a 4-isosceles planar set if and only if its successive interpoint distances have one of the two patterns: pattern 1, $a : a : *$ or $* : a : a$; pattern 2, $a : (a - *) : *$ or $* : (a - *) : a$. Let F be a 4-isosceles planar 5-set, its five points are 1, 2, 3, 4, 5 on a horizontal line from left to right. Then the successive distance pattern of each 4-point subset of F must be of pattern 1 or pattern 2. If the successive distance pattern of $\{1, 2, 3, 4\}$ is of pattern 1, consider the other four 4-point subsets of $\{1, 2, 3, 4, 5\}$. It is easy to see that the successive distance pattern of $\{1, 2, 3, 4, 5\}$ is $a : a : a : 3a$ or $2a : a : a : 2a$ or $a : a : 2a : 3a$ or $a : a : a/2 : 3a/2$, and so we have patterns $1 : 1 : 1 : 3$ or $2 : 1 : 1 : 2$ or $1 : 1 : 2 : 3$ or $2 : 2 : 1 : 3$. If the successive distance pattern of $\{1, 2, 3, 4\}$ is of pattern 2, discuss the other 4-point subsets of $\{1, 2, 3, 4, 5\}$. We get the successive distance patterns $a/2 : a/2 : a : 3a/2$ or $a : a/2 : a/2 : a$ or $a : 2a/3 : a/3 : a/3$ or $a : a/3 : 2a/3 : 2a/3$, hence we get three patterns: $1 : 1 : 2 : 3$, $2 : 1 : 1 : 2$, and $3 : 1 : 2 : 2$. So if F is a 4-isosceles planar 5-set on a line, then its linear successive distance pattern is $2 : 1 : 1 : 2$ or $1 : 1 : 1 : 3$ or $1 : 1 : 2 : 3$ or $2 : 2 : 1 : 3$.

The sufficiency is obvious. $\square$

Adding the new linear successive distance pattern $2 : 2 : 1 : 3$, we can show that the other conclusions of Theorem 1 in [1] are still true. There is one more maximum 4-isosceles planar set with four points $\{1, 2, 3, 4\}$ on a line whose linear successive distance pattern is $1 : 1 : 2$ except the one as shown in [1]. The other 4-isosceles 7-set has $\{1, 2, 3, 4\}$, one point on $\perp_{13}$ distance 1 from point 2, two points on $\perp_{14}$ distance 1 from point 3. We can show that there is no 8-point 4-isosceles set in conjunction with this 7-set.

3. Circle and Linear Restriction

In [1] Fishburn proposed two open questions:

Let F denote a 4-isosceles planar set. Is there a 6-point F with no four points on a circle and no three points on a line? Is there a 7-point F with no four points on a circle?

We give affirmative answers to the two questions.

Theorem 2. *There exists a 6-point set F with no four points on a circle and no three points on a line.*

Proof. First construct a 6-point set F : Starting from $\{1, 2, 3, 4\}$ in which $\{1, 2, 3\}$ is the vertex set of an equilateral triangle with edge length 1, 4 is the center of $\triangle 123$. Establish a coordinate system such that 2, 3 are on the x -axis, 3 is on the positive x -semiaxis, and 1 is on the positive y -semiaxis. Then we have $1(0, \sqrt{3}/2)$, $2(-1/2, 0)$, $3(1/2, 0)$, and $4(0, \sqrt{3}/6)$. Choose a point 5 on $\perp_{14}$ and a point 6 on the circle centered at 2 with radius 1 such that $d(1, 5) = d(1, 6)$ and $d(4, 5) = d(5, 6)$. Solving a system of equations in x , y , and z we find that such two points do exist. For example, we may take

$5(z, \sqrt{3}/3)$, $6(x, y)$, where $x = -0.7287\dots$, $y = -0.9734\dots$, and $z = -1.9574\dots$. Let $F = \{1, 2, 3, 4, 5, 6\}$.

