Canonical Modules of Semigroup Rings and a Conjecture of Reiner

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Abstract. We prove that the homotopy types of two naturally defined simplicial complexes are related in the following way: one is homotopy equivalent to a multiple suspension of the canonical Alexander dual of the other. These simplicial complexes arise in free resolutions of semigroup rings and modules. The relation between their homotopy types was conjectured by Reiner, and was suggested by a homological consequence of a result due independently to Danilov and Stanley on canonical modules of normal semigroup rings. Our proof is purely topological, and gives an alternative proof of their result. We also prove a generalization of a result of Hochster saying that these rings are Cohen–Macaulay, and indicate a new proof of Ehrhart’s reciprocity law for rational polytopes.

1. Introduction

In this note we prove that the homotopy types of two naturally defined simplicial complexes are related in the following way: one is homotopy equivalent to a multiple suspension of the canonical Alexander dual of the other. These simplicial complexes arise in free resolutions of semigroup rings and modules. The relation between their homotopy types was conjectured by Reiner, and was suggested by a homological consequence of a result due independently to Danilov and Stanley on canonical modules of normal semigroup rings. Our proof is purely topological, and gives an alternative proof of their result. We also prove a generalization of a result of Hochster saying that these rings are Cohen–Macaulay, and indicate a new proof of Ehrhart’s reciprocity law for rational polytopes.

The paper is organized as follows. In this section we formulate our main result Theorem 1.1 in an elementary fashion. In Section 2 we explain our motivations from normal semigroup rings and state Reiner’s Conjecture (Conjecture 2.5). In Section 3 we prove Theorem 1.1, and as a direct corollary prove Conjecture 2.5 (Theorem 3.9). Finally we show a few applications of Theorem 3.9 in Section 4 as promised.

Let P be a *pointed* polyhedral cone in $\mathbb{R}^d$ (see [13]), with its unique vertex at the origin. Without loss of generality, we assume that $d = \dim P$. Let $\{f_i(\mathbf{x}) = 0\}_{i=1}^m$ be the defining hyperplanes of all the facets of P . Thus

$$P = \{\mathbf{x} \in \mathbb{R}^d: f_i(\mathbf{x}) \geq 0, i \in [m]\},$$

where $[m] := \{1, \dots, m\}$. Fix a point $\mu \in P$, and define two subsets of P ,

$$\mathcal{A}_\mu := \{\mathbf{x} \in P: \forall i \in [m], f_i(\mathbf{x}) \leq f_i(\mu)\},$$

$$\mathcal{B}_\mu := \{\mathbf{x} \in P: \exists i \in [m], f_i(\mathbf{x}) \leq f_i(\mu)\}.$$

Given a set of points $S = \{\alpha_1, \dots, \alpha_n\}$ in P , such that *there is at least one α_i on each extreme ray of P* , define two abstract simplicial complexes on a vertex set indexed by S ,

$$\Delta_{S,\mu} := \left\{ F \subseteq S: \sum_{\gamma \in F} \gamma \in \mathcal{A}_\mu \right\},$$

$$\Gamma_{S,\mu} := \left\{ F \subseteq S: \sum_{\gamma \in F} \gamma \in \mathcal{B}_\mu \right\}.$$

If there is no confusion involved, we usually omit the index μ and write them as $\mathcal{A}$, $\mathcal{B}$, Δ_S and Γ_S . It is clear that Δ_S and Γ_S are indeed simplicial complexes, and moreover that Δ_S is a subcomplex of Γ_S .

In what follows we draw no distinction between a simplicial complex Δ and the topological space which is its geometric realization. If two topological spaces X and Y are homotopy equivalent, we denote it by $X \simeq Y$.

Theorem 1.1. *We have $\Gamma_S \simeq \text{Susp}^{d-1} \Delta_S$, where $\text{Susp}^{d-1} \Delta_S$ is the $(d-1)$ -fold suspension of Δ_S .*

Example 1.2. Let

$$P = \{(x, y) \in \mathbb{R}^2 | x - y \geq 0, -x + 3y \geq 0\},$$

$$\mu = (12, 4),$$

$$\alpha_1 = (3, -1), \alpha_2 = (3, 3), \alpha_3 = (6, -1), \alpha_4 = (6, 5),$$

$$S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$$

See Fig. 1. Then

$$\Delta_S = \{\emptyset, \{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}, \{\alpha_1\alpha_2\}, \{\alpha_1\alpha_4\}, \{\alpha_2\alpha_3\}, \{\alpha_3\alpha_4\}\}.$$

Clearly Δ_S is a 4-cycle and hence homeomorphic to $\mathbb{S}^1$. Similarly one sees that any proper subset of S is in Γ_S . Hence Γ_S is the boundary of a tetrahedron and is homeomorphic to $\mathbb{S}^2$.

Fig. 1. (Example 1.2) The shadowed region is $\mathcal{B}$, whereas the darker subregion is $\mathcal{A}$.

