Lower Bounds on the Transversal Numbers of d -Intervals*

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Abstract. Let $\ell_1, \ell_2, \dots, \ell_d$ be disjoint parallel lines in the plane. A d -interval is a subset of their union that intersects each ℓ_i in a closed interval. Kaiser [4] showed that any system of d -intervals containing no subsystem of $k + 1$ pairwise disjoint d -intervals can be pierced by at most $(d^2 - d)k$ points. We show that this bound is close to being optimal, by proving a lower bound of $\text{const}(d^2/\log^2 d)k$. The construction involves an extension of a construction due to Sgall [8] of certain systems of set pairs.

1. Introduction

Let $\ell_1, \ell_2, \dots, \ell_d$ be disjoint parallel lines in the plane. A set $J \subset \bigcup_{j=1}^d \ell_j$ is a d -interval if it intersects each ℓ_i in a closed interval. A *homogeneous d -interval* is the union of at most d closed intervals on the real line.

Let $\mathcal{S}$ be a system of subsets of a ground set X . The *matching number* $\nu(\mathcal{S})$ is the maximum cardinality of a subsystem $\mathcal{M} \subseteq \mathcal{S}$ consisting of pairwise disjoint sets. The *transversal number* $\tau(\mathcal{S})$ is the smallest possible cardinality of a transversal of $\mathcal{S}$, i.e., of a subset $T \subseteq X$ with $S \cap T \neq \emptyset$ for all $S \in \mathcal{S}$. For d -intervals, the matching number and transversal number are taken in this set-theoretical sense.

Gyárfás and Lehel [2] (also see [3]) initiated the investigation of the following question. Let $\mathcal{F}$ be a family of d -intervals with $\nu(\mathcal{F}) \leq k$. What is the maximum possible value of $\tau(\mathcal{F})$, in terms of k and d ? (Note that for $d = 1$, we have $\nu(\mathcal{F}) = \tau(\mathcal{F})$ for any family of 1-intervals, by a well-known property of systems of intervals.) They proved that there is a finite bound for all k and d ; their estimate was rather large, about $k^{d!}$ for a fixed d . Károlyi and Tardos [6] obtained a bound of $O(k^d)$ for every fixed d . A major breakthrough was achieved by Tardos [9], who proved $\tau(\mathcal{F}) \leq 2\nu(\mathcal{F})$ for any family

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of 2-intervals (which is tight in the worst case) by an ingenious topological method. By a somewhat different topological approach, Kaiser [4] proved that for any system of d intervals with $v(\mathcal{F}) \leq k$, we have $\tau(\mathcal{F}) \leq (d^2 - d)k$. Later Alon [1] found a short nontopological proof of the slightly weaker bound $\tau(\mathcal{F}) \leq 2d^2k$.

Homogeneous d -intervals are strictly more general than d -intervals, and the upper bounds obtained for them are slightly worse: Kaiser [4] showed $\tau(\mathcal{F}) \leq 3k$ for any family of homogeneous 2-intervals with $v(\mathcal{F}) \leq k$, which is the best possible, and $\tau(\mathcal{F}) \leq (d^2 - d + 1)k$ for an arbitrary family of homogeneous d -intervals with $v(\mathcal{F}) \leq k$, for any $d \geq 3$. Alon's upper bound applies for homogeneous d -intervals as well.

The best lower bound up until now was given by Kaiser [5], who constructed an example with $v = k$ and $\tau \geq 2(d - 1)k$, for all d and k , improving a bound of dk of Károlyi and Tardos [6]. Here we give lower bounds which are near-quadratic in d and almost match Kaiser's upper bounds.

