Edge Close Ball Packings*

K. Böröczky

Department of Geometry, Lorand Eötvös University,
Budapest, Hungary
boroczky@cs.elte.hu

Abstract. For a packing of unit balls in $\mathbb{R}^3$, an edge-line is the intersection of the planes of two intersecting faces of some Dirichlet–Voronoi cell, like for example the line of an edge. Now the edge closeness of the arrangement is the maximal distance between an edge-line for a Dirichlet–Voronoi cell and the centre of the corresponding unit ball. We show that the unique packing with minimal edge closeness is generated by the space centred cubic lattice.

1. Introduction

The notion of closeness of a ball packing P in spaces of constant curvature originates from L. Fejes Tóth (see [11]). This notion provides a way to characterize dense infinite arrangements of balls. Unlike the usual notion of density, closeness can be defined also for arrangements in hyperbolic space, and it is sensitive to local modifications.

We denote the supremum of the radii of the balls contained in the complement of the union of the balls in a packing by ϱ . If ρ is finite then L. Fejes Tóth considered $1/\rho$ as a measure of closeness. We deal with packings of unit balls, and define *closeness* as $1 + \varrho$. This notion was also considered in the early paper [15] of Ryškov for lattice packings in the Euclidean space. He verified (see [16]) that in $\mathbb{R}^n$ the optimal lattice arrangement over all first Voronoi type lattices is the first basic Voronoi lattice. It follows that in $\mathbb{R}^3$ the unique lattice packing of unit balls with minimal closeness is generated by the space centred cubic lattice.

We denote by $\{o_i\}$ the family of the centres in the packing. In the study of an arrangement of balls in a space, the Dirichlet–Voronoi tiling and the Delone tiling play an important role. The Dirichlet–Voronoi cell associated with a centre o_i consists of the

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points in the space which are closer to o_i than to any other centre. This Dirichlet–Voronoi cell is denoted by DV_i .

A ball is called a supporting ball if it is spanned by four non-coplanar centres $\{o_i\}$ and it contains no centre o_i in its interior. The convex hull of the centres contained in a supporting ball is called a Delone cell. The Delone cells form a tiling, and the vertices of the Dirichlet–Voronoi tiling are the circumcentres of the Delone tiles (see [10]). For some packings, there exist Delone tiles which are not tetrahedra. Therefore a triangulation of the Delone tiling is called a Delone triangulation. Note that both the Dirichlet–Voronoi tiling and the Delone tiling can be defined in any space of constant curvature.

Next, a circle is called a *supporting circle in the space* if it lies on the boundary of a supporting ball, and is spanned by three centres $\{o_i\}$. Then for any supporting circle there exists a Delone triangulation where the supporting circle is circumscribed around a face of the triangulation. In this paper the circumscribed ball (or circle) of a tetrahedron (or triangle) is the one whose boundary contains the vertices.

Observe that the closeness is the supremum of the radii of the supporting balls. Define the *edge closeness* to be the supremum of the radii of the supporting circles in the space. Then the closeness (the edge closeness) is the supremum of the circumradii of the tetrahedra (the faces of the tetrahedra) in any Delone triangulation. In a similar manner, the notion of lower-dimensional closeness can be defined in any space of constant curvature (see [3]).

The geometric meaning of edge closeness is made more clear by considering Dirichlet–Voronoi cells and edge-lines. In general, let P be a polytope. Then an *edge-line* is the intersection of the affine hulls of two intersecting faces. A typical example for an edge-line is the affine hull of an edge. On the other hand, say a vertex of valence 4 generates two additional edge-lines, and vertices with valence at least 4 count as degenerate edges.

Since the vertices of Dirichlet–Voronoi cells are the centres of supporting balls, the original notion of closeness is the supremum of the distances between a vertex of a Dirichlet–Voronoi cell and the corresponding centre o_i . On the other hand, edge closeness is the supremum of the distances between an edge-line of some DV_i and the corresponding o_i . Note that the notion of edge closeness is continuous.

L. Fejes Tóth [11] conjectured that the space centred cubic lattice generates the packing of unit balls with the minimal closeness, and this is the unique minimal arrangement. The conjecture is verified in [4].

In the optimal arrangement, the edge length of the corresponding cube is $4/\sqrt{3}$, and the Dirichlet–Voronoi cells are octahedra truncated in a way that the hexagonal faces touch the corresponding ball. The optimal closeness is $\sqrt{5/3}$, which is the circumradius of a Dirichlet–Voronoi cell. The cells of the Delone tiling are tetrahedra whose two opposite edges have length $4/\sqrt{3}$, and the other four edges have length 2.

With respect to edge closeness, the same arrangement is proved to be optimal.

Theorem 1. *The minimal edge closeness of a packing of unit balls is $\sqrt{\frac{3}{2}} = 1.2247$. The unique optimal arrangement is the space centred cubic lattice.*

The theorem was proved for lattice packings in [3].

Besides statements about the combinatorial structure of tilings, the essential part of the proof of Theorem 1 is Theorem 2 about the circumradius of a tetrahedron with small dihedral angle at some edge.

A few words about notation: We work in $\mathbb{R}^3$, and the origin is denoted by o . The Euclidean distance is denoted by $d(\cdot, \cdot)$. We use ab to denote the segment connecting a and b , and if neither a or b is the origin then $\angle(a, b)$ denotes the angle between the vectors.

2. Polytopes with at Most 10 Faces

The following statement which is needed in the proof of Lemma 1 is contained in Lemma 1 in [5], but we present a short proof for sake of completeness.

Proposition 1. *Let σ be a face-to-face tiling of S^2 with possibly non-convex polygons such that each face has at least three sides and lies in an open hemisphere. In addition, assume that there exists a $c > 0$ such that to each face one can assign a regular polygon contained