Remark 1.3. Theorem 1.1 turns out to be very easy when P is simplicial, i.e. the cross section of P is a simplex. In this case let

$$X_i := \left\{ F \in \Gamma_S : f_i \left(\sum_{\gamma \in F} \gamma \right) \leq f_i(\mu) \right\}.$$

Then $\Gamma_S = \bigcup_{i=1}^m X_i$ and $\Delta_S = \bigcap_{i=1}^m X_i$. It is easy to show that $\bigcap_{i \in A} X_i$ is contractible for $A \subsetneq [m]$. Theorem 1.1 then follows from a special case of Wedge Lemma 1.8 of [14]. Unfortunately this method does not work in general.

Remark 1.4. It is shown in [3] that for any simplicial complex X , one can find suitable P , S and μ such that $\Delta_S \simeq X$. Hence homotopically Δ_S can be arbitrarily complicated.

2. Normal Semigroup Rings and Simplicial Complexes

In this section we introduce Conjecture 2.5 and its background. The conjecture concerns certain simplicial complexes arising in resolutions of normal semigroup rings.

Normal Semigroup Rings

Here we only introduce some basic facts about normal semigroup rings. We refer the reader to [4] for a detailed treatment.

Let L be a lattice (i.e. an abelian group isomorphic to $\mathbb{Z}^N$), and let Λ be a finitely generated subsemigroup of L . Then Λ is called an *affine semigroup*. The *affine semigroup*

ring $k[\Lambda]$ over a field k is the subring of $k[x_1, \dots, x_N, x_1^{-1}, \dots, x_N^{-1}]$ generated by those monomials $\mathbf{x}^\gamma$ with $\gamma \in \Lambda$. Let P be a d -dimensional *rational polyhedral cone* in $L \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $C = P \cap L$ is called a *normal (affine) semigroup* of rank d . It is affine because P is rational implies C is finitely generated. It is normal in the sense that if $m\gamma \in C$ for some $\gamma \in \mathbb{Z}C$ and $m \in \mathbb{N}, m > 0$, then $\gamma \in C$. The *normal semigroup ring* $k[C]$ is a Noetherian normal domain of dimension d . We use $\text{relint } C$ to denote the intersection of L with the relative interior of P .

Theorem 2.1. *Let C be a normal semigroup, and let k be a field. Then*

- (a) (Hochster) $k[C]$ is a Cohen–Macaulay ring,
- (b) (Danilov, Stanley) the ideal I generated by the monomials $\mathbf{x}^\gamma$ with $\gamma \in \text{relint } C$ is the canonical module of $k[C]$.

Remark 2.2. The invertible elements in a semigroup C form a group C_0 . If $C_0 = 0$, we say that C is *positive*. If C is normal, then $C \cong C_0 \oplus C'$ with C' being a positive normal semigroup and $C_0 \cong \mathbb{Z}^l$ for some $l \geq 0$. Therefore one has $k[C] \cong k[C_0] \otimes_k k[C'] \cong k[\mathbb{Z}^l] \otimes_k k[C']$. It then follows from Theorem 2.1.9 and Corollary 3.3.21 of [4], that to prove Theorem 2.1 for C , it is enough to prove it for C' .

For simplicity, from now on we assume C is a positive normal semigroup, thus P is a *pointed* rational polyhedral cone. We also use R to denote $k[C]$.

Let $\{\alpha_1, \dots, \alpha_n\} \subset C$ be such that C is a finitely generated semigroup module over the subsemigroup Λ generated by $\alpha_1, \dots, \alpha_n$. Equivalently, there exists at least one α_i on each extreme ray of P . Let $A = k[u_1, \dots, u_n]$ be a polynomial ring with n indeterminates. The map of k -algebras $\pi: A \rightarrow R$ with $u_i \mapsto \mathbf{x}^{\alpha_i}$ makes R a finitely generated A -module. The polynomial ring A inherits a $\mathbb{Z}^n$ -grading from R (which in general is different from the usual $\mathbb{Z}^n$ -grading of A). We take a minimal graded free resolution $\mathcal{F}$ of R as an A -module,

$$\mathcal{F}: 0 \rightarrow \bigoplus_{\mu \in \mathbb{Z}^N} A(-\mu)^{\beta_{h\mu}} \rightarrow \dots \rightarrow \bigoplus_{\mu \in \mathbb{Z}^N} A(-\mu)^{\beta_{0\mu}} \rightarrow R \rightarrow 0, \quad (1)$$

where $\sum_{\mu \in \mathbb{Z}^N} \beta_{h\mu} \neq 0$ and $h = n - \text{depth}_A(R)$ by the Auslander–Buchsbaum theorem. The boundary maps in the resolution can be written as matrices whose nonzero entries are homogeneous of nonzero degrees. Thus by tensoring with k all the boundary maps become trivial. This shows that $\beta_{i\mu} = \dim_k \text{Tor}_i^A(R, k)_\mu$, where we use $\text{Tor}_i^A(R, k)_\mu$ to denote the μ -graded piece of $\text{Tor}_i^A(R, k)$.