It is easy to see that F is a 4-isosceles 6-set, and no three points are on a line. Next we prove that no four points of F are on a circle. A set is said to be nonconcyclic if the points of the set are not on a circle. It is obvious that the convex hull of each of the following 4-point subsets is a triangle:

$$\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\}.$$

So each of the above subsets is nonconcyclic. Each of the other nine 4-point subsets is also nonconcyclic. The reasoning is as follows.

$$\begin{aligned} \angle 126 > 150^\circ &\implies \{1, 2, 3, 6\}, \{1, 2, 4, 6\} \text{ nonconcyclic;} \\ \angle 315 > 60^\circ, \angle 325 > 120^\circ &\implies \{1, 2, 3, 5\} \text{ nonconcyclic;} \\ \angle 142 = 120^\circ, \angle 152 \neq 60^\circ &\implies \{1, 2, 4, 5\} \text{ nonconcyclic;} \\ \angle 156 = 60^\circ, \angle 136 \neq 120^\circ, \angle 146 \neq 120^\circ &\implies \{1, 3, 5, 6\}, \{1, 4, 5, 6\} \text{ nonconcyclic;} \\ \angle 345 > 120^\circ, \angle 325 > 60^\circ, \angle 365 > 60^\circ &\implies \{2, 3, 4, 5\}, \{3, 4, 5, 6\} \text{ nonconcyclic;} \\ \angle 436 > 60^\circ, \angle 426 > 120^\circ &\implies \{2, 3, 4, 6\} \text{ nonconcyclic.} \end{aligned}$$

The proof is complete. $\square$

Theorem 3. *There exists a 4-isosceles 7-set F with no four points on a circle.*

Proof. First construct a 7-set F : Starting from $\{1, 2, 3, 4\}$ left to right on a horizontal line with its successive distance pattern: $1 : 1 : 1$, the other three points 5, 6, 7 are on $\perp_{14}$, point 5 is at distance 2 from point 2 above line 14, point 6 is at distance 1 from point 2 above line 14, and point 7 is at distance 2 from point 2 below line 14.

It is easy to show that F is a 4-isosceles set. Next we will show that no four points in F are on a circle. If a 4-point subset has at least three points on a line, then the four points are not on a circle. So we need only check that each 4-point subset with no three points on a line is not on a circle. By duality, we need only check the following 4-point subsets:

$$\begin{aligned} &\{1, 2, 5, 6\}, \{1, 2, 5, 7\}, \{1, 2, 6, 7\}, \{1, 3, 5, 6\}, \{1, 3, 5, 7\}, \{1, 3, 6, 7\}, \\ &\{1, 4, 5, 6\}, \{1, 4, 5, 7\}, \{1, 4, 6, 7\}, \{2, 3, 5, 6\}, \{2, 3, 5, 7\}, \{2, 3, 6, 7\}. \end{aligned}$$

Each convex hull of subsets $\{1, 2, 5, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 3, 5, 6\}$, $\{2, 3, 5, 6\}$, $\{1, 4, 5, 6\}$ is a triangle, so no four points are on a circle. $\angle 126 = 120^\circ$ while $\angle 156 \neq 60^\circ$, hence $\{1, 2, 5, 6\}$ is not on a circle. It is easy to see that $\angle 637 > 120^\circ$ and $\angle 617 > 60^\circ$, so $\{1, 3, 6, 7\}$ is not on a circle. Similarly, $\{1, 3, 5, 7\}$ is not on a circle. As 1547 and 2537 are rhombuses but not squares, the four points in each subset are not on a circle. $\angle 263 = 60^\circ$ and $\angle 273 \neq 120^\circ$ imply that $\{2, 3, 6, 7\}$ is not on a circle, and $\{1, 4, 6, 7\}$ is not on a circle noticing that $\angle 164 = 120^\circ$, $\angle 174 \neq 60^\circ$. $\square$

From the above discussion we get F is a 4-isosceles 7-set with no four points on a circle.

Reference

1. P. Fishburn, Isosceles planar subsets, *Discrete Comput. Geom.*, **19** (1998), 391–398.

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