Theorem 1.1. *There is a positive constant $c > 0$ such that:*

- (i) *For any $d \geq 1$ and $k \geq 1$, there is a family $\mathcal{F}$ of homogeneous d -intervals with $v(\mathcal{F}) = k$ and*

$$\tau(\mathcal{F}) \geq c \frac{d^2}{\log d} k.$$

- (ii) *For any $d \geq 1$ and $k \geq 1$, there is a family $\mathcal{F}$ of d -intervals with $v(\mathcal{F}) = k$ and*

$$\tau(\mathcal{F}) \geq c \frac{d^2}{(\log d)^2} k.$$

Part (i) follows in a simple way from an ingenious construction due to Sgall [8] of certain systems of set pairs. For part (ii), we need to modify Sgall's construction to yield d -partite set systems. The families of (homogeneous) d -intervals constructed in the proof of Theorem 1.1 are of a fairly restricted type: each d -interval consists of points and of unit-length intervals, and any two unit-length intervals appearing as a component of a d -interval in the construction are either disjoint or identical.

The most interesting related open problem, in our opinion, is to estimate the maximum possible transversal number of a system of (homogeneous) d -intervals in which every three d -intervals have a common point, as a function of d . Is it (almost) linear, near-quadratic, or somewhere in between (perhaps like $d^{3/2}$)?

2. Sgall's Construction and Homogeneous d -Intervals

Wigderson raised a problem in 1985 which can be reformulated as follows. Suppose that $A_1, A_2, \dots, A_N$ and $B_1, B_2, \dots, B_N$ are sets with the following properties:

- (C1) for all $i = 1, 2, \dots, N$, $\emptyset \neq B_i \subseteq A_i$ and $|A_i| \leq a$, and
 (C2) for all i_1, i_2 , $1 \leq i_1 < i_2 \leq N$, we have $A_{i_1} \cap B_{i_2} \neq \emptyset$ or $A_{i_2} \cap B_{i_1} \neq \emptyset$.

Let $\mathcal{B} = \{B_i : i = 1, 2, \dots, N\}$. What is the maximum possible value of $\tau(\mathcal{B})$, in terms of a ?

It is not difficult to see that $\tau(\mathcal{B}) \leq a(a+1)/2$: Put all points of A_1 into the transversal, remove them from all the A_i and B_i , and discard all i with $B_i \cap A_1 \neq \emptyset$; for the remaining i , we now have $|A_i| \leq a-1$ and we can use induction on a . A clever construction by Sgall [8] shows that this simple bound cannot be improved much and that $\tau(\mathcal{B})$ can be as large as $\text{const} \cdot a^2/\log a$. (The title of Sgall's paper refers to a different, but essentially equivalent, formulation of the problem dealing with labeled tournaments.) For completeness and for the reader's delight, we briefly recall the construction, although we do not need it.

Set $b = \lfloor a/3 \rfloor$ and $n = cb^2/\log b$ for a sufficiently small constant $c > 0$. Let V be an n -element ground set. Let $G = (V, E)$ be a graph of maximum degree b with the following expander-type property: for any two disjoint b -element subsets $A, B \subseteq V$, there is at least one edge $e \in E$ connecting a vertex of A to a vertex of B (the existence of such a graph can be easily shown by the probabilistic method; see [8] for references). Let $\mathcal{B} = \{B_1, B_2, \dots, B_N\}$ be a system of b -element subsets of V with $\tau(\mathcal{B}) = t$ (taking all b -tuples for $\mathcal{B}$, we have $t = n - b + 1$). For each i , let v_i be an (arbitrary) element of B_i , and let $A_i = B_i \cup N(v_i) \cup (V \setminus \bigcup_{u \in B_i} N(u))$, where $N(v)$ denotes the set of neighbors in G of a vertex $v \in V$. It is easy to check that $|A_i| \leq 3b$, and some thought reveals that condition (C2) is satisfied.

Proof of part (i) of Theorem 1.1. Given d and k , we want to construct a system $\mathcal{F}$ of homogeneous d -intervals. Clearly, it suffices to consider the case $k = 1$, since for larger k , we can take k disjoint copies of the $\mathcal{F}$ constructed for $k = 1$. Thus, we want an $\mathcal{F}$ in which every two d -intervals intersect and with $\tau(\mathcal{F})$ large.

Let $A_1, A_2, \dots, A_N$ and $B_1, B_2, \dots, B_N$ be sets satisfying (C1) with $a = d$ and (C2) and such that $\tau(\mathcal{B}) \geq t = cd^2/\log d$ (where, as before, $\mathcal{B} = \{B_1, \dots, B_N\}$). We note that if we remove any at most $t/2$ sets of $\mathcal{B}$, the resulting set system still has a transversal number at least $t/2$.