Let $\Omega(A)$ be the canonical module of the polynomial ring A and let $\alpha := \sum_{i=1}^n \alpha_i$. Then $\Omega(A) \cong A(-\alpha)$ (see Section 3 of [3]). Since R is Cohen–Macaulay, applying the functor $\text{Hom}_A(\cdot, \Omega(A))$ to (1) gives a minimal free resolution of the canonical module $\Omega(R)$ of R :

$$0 \leftarrow \Omega(R) \leftarrow \bigoplus_{\mu \in \mathbb{Z}^N} A(\mu - \alpha)^{\beta_{h\mu}} \leftarrow \dots \leftarrow \bigoplus_{\mu \in \mathbb{Z}^N} A(\mu - \alpha)^{\beta_{0\mu}} \leftarrow 0. \quad (2)$$

Now $\Omega(R) \cong I$ by Theorem 2.1, where I is the ideal of R generated by the monomials $\mathbf{x}^\gamma$ with $\gamma \in \text{relint } C$. Thus we conclude from this resolution that $\text{Tor}_i^A(R, k)_\mu$ and

$\text{Tor}_{h-i}^A(I, k)_{\alpha-\mu}$ are dual to each other as k -vector spaces for all i :

$$\text{Tor}_i^A(R, k)_\mu^* \cong \text{Tor}_{h-i}^A(I, k)_{\alpha-\mu}. \quad (3)$$

Squarefree Divisor Complexes

Next we identify these Tor modules with the homology of certain simplicial complexes. Let Λ be an arbitrary affine semigroup with a set of generators $\alpha_1, \dots, \alpha_n$, and let M be a finitely generated semigroup module over $k[\Lambda]$. Now for any $\mu \in \Lambda$, define an abstract simplicial complex K_μ^M on vertex set $S := \{\alpha_1, \dots, \alpha_n\}$ as follows:

$$K_\mu^M := \left\{ F \subseteq S : \frac{\mathbf{x}^\mu}{\prod_{\gamma \in F} \mathbf{x}^\gamma} \in M \right\}.$$

This complex is called the *squarefree divisor complex* of M (see [3]). The crucial fact is the next proposition which has been frequently rediscovered. See Proposition 3.1 of [7] for a proof of it.

For a simplicial complex K , we use $\tilde{H}_i(K; k)$ to denote the i th reduced homology group with coefficients k . Note that if $K = \{\emptyset\}$, then

$$\tilde{H}_i(K; k) \cong \begin{cases} k, & \text{if } i = -1, \\ 0, & \text{otherwise.} \end{cases}$$

If on the other hand $K = \emptyset$, then $\tilde{H}_i(X; k) = 0$ for all i .

Proposition 2.3. *We have $\text{Tor}_i^A(M, k)_\mu \cong \tilde{H}_{i-1}(K_\mu^M; k)$ for $\mathbf{x}^\mu \in M$, and all other graded pieces for $\mathbf{x}^\mu \notin M$ vanish.*

The fact that R is Cohen–Macaulay is equivalent to saying that its depth is equal to its dimension, i.e. $h = n - d$. Thus in view of the above proposition, isomorphism (3) becomes

$$\tilde{H}_{i-1}(K_\mu^R; k)^* \cong \tilde{H}_{n-d-i-1}(K_{\alpha-\mu}^I; k). \quad (4)$$

This suggests that there is some dual relation between the two complexes K_μ^R and $K_{\alpha-\mu}^I$. To make it more precise, we introduce the following well-known lemma. We refer the reader to Proposition 10.4 of [2] for a proof of it.

Lemma 2.4 (Combinatorial Alexander-Duality). *Let Δ be a finite simplicial complex on vertex set V and define the dual complex $\Delta^\vee := \{F \subseteq V : V \setminus F \notin \Delta\}$. Then*

$$\tilde{H}_i(\Delta; \mathbb{Z}) \cong \tilde{H}^{|V|-i-3}(\Delta^\vee; \mathbb{Z}).$$

In particular, when the coefficient is a field k we have

$$\tilde{H}_i(\Delta; k) \cong \tilde{H}^{|V|-i-3}(\Delta^\vee; k) \cong \tilde{H}_{|V|-i-3}(\Delta^\vee; k)^*.$$

It follows from (4) and the above lemma that

$$\tilde{H}_{i-1}(K_\mu^R; k)^* \cong \tilde{H}_{d+i-2}((K_{\alpha-\mu}^I)^\vee; k).$$

This suggests the following conjecture of Reiner. We prove it in Theorem 3.9.

