A Tight Bound on the Number of Geometric Permutations of Convex Fat Objects in $\mathbb{R}^{d*}$

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Abstract. We show that the maximum number of geometric permutations of a set of n pairwise-disjoint convex and fat objects in $\mathbb{R}^d$ is $O(n^{d-1})$. This generalizes the bound of $\Theta(n^{d-1})$ obtained by Smorodinsky et al. [5] on the number of geometric permutations of n pairwise-disjoint balls.

1. Introduction

Let $\mathcal{A}$ be a family of n pairwise-disjoint convex sets in $\mathbb{R}^d$. An oriented line ℓ is said to be a *transversal* of $\mathcal{A}$ if it intersects all the convex sets in $\mathcal{A}$. Any transversal ℓ of $\mathcal{A}$ induces an ordering of $\mathcal{A}$, which is the order in which ℓ meets the convex sets in $\mathcal{A}$. We call such an ordering a *geometric permutation* of $\mathcal{A}$. Let $g_d(\mathcal{A})$ denote the number of distinct geometric permutations of $\mathcal{A}$, and let $g_d(n)$ denote the maximum of $g_d(\mathcal{A})$ over all families $\mathcal{A}$ of pairwise-disjoint convex sets. The study of geometric permutations was initiated by Katchalski et al. [3].

The following bounds are known for $g_d(n)$. For a more detailed summary of the literature, the reader is referred to [5].

1. $g_2(n) = 2n - 2$ [1].

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2. $g_d(n) = \Omega(n^{d-1})$ [4].
3. $g_d(n) = O(n^{2d-2})$ [6].

Thus for $d > 2$, there is a large gap between the upper and lower bounds, and bridging the gap seems challenging. Smorodinsky et al. [5] considered the case where $\mathcal{A}$ consists of a set of pairwise-disjoint balls. They showed that the number of geometric permutations in this case is $O(n^{d-1})$. They also showed that this bound is tight by presenting instances where the number of geometric permutations is $\Omega(n^{d-1})$. In this paper we generalize the upper bound of Smorodinsky et al. [5] to the case of fat objects.

We say that a convex set (i.e., object) O in $\mathbb{R}^d$ is γ -fat, for some $\gamma \geq 1$, if the ratio between the radius of the smallest ball containing O and the radius of the largest ball contained in O is at most γ . Let $\mathcal{O} = \{O_1, \dots, O_n\}$ be a set of n disjoint convex γ -fat objects in $\mathbb{R}^d$. We show that $g_d(\mathcal{O}) = O(n^{d-1})$. The constant depends on d and γ . This bound is clearly tight because of the $\Omega(n^{d-1})$ lower bound for the case of balls.

2. Preliminaries

Like many of the earlier results (see [5] and [6]), our proof uses the notion of a *separating set* of hyperplanes. Given a set $\mathcal{A}$ of pairwise-disjoint convex sets in $\mathbb{R}^d$, a set $\mathcal{H}$ of hyperplanes is said to be a *separating set* for $\mathcal{A}$ if for each pair of convex sets $A_i, A_j \in \mathcal{A}$, there is a hyperplane $h \in \mathcal{H}$ that *separates* them, that is, A_i and A_j lie in distinct open half-spaces bounded by h .

The connection between the size of a separating set for $\mathcal{A}$ and the number of geometric permutations of $\mathcal{A}$ is the following. See [5] and [6] for a proof.

Lemma 1. *Let $\mathcal{A}$ be a family of pairwise-disjoint convex sets in $\mathbb{R}^d$ and let $\mathcal{H}$ be a separating set for $\mathcal{A}$. Then $g_d(\mathcal{A})$, the number of geometric permutations of $\mathcal{A}$, is $O(|\mathcal{H}|^{d-1})$.*

For a set $\mathcal{O}$ of n disjoint convex γ -fat objects in $\mathbb{R}^d$, we construct a separating set of size $O(n)$. By Lemma 1, it then follows that $g_d(\mathcal{O})$ is $O(n^{d-1})$.

3. A Linear-Sized Separation Set

Let $\mathcal{O} = \{O_1, \dots, O_n\}$ be a set of n disjoint convex γ -fat objects in $\mathbb{R}^d$, for some constant $\gamma \geq 1$. We denote by B_i the largest ball contained in O_i . Let p_i and r_i be the center point and radius of B_i , respectively. We refer to p_i as the center point (or simply center) of O_i as well. Assume that the objects are ordered so that $r_1 > r_2 > \dots > r_n$. We denote by $\bar{B}_i$ the smallest ball containing O_i , and let $\bar{r}_i$ be the radius of $\bar{B}_i$. We have $\bar{r}_i/r_i \leq \gamma$.