Let $V = \bigcup_{i=1}^N A_i$ be the ground set. For each $v \in V$, choose a unit interval $I_v \subset \mathbf{R}$, in such a way that $I_v \cap I_{v'} = \emptyset$ for $v \neq v'$. For each $v \in V$ and $i = 1, 2, \dots, N$, choose a point $x_{v,i} \in I_v$, taking distinct points for distinct i . Define a system $\mathcal{J}$ of N homogeneous d -intervals $J_1, J_2, \dots, J_N$ by setting

$$J_i = \left(\bigcup_{v \in B_i} I_v \right) \cup \{x_{u,i} : u \in A_i \setminus B_i\}.$$

Condition (C2) implies that $\mathcal{J}$ is intersecting: if, for example, $u \in A_{i_1} \cap B_{i_2}$, then $x_{u,i_1} \in J_{i_1} \cap J_{i_2}$.

Suppose that T is a transversal of $\mathcal{J}$. At most $|T|$ of the J_i are intersected by T in their point components, and so the set $\{v \in V : T \cap I_v \neq \emptyset\}$ is a transversal for some system of all but at most $|T|$ sets among the B_i . It follows that $2|T| \geq t = cd^2/\log d$. $\square$

Remark. It would be interesting to narrow the gap of $\log d$ between the lower and upper bounds, at least in Wigderson's problem about set pairs. The number of vertices $n = \text{const} \cdot b^2/\log b$ for the expander graph in Sgall's construction is asymptotically the largest possible, as was communicated to me by Noga Alon. This follows by the Kővári–Sós–Turán bound [7] for the maximum number of edges of a $K_{b,b}$ -free graph on n vertices applied to the complement of G .

3. Making the Construction d -Partite

Call a set system $\mathcal{S}$ on a ground set V d -partite if V can be partitioned into disjoint subsets $V_1, V_2, \dots, V_d$ such that $|V_j \cap S| \leq 1$ for all $S \in \mathcal{S}$ and all j . Part (ii) of Theorem 1.1 is proved by constructing set systems $\bar{\mathcal{A}} = \{\bar{A}_1, \bar{A}_2, \dots, \bar{A}_N\}$ and $\bar{\mathcal{B}} = \{\bar{B}_1, \bar{B}_2, \dots, \bar{B}_N\}$ with properties as in Sgall's construction but also d -partite, for a suitable d not much larger than a . We transform the sets from Sgall's construction using the following lemma.

Lemma 3.1 (On m -Partite Representation). *Let V be an n -element set, and let $a \leq n$ be a parameter. Then there are integers $m = O(a \log n)$ and $r = O(\log n)$ such that the following exist:*

- a set W partitioned into disjoint subsets $W_1, W_2, \dots, W_m$,
- for each $v \in V$, an at most $(2r - 1)$ -element set $U_v \subseteq W$ such that $U_v \cap U_{v'} = \emptyset$ for $v \neq v'$,
- for each $A \subseteq V$ with $|A| \leq a$, a set $f(A) \subseteq \bigcup_{v \in A} U_v$, such that $|f(A) \cap W_j| \leq 1$ for all j and such that, for any $v \in A$, $|f(A) \cap U_v| \geq r$.

Proof of part (ii) of Theorem 1.1 (assuming Lemma 3.1). As in part (i), it suffices to consider the case $k = 1$. Set $a = \lfloor c_1 d / \log d \rfloor$ for a sufficiently small constant $c_1 > 0$, let $b = \lfloor a/3 \rfloor$, and let $n = cb^2 / \log b$. Let $B_1, B_2, \dots, B_N$ be all the b -tuples on an n -element ground set V , and let $A_1, A_2, \dots, A_N$ be the at most a -element sets as in Sgall's construction.