Conjecture 2.5. *Let $\alpha = \sum_{i=1}^n \alpha_i$, then for any $\mu \in C$ we have*

$$(K_{\alpha-\mu}^I)^\vee \simeq \text{Susp}^{d-1}(K_\mu^R).$$

3. Proof of the Conjecture

We prove Theorem 1.1 in this section, from which Conjecture 2.5 follows. We need the following two lemmas (see Section 10 of [2]).

Lemma 3.1 (Gluing Lemma). *Let X and Y be two regular cell complexes. If X and $X \cap Y$ are contractible, then $X \cup Y \simeq Y$.*

For a poset (partially ordered set) $\mathcal{P}$, the *order complex* of $\mathcal{P}$ is the simplicial complex on vertex set $\mathcal{P}$, whose faces are the chains of the form $x_0 < x_1 < \cdots < x_i$ in $\mathcal{P}$.

Lemma 3.2 (Nerve Lemma). *Let X be a regular cell complex and let $\{X_1, \dots, X_n\}$ be a family of closed subcomplexes such that $X = \bigcup_{i=1}^n X_i$. Let $\mathcal{P}$ be the poset of all nonempty intersections of X_i 's ordered by inclusion. If all the nonempty intersections of X_i 's are contractible, then X is homotopy equivalent to the order complex of $\mathcal{P}$.*

Let P , μ and $S = \{\alpha_1, \dots, \alpha_n\}$ be as defined in Section 1, so P is a d -dimensional pointed polyhedral cone in $\mathbb{R}^d$, but not necessarily rational. We say that a point β in P is *far enough* (from the origin), if $\beta \neq 0$ and, for any $i = 1, \dots, m$, either $f_i(\beta) = 0$ or else $f_i(\beta) > f_i(\mu)$. Clearly for any point $\mathbf{x} \in P$, $\mathbf{x} \neq 0$, we can find a positive number a such that $a\mathbf{x}$ is far enough.

Lemma 3.3. *Let β be a far enough point on an extreme ray of P , then*

$$\Gamma_{S \cup \{\beta\}} \simeq \Gamma_S.$$

Proof. There exists $\alpha_k \in S$ such that α_k and β are on the same extreme ray of P . Note that

$$\begin{aligned} \Gamma_S \cup \text{star}(\beta, \Gamma_{S \cup \{\beta\}}) &= \Gamma_{S \cup \{\beta\}}, \\ \Gamma_S \cap \text{star}(\beta, \Gamma_{S \cup \{\beta\}}) &= \text{link}(\beta, \Gamma_{S \cup \{\beta\}}). \end{aligned}$$

In view of Lemma 3.1, it is enough for us to show that $\text{link}(\beta, \Gamma_{S \cup \{\beta\}})$ is contractible. Let $F \in \text{link}(\beta, \Gamma_{S \cup \{\beta\}})$, so that $\sum_{\gamma \in F} \gamma + \beta \in \mathcal{B}$. Hence there exists j in $[m]$ such that $f_j(\sum_{\gamma \in F} \gamma + \beta) \leq f_j(\mu)$. Thus $f_j(\beta) \leq f_j(\mu)$, and consequently $f_j(\beta) = 0$ since β is a far enough point. It follows that $f_j(\alpha_k) = 0$ since α_k and β are on the same extreme ray

of P . Hence $f_j(\sum_{\gamma \in F \cup \{\alpha_k\}} \gamma) \leq f_j(\mu)$, i.e. $F \cup \{\alpha_k\} \in \Gamma_{S \cup \{\beta\}}$. Therefore $\text{link}(\beta, \Gamma_{S \cup \{\beta\}})$ is contractible since it has a cone vertex $\{\alpha_k\}$. $\square$

We pick one far enough point from each extreme ray of P , and denote this finite set of points by T . Without loss of generality, we assume that all points in T lie on a cross section Q of P by some affine hyperplane. Thus T is exactly the vertex set of the $(d-1)$ -dimensional polytope Q . We may also assume that $S \cap T = \emptyset$.

Corollary 3.4. $\Gamma_{S \cup T} \simeq \Gamma_S$.

Proof. Apply Lemma 3.3 to S and proceed by induction. $\square$

Lemma 3.5. *Let T be as above. Then $\Gamma_T \simeq \mathbb{S}^{d-2}$.*

Proof. Let $T = \{\beta_1, \dots, \beta_i\}$. Then for any subset $F \subseteq T$, $F \in \Gamma_T$ if and only if there exists some $j \in [m]$ such that $f_j(\sum_{\gamma \in F} \gamma) \leq f_j(\mu)$. Consequently $f_j(\gamma) = 0$ for any $\gamma \in F$. In other words, $F \in \Gamma_T$ if and only if all the points in F lie on the same facet of Q . Therefore a facet of Γ_T is just the vertex set of some facet of Q . If we consider the covering of Γ_T by its facets, the intersection poset of this covering is isomorphic to the face lattice of ∂Q . Applying Lemma 3.2 we conclude that $\Gamma_T \simeq \partial Q$. $\square$