Let $\tilde{B}_i$ be the ball of radius $\tilde{r}_i = 2\bar{r}_i$ centered at p_i . We have $B_i \subseteq O_i \subseteq \bar{B}_i \subseteq \tilde{B}_i$ (see Fig. 1). Put $\alpha = 2\gamma$; note that $\tilde{r}_i/r_i \leq \alpha$. For any $x > 0$, we denote by $\tilde{B}_i^x$ the ball of radius $x \cdot \tilde{r}_i$ centered at p_i . Let $\hat{B}_i$ denote the ball $\tilde{B}_i^{10\alpha}$. Finally, let $d(p, q)$ denote the Euclidean distance between two points p and q .

Fig. 1. The balls defined for O_i .

Claim 2. *The number of objects O_j with $j < i$ whose center p_j lies within the ball $\hat{B}_i = \tilde{B}_i^{10\alpha}$ is bounded by some constant $c = c(d, \alpha)$.*

□

3.1. The Construction

We construct a set $\mathcal{H}$ of $O(n)$ hyperplanes as follows. For each $O_i \in \mathcal{O}$ we perform the following two steps:

1. For each O_j with $j < i$ whose center p_j lies in $\hat{B}_i$, let $h_{i,j}$ be any hyperplane that separates O_i and O_j . We add $h_{i,j}$ to $\mathcal{H}$.
2. Let $\mathcal{C} = \{c_1(p_i), \dots, c_K(p_i)\}$ be a set of K infinite cones, for an appropriate constant K , such that (i) the apex of each $c_k(p_i) \in \mathcal{C}$ is p_i and its associated angle is θ , where $\theta = \sin^{-1}(1/4\alpha)$, and (ii) the unit sphere centered at p_i is covered by the union of the caps of the unit sphere defined by the cones in $\mathcal{C}$.
 - (a) For each cone $c_k(p_i) \in \mathcal{C}$, let ρ_k be any ray emanating from p_i and contained in $c_k(p_i)$, and let h_k be the hyperplane supporting $\hat{B}_i^\alpha$ at the point where ρ_k emerges out of $\hat{B}_i^\alpha$. We add h_k to our set $\mathcal{H}$, see Fig. 2.
 - (b) For each cone $c_k(p_i) \in \mathcal{C}$, if there exists in $c_k(p_i)$ a center point $p_{j'}$ (of a ball $B_{j'}$) with $j' < i$, such that $p_{j'} \notin \hat{B}_i$ but $O_{j'}$ intersects h_k , then let p_j be the closest to p_i among all such center points. Let $h_{i,j}$ be any hyperplane that separates O_i and O_j . We add $h_{i,j}$ to $\mathcal{H}$.

The total number of hyperplanes that were added during the construction process is only $O(n)$. This is true since for each object O_i , the number of hyperplanes that were added for O_i is bounded by some constant; the number added in step 1 is a constant by

Fig. 2. The construction.

Claim 2, and the number added in step 2 is a constant since the number of cones K is a constant.

3.2. The Proof

In this section we prove that the set $\mathcal{H}$ constructed in the previous section is a separating set for $\mathcal{O}$.

For each object O_i , we prove that O_i is separated from each O_j with $j < i$. The proof is by induction. For O_1 this is trivially true. Assume this is true for each $O_{i'}$, $i' \leq i$. We shall prove that this is also true for $O = O_{i+1}$. Let $O' = O_j$ with $j < i + 1$. We need to show that there exists a hyperplane in $\mathcal{H}$ that separates O and O' .

If p' , the center of O' , lies in $\hat{B} = \hat{B}_{i+1}$, then a hyperplane separating O and O' was added to $\mathcal{H}$ in step 1 of the construction for O . Assume therefore that p' does not lie in $\hat{B}$. Let $c_k(p) \in C$ be a cone such that $p' \in c_k(p)$ (where $p = p_{i+1}$).

It is easy to see that the hyperplane h_k added in step 2(a) of the construction for O separates $\tilde{B}^\alpha$ from the portion of the cone $c_k(p)$ that lies outside $\hat{B}$; see Fig. 2. It follows that h_k separates $\tilde{B}^\alpha$ from p' , and, therefore, if O' does not intersect h_k , then h_k separates $\tilde{B}^\alpha$ and O' (hence O and O'). Assume therefore that O' does intersect h_k .