We consider $W = W_1 \dot{\cup} W_2 \dot{\cup} \dots \dot{\cup} W_m$, the U_v , and the $f(A)$ as in Lemma 3.1; we may assume $m = d$. Let $\bar{A}_i = f(A_i)$ and $\bar{B}_i = \bar{A}_i \cap (\bigcup_{v \in B_i} U_v) \subseteq \bar{A}_i$. The system $\bar{\mathcal{A}} = \{\bar{A}_i : i = 1, 2, \dots, N\}$ is d -partite. If $v \in A_{i_1} \cap B_{i_2}$, then $|\bar{A}_{i_1} \cap U_v| \geq r$ and $|\bar{B}_{i_2} \cap U_v| \geq r$, and since $|U_v| \leq 2r - 1$, we have $\bar{A}_{i_1} \cap \bar{B}_{i_2} \neq \emptyset$. So (C2) holds for the $\bar{A}_i$ and $\bar{B}_i$. Moreover, the transversal number for any subsystem of $\bar{\mathcal{B}}$ is no smaller than the transversal number for the corresponding subsystem of $\mathcal{B}$.

It remains to construct the system $\mathcal{J} = \{J_1, J_2, \dots, J_N\}$ of d -intervals. This is the same as in the proof of part (i), only this time we choose disjoint unit intervals $I_w \subset \ell_j$ for $w \in W_j$, $j = 1, 2, \dots, d$. The resulting $\tau(\mathcal{J})$ is at least $\text{const} \cdot b^2 / \log b = \text{const} \cdot d^2 / \log^2 d$. Part (ii) of Theorem 1.1 is proved. $\square$

Proof of Lemma 3.1. Choose m and r so that there exists a bipartite graph H (expander) with color classes V and $[m] = \{1, 2, \dots, m\}$ with the following properties:

- (a) Each vertex $v \in V$ has at most $2r - 1$ neighbors in $[m]$.
- (b) For any subset $A \subseteq V$ with $|A| \leq a$, the number of (distinct) neighbors of A in $[m]$ is at least $r|A|$.

The existence of such an expander with $r = O(\log n)$ and $m = O(ar)$ follows by a standard probabilistic argument, and it might very well be mentioned in the literature. Since we have no explicit reference, the calculation is sketched here.

For each $v \in V$, pick its neighbors in $[m]$ by $(2r - 1)$ independent random draws (with repetitions allowed, so we have at most $(2r - 1)$ -neighbors but possibly fewer).

For a fixed s -element $A \subseteq V$ and an (sr) -element $T \subseteq [m]$, the probability that all neighbors of A live in T is $(sr/m)^{(2r-1)s}$. Thus, the probability that a random H violates (b) is at most

$$\begin{aligned} \sum_{s=1}^a \binom{n}{s} \binom{m}{sr} \left(\frac{sr}{m}\right)^{(2r-1)s} &\leq \sum_{s=1}^a n^s \left(\frac{em}{sr}\right)^{sr} \left(\frac{sr}{m}\right)^{(2r-1)s} \\ &= \sum_{s=1}^a \left[en \left(\frac{esr}{m}\right)^{r-1} \right]^s. \end{aligned}$$

With $m = \lceil 2ear \rceil$ and $r = 4 + \lceil \log_2 n \rceil$, say, the expression in square brackets is below $\frac{1}{2}$ and so random H works with a positive probability.

Given a graph H satisfying (a) and (b), we let W be the set of edges of H . For $j \in [m]$, W_j are the edges incident to j , and for $v \in V$, U_v are the at most $2r - 1$ edges incident to v .

Consider an $A \subseteq V$ of size at most a . By condition (b), any s vertices in A have together at least rs neighbors in $[m]$. By an r -fold application of Hall's marriage theorem, we can select an r -fold system of distinct representatives for the a neighborhoods of the vertices $v \in A$. That is, for each $v \in A$, we choose a subset $U_{v,A} \subseteq U_v$ of r edges incident to v , in such a way that no two edges in $\bigcup_{v \in A} U_{v,A}$ share an endpoint in $[m]$. We set $f(A) = \bigcup_{v \in A} U_{v,A}$. The conditions of Lemma 3.1 are satisfied. $\square$

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