Let X and Y be two simplicial complexes on disjoint vertex sets. The *join* of X and Y is the simplicial complex

$$X * Y := \{F \cup G : F \in X, G \in Y\}.$$

Lemma 3.6. *Let S and T be as before. Then $\Delta_S * \Gamma_T \subseteq \Gamma_{S \cup T}$.*

Proof. For any $F \in \Delta_S$ and $G \in \Gamma_T$, we want to show that $F \cup G \in \Gamma_{S \cup T}$. Note $G \in \Gamma_T$ implies there is some $k \in [m]$ such that $f_k(\sum_{\gamma \in G} \gamma) \leq f_k(\mu)$, hence $f_k(\gamma) = 0$ for any $\gamma \in G$. Now $F \in \Delta_S$ implies that $f_j(\sum_{\gamma \in F} \gamma) \leq f_j(\mu)$ for any $j \in [m]$. Therefore $f_k(\sum_{\gamma \in F \cup G} \gamma) \leq f_k(\mu)$. Hence $F \cup G \in \Gamma_{S \cup T}$. $\square$

Let X be any simplicial complex with vertex set $V \uplus W$. Let X_V and X_W be the induced subcomplexes on vertex sets V and W , respectively. For a face $F \in X_V$, define

$$\text{link}(F, X_W) := \{G \in X_W : F \cup G \in X\}.$$

Define $K := \{F \in X_V : \text{link}(F, X_W) = X_W\}$. Clearly, K is a simplicial complex and $K * X_W \subseteq X$.

Lemma 3.7. *Let X be a simplicial complex with vertex set $V \uplus W$, and let $K \subseteq X_V$ be as defined above. If for any face $F \in X_V \setminus K$, we have that $\text{link}(F, X_W)$ is contractible, then $X \simeq K * X_W$.*

Proof. Induct on the number of faces in $X_V \setminus K$. If $X_V \setminus K = \emptyset$, then clearly $X = K * X_W$. Otherwise let F be a facet in $X_V \setminus K$, and $X' := X \setminus \{G \in X: F \subseteq G\}$. It is easy to see that $X'_W = X_W$ and $X'_V = X_V \setminus \{F\}$. By induction hypothesis we get $X' \simeq K * X_W$. Thus to complete the proof we only need to show that $X \simeq X'$. Note that

$$\begin{aligned} X' \cup \text{star}(F, X) &= X, \\ X' \cap \text{star}(F, X) &= \{G: G \subset F\} * \text{link}(F, X). \end{aligned}$$

Since F is a facet of X_V we have $\text{link}(F, X) = \text{link}(F, X_W)$, which is contractible. Hence $X' \cap \text{star}(F, X)$ is also contractible. Therefore $X \simeq X'$. $\square$

For a point $\gamma \in \mathbb{R}^d$, we say a facet F of P is *visible* from γ , if for any point $\mathbf{x} \in F$ the line segment between γ and $\mathbf{x}$ intersects P only at $\mathbf{x}$. By abuse of notation, we say that a facet of the cross section Q is visible from a point, if the corresponding facet of P is visible from that point. Those facets of Q that are visible from γ form an initial segment of a shelling of ∂Q , and hence form a contractible subcomplex, unless every facet is visible from γ (see Section 8.2 of [13]).

Lemma 3.8. *Let S and T be as before. Then $\Gamma_{S \cup T} \simeq \Delta_S * \Gamma_T$.*

Proof. For a face $F \in \Gamma_S$, consider $\text{link}(F, \Gamma_T)$. A face $G \in \text{link}(F, \Gamma_T)$ means that there exists $j \in [m]$ such that $f_j(\sum_{\gamma \in F \cup G} \gamma) \leq f_j(\mu)$. This implies $f_j(\gamma) = 0$ for any $\gamma \in G$, and $f_j(\sum_{\gamma \in F} \gamma) \leq f_j(\mu)$. In other words, $G \in \text{link}(F, \Gamma_T)$ if and only if there is a facet of P (defined by f_j) which is visible from the point $(\sum_{\gamma \in F} \gamma - \mu)$, such that G is a subset of the vertex set of the corresponding facet of Q . Now the facets of $\text{link}(F, \Gamma_T)$ are the vertex sets of the facets of Q that are visible from $(\sum_{\gamma \in F} \gamma - \mu)$. Consider the covering of $\text{link}(F, \Gamma_T)$ by its facets. By an application of Lemma 3.2, we conclude that $\text{link}(F, \Gamma_T)$ is homotopy equivalent to the union of the facets of Q that are visible from $(\sum_{i \in \sigma} \alpha_i - \mu)$ since they have isomorphic intersection posets. Hence $\text{link}(\sigma, \Gamma_T)$ is either contractible, or else every facet of Q is visible from $(\sum_{\gamma \in F} \gamma - \mu)$, which means $f_j(\sum_{\gamma \in F} \gamma - \mu) \leq 0$ for any $j \in [m]$, i.e. $F \in \Delta_S$. In summary, we have $\Delta_S = \{F \in \Gamma_S: \text{link}(F, \Gamma_T) = \Gamma_T\}$, and $\text{link}(F, \Gamma_S)$ is contractible for any $F \in \Gamma_S \setminus \Delta_S$. In view of Lemma 3.7, this implies $\Gamma_{S \cup T} \simeq \Delta_S * \Gamma_T$. $\square$