Let $O'' = O_m$, with center p'' , denote the object that was picked for $c_k(p)$ in the second phase of the construction. That is, among all objects preceding O that intersect h_k and whose centers lie in $c_k(p)$ (and outside $\hat{B}$), O'' is the one whose center p'' is closest to p . If $O' = O''$, then we are done since our construction for O explicitly adds (in step 2(b)) a hyperplane separating O and O'' . So assume that O'' is different from O' . We have that p'' is closer to p than p' .

By the induction hypothesis there exists a hyperplane $h \in \mathcal{H}$ that separates O' and O'' . We show that h also separates O' and O .

Recall that $O = O_{i+1}$ and $O'' = O_m$. Let $\tilde{r}$, $\tilde{B}$, and $\tilde{B}^\alpha$ denote $\tilde{r}_{i+1}$, $\tilde{B}_{i+1}$, and $\tilde{B}_{i+1}^\alpha$, respectively. Let r'' , $\tilde{r}''$, B'' , and $\tilde{B}''$ denote r_m , $\tilde{r}_m$, B_m , and $\tilde{B}_m$, respectively.

Fig. 3. The object O'' .

Claim 3. $d(p, p'') \leq 2\alpha\tilde{r} + 2\tilde{r}''$.

Proof. Follows because both $\tilde{B}''$ and $\tilde{B}''$ intersect h_k and the angle between the normal to h_k and the segment pp'' is at most θ , and $\cos \theta \geq \frac{1}{2}$. See Fig. 3. $\square$

Claim 4. $r'' > 4\tilde{r}$.

Proof. If $r'' \leq 4\tilde{r}$, then $d(p, p'') \leq 2\alpha\tilde{r} + 2\tilde{r}'' \leq 2\alpha\tilde{r} + 2\alpha r'' \leq 2\alpha\tilde{r} + 8\alpha\tilde{r} = 10\alpha\tilde{r}$. So p'' lies in $\tilde{B}$ which contradicts our assumption. $\square$

Using the above claim, we can bound the distance $d(p, p'')$. $d(p, p'') \leq 2\alpha\tilde{r} + 2\tilde{r}'' < 2\alpha(r''/4) + 2\alpha r'' = \frac{5}{2}\alpha r'' < 3\alpha r''$.

Consider the line ℓ through p and p' , and let q be the point on ℓ closest to p'' (see Fig. 4). Then $d(p'', q) \leq d(p'', p) \sin \theta \leq 3r''/4$. Thus there is a ball $B(q, r''/4)$ centered at q with radius $r''/4$ such that $B(q, r''/4) \subseteq B''$. Also, observe that q lies between p and p' on the line ℓ . This follows because $d(p, q) \leq d(p, p'')$ whereas $d(p, p') \geq d(p, p'')$.

Fig. 4. $\tilde{B}$ does not intersect h' .

Since the hyperplane h' separates O'' and O' , it also separates $B(q, r''/4)$ and p' . Thus h' intersects the segment qp' , and the distance from q to h' is at least $r''/4$.

As we move on line ℓ away from its intersection point with hyperplane h' , it is easy to see that the distance to h' monotonically increases. Hence $d(p, h') > d(q, h') \geq r''/4 > \tilde{r}$. Thus $\tilde{B}$ does not intersect h' , and so O does not intersect h' .

We also have that O' does not intersect h' since h' was assumed to separate O' and O'' . Since h' intersects the segment qp' , it also intersects the segment pp' , which means that p and p' lie on opposite sides of h' . However, this implies that O and O' lie on opposite sides of h' , completing the proof.

Theorem 5. *The set $\mathcal{H}$ is a separating set for $\mathcal{O}$.*

Theorem 6. *Let $\mathcal{O}$ be a set of n pairwise-disjoint convex γ -fat objects in $\mathbb{R}^d$. The number of geometric permutations of $\mathcal{O}$ is $O(n^{d-1})$; the constant of proportionality depends on d and γ .*

4. Conclusions

Recent results [2], [7] show that the number of geometric permutations of n disjoint *unit* balls in $\mathbb{R}^d$ is a constant independent of n . (Previously, this result was known for $d = 2$.) This result also generalizes to the case where the ratio of the radius of the largest ball to the radius of the smallest ball is bounded. It would be interesting to see if these results generalize to the case of fat objects.

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