Proof of Theorem 1.1. Combining Corollary 3.4 and Lemmas 3.5 and 3.8, we get

$$\Gamma_S \simeq \Delta_S * \Gamma_T \simeq \Delta_S * \mathbb{S}^{d-2} \cong \text{Susp}^{d-1} \Delta_S. \quad \square$$

As a direct corollary of Theorem 1.1, we prove Conjecture 2.5. Let K_μ^R and $K_{\alpha-\mu}^I$ be the two squarefree divisor complexes as defined in Section 2.

Theorem 3.9. $(K_{\alpha-\mu}^I)^\vee \simeq \text{Susp}^{d-1}(K_\mu^R)$.

Proof. As a special case of Theorem 1.1, we may take P to be the rational cone, and $\mu, \alpha_1, \dots, \alpha_n$ to be the lattice points in $C = P \cap L$ as defined in Section 2. Recall that

$\alpha = \sum_{i=1}^n \alpha_i$. We have

$$K_{\alpha-\mu}^I = \left\{ F \subseteq S: \frac{\mathbf{x}^{\alpha-\mu}}{\prod_{\gamma \in F} \mathbf{x}^\gamma} \in I \right\}.$$

Thus

$$\begin{aligned} (K_{\alpha-\mu}^I)^\vee &= \left\{ F \subseteq S: \frac{\mathbf{x}^{\alpha-\mu}}{\prod_{\gamma \in S \setminus F} \mathbf{x}^\gamma} \notin I \right\} \\ &= \left\{ F \subseteq S: \sum_{\gamma \in F} \gamma - \mu \notin \text{relint } C \right\} \\ &= \left\{ F \subseteq S: f_j \left(\sum_{\gamma \in F} \gamma \right) \leq f_j(\mu) \text{ for some } j \in [m] \right\} \\ &= \Gamma_S \end{aligned}$$

and

$$\begin{aligned} K_\mu^R &= \left\{ F \subseteq S: \frac{\mathbf{x}^\mu}{\prod_{\gamma \in F} \mathbf{x}^\gamma} \in R \right\} \\ &= \left\{ F \subseteq S: \mu - \sum_{\gamma \in F} \gamma \in C \right\} \\ &= \left\{ F \subseteq S: f_j \left(\sum_{\gamma \in F} \gamma \right) \leq f_j(\mu), \forall j \in [m] \right\} \\ &= \Delta_S. \end{aligned}$$

Now apply Theorem 1.1. □

Note that (4) in Section 2 is a direct corollary of the above theorem, and hence independent of Theorem 2.1:

Corollary 3.10. *We have $\tilde{H}_{i-1}(K_\mu^R; k)^* \cong \tilde{H}_{n-d-i-1}(K_{\alpha-\mu}^I; k)$ for any field k .*

4. Remarks and Further Applications

Instead of taking the free resolution of a normal semigroup ring R as an A -module and studying the Tor modules, one can study the *local cohomology* of R with respect to its irrelevant maximal ideal. This was the approach taken by Stanley [10]. In that paper Stanley identified the graded pieces of the local cohomology of R with the homology of certain simplicial complexes. He then showed that on the homology level these simplicial complexes may be replaced by certain polyhedral complexes that are relatively easier to

work with (see also [6]). As applications he proved various results including Theorem 2.1 and a general reciprocity theorem. Our intention here is to show some similar applications of Theorem 3.9 (or the more general Theorem 1.1). The difference is that while Stanley's proofs depend on the topology of his polyhedral complexes being trivial under certain conditions, our proofs rely on the relation between Δ_S and Γ_S .

It is worthwhile mentioning that Stanley made a conjecture [10, Conjecture 2.9] which is very similar to Conjecture 2.5 in spirit. Stanley's conjecture was proved by the author in [5]. However, the author has not been able to find any direct relations between these two conjectures as well as their proofs.

We start with the Cohen–Macaulayness of normal semigroup rings and some semigroup modules over them. In view of Remark 2.2, we restrict our attention to positive normal semigroup rings. Let P be a d -dimensional rational cone in $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^N$, let $\alpha_1, \dots, \alpha_n$ be lattice points in $P \cap L$ chosen as in Section 2, and let $\beta \in L \otimes_{\mathbb{Z}} \mathbb{R}$ be an arbitrary point. We use M to denote the semigroup module over R generated by those monomials $\mathbf{x}^\gamma$ with $\gamma \in L \cap (\beta + P)$, where $\beta + P := \{\beta + \mathbf{x} : \mathbf{x} \in P\}$.

Theorem 4.1. *M is finitely generated as an R -module, and it is Cohen–Macaulay.*

Proof. Without loss of generality, we assume that P is full-dimensional, i.e. $d = N$. Let $\{f_i(\mathbf{x}) = 0\}_{i=1}^m$ be the defining hyperplanes of P as before, and let $\alpha = \sum_{i=1}^n \alpha_i$. Then M as an R -module is generated by

$$\{\mathbf{x}^\gamma : \gamma \in L \cap (\beta + P), 0 \leq f_i(\gamma - \beta) \leq f_i(\alpha), \forall i \in [m]\},$$

which is a finite set.

Note that $\dim_A M = d$. To prove that M is Cohen–Macaulay, it is enough to show that $\text{depth}_A M = d$. We take a minimal L -graded free resolution of M as an A -module,

$$0 \rightarrow A^{\beta_h} \rightarrow \dots \rightarrow A^{\beta_0} \rightarrow M \rightarrow 0,$$

where $\beta_h = n - \text{depth}_A M \neq 0$. Thus we only need to show that $\text{Tor}_i^A(M; k)_\mu = 0$ for $i > n - d$ and any $\mu \in L$. By Proposition 2.3 $\text{Tor}_i^A(M; k)_\mu \cong \tilde{H}_{i-1}(K_\mu^M; k)$, where K_μ^M is the squarefree divisor complex. We have

$$\begin{aligned} K_\mu^M &= \left\{ F \subseteq S : \frac{\mathbf{x}^\mu}{\prod_{\gamma \in F} \mathbf{x}^\gamma} \in M \right\} \\ &= \left\{ F \subseteq S : \mu - \sum_{\gamma \in F} \gamma \in L \cap (\beta + P) \right\} \\ &= \left\{ F \subseteq S : f_j \left(\sum_{\gamma \in F} \gamma \right) \leq f_j(\mu - \beta), \forall j \in [m] \right\} \\ &= \Delta_{S, \mu - \beta}. \end{aligned}$$

It follows from Theorem 1.1 that $\text{Susp}^{d-1} \Delta_{S, \mu - \beta} = \Gamma_{S, \mu - \beta}$. Now since $\Gamma_{S, \mu - \beta}$ is a simplicial complex on n vertices, we either have $\dim \Gamma_{S, \mu - \beta} \leq n - 2$ or $\Gamma_{S, \mu - \beta}$ is a

simplex. In either case we have

$$\tilde{H}_{i-1}(\Delta_{S, \mu-\beta}; k) = \tilde{H}_{i-1+d-1}(\Gamma_{S, \mu-\beta}; k) = 0$$

for $i - 1 + d - 1 > n - 2$, i.e. $i > n - d$. $\square$

If we take β to be the zero vector, we recover Hochster's result, Theorem 2.1(a). Our theorem is also somewhat more general than Theorem 3.5 of [9] and Theorem 3.2 of [10], as we now explain. Let Φ be an $r \times N$ matrix of integers, and let $\alpha \in \mathbb{Z}^r$. Define

$$E := \{\gamma \in \mathbb{N}^N : \Phi\gamma = 0\},$$

$$E^\alpha := \{\gamma \in \mathbb{N}^N : \Phi\gamma = \alpha\}.$$

Then $k[E^\alpha]$ is a finitely generated semigroup module over the normal semigroup ring $k[E]$.

Corollary 4.2 [9], [10]. *If the system of linear equations $\Phi\mathbf{x} = \alpha$ has a rational solution $\beta = (\beta_1, \dots, \beta_N)$ with the following two properties:*

- (1) $\beta_i \leq 0$ for $1 \leq i \leq N$,
- (2) if $(b_1, \dots, b_N)$ is an integer solution to $\Phi\mathbf{x} = \alpha$ such that $b_i \geq \beta_i$ for all i , then $b_i \geq 0$ for all i ,

then $k[E^\alpha]$ is Cohen–Macaulay.

Proof. Let P denote the rational polyhedral cone

$$P := \{\mathbf{x} \in \mathbb{R}^N : \Phi\mathbf{x} = 0, x_i \geq 0, \forall i\}.$$

Then $E = \mathbb{Z}^N \cap P$, and the assumptions on E^α in our corollary is equivalent to saying that $E^\alpha = \mathbb{Z}^N \cap (\beta + P)$. $\square$

Let A be a $\mathbb{Z}^N$ -graded k -algebra and let M be a finitely generated $\mathbb{Z}^N$ -graded module over A . Recall that the *Hilbert function* of M is $H(M, \gamma) := \dim_k(M_\gamma) < \infty$, for any $\gamma \in \mathbb{Z}^N$. The *Hilbert series* of M is $F(M, \lambda) := \sum_{\gamma \in \mathbb{Z}^N} H(M, \gamma) \lambda^\gamma$. Here $\lambda = (\lambda_1, \dots, \lambda_N)$, $\gamma = (\gamma_1, \dots, \gamma_N)$ and $\lambda^\gamma = \lambda_1^{\gamma_1} \cdots \lambda_N^{\gamma_N}$. Now let A be the polynomial ring graded as in Section 2. Thus $A = k[\mathbf{x}] = k[x_1, \dots, x_n]$ with α_i being the degree of x_i . Then we have

$$F(A, \lambda) = \frac{1}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})}.$$

Let M be any finitely generated $\mathbb{Z}^N$ -graded module over A . Then by looking at a minimal resolution of M that is analogous to (1), one can conclude that

$$F(M, \lambda) = \frac{1}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})} \sum_{j=0}^h (-1)^j \sum_{\mu \in \mathbb{Z}^N} \dim_k \tilde{H}_{j-1}(K_\mu^M; k) \lambda^\mu. \quad (5)$$

The following lemma was proved in [12, Theorem 4.6.14] by combinatorial methods. A ring-theoretic argument can also be given using Theorem 2.1(b). As an alternative, here we use Corollary 3.10 to prove the lemma.

Lemma 4.3. $F(R, \lambda) = (-1)^d F(I, \lambda^{-1})$, where $\lambda^{-1} := (\lambda_1^{-1}, \dots, \lambda_N^{-1})$.

Proof.

$$\begin{aligned}
F(R, \lambda) &= \frac{1}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})} \sum_{j=0}^{n-d} (-1)^j \sum_{\mu \in \mathbb{Z}^N} \dim_k \tilde{H}_{j-1}(K_\mu^R; k) \lambda^\mu \\
&= \frac{1}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})} \sum_{j=0}^{n-d} (-1)^j \sum_{\mu \in \mathbb{Z}^N} \dim_k \tilde{H}_{n-d-j-1}(K_{\alpha-\mu}^I; k) \lambda^\mu \\
&= \frac{(-1)^{d-n} \lambda^\alpha}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})} \sum_{j=0}^{n-d} (-1)^{n-d-j} \sum_{\mu \in \mathbb{Z}^N} \dim_k \tilde{H}_{n-d-j-1}(K_{\alpha-\mu}^I; k) \lambda^{\mu-\alpha} \\
&= \frac{(-1)^d}{\prod_{i=1}^n (1 - \lambda^{-\alpha_i})} \sum_{l=0}^{n-d} (-1)^l \sum_{\mu \in \mathbb{Z}^N} \dim_k \tilde{H}_{l-1}(K_{\alpha-\mu}^I; k) \lambda^{\mu-\alpha} \\
&= (-1)^d F(I, \lambda^{-1}),
\end{aligned}$$

where the first and last equalities use (5). $\square$

An elegant and useful application of Lemma 4.3 is to prove Ehrhart's reciprocity law for rational polytopes. For a statement of it and how it follows from Lemma 4.3, we refer the reader to Section 4.6 of [12] and Theorem 6.3.11 of [4].

It remains to show that Theorem 2.1(b) can also be derived from Corollary 3.10.

Proof of Theorem 2.1(b). The idea is essentially that of Stanley [10, Theorem 4.4]. We first invoke the result that since R is a normal domain, $\Omega(R)$ is isomorphic to an ideal of R (see also Corollary 3.3.19 of [4]). Next we notice that combining the minimal resolution (2) and Corollary 3.10 one gets

$$\begin{aligned}
F(\Omega(R), \lambda) &= \frac{1}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})} \sum_{j=0}^{n-d} (-1)^{h-j} \sum_{\mu \in \mathbb{Z}^N} \beta_{i\mu} \lambda^{\alpha-\mu} \\
&= \frac{1}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})} \sum_{j=0}^{n-d} (-1)^{n-d-j} \sum_{\mu \in \mathbb{Z}^N} \dim_k \tilde{H}_{j-1}(K_\mu^R; k) \lambda^{\alpha-\mu} \\
&= \frac{1}{\prod_{i=1}^n (1 - \lambda^{\alpha_i})} \sum_{j=0}^{n-d} (-1)^{n-d-j} \sum_{\mu \in \mathbb{Z}^N} \dim_k \tilde{H}_{n-d-j-1}(K_{\alpha-\mu}^I; k) \lambda^{\alpha-\mu} \\
&= F(I, \lambda).
\end{aligned}$$

Equivalently $\dim_k \Omega(R)_\gamma = \dim_k I_\gamma$ for any $\gamma \in \mathbb{Z}^N$. The fact that $\dim_k R_\gamma \leq 1$ for any $\gamma \in \mathbb{Z}^N$, then implies $\Omega(R) \cong I$. $\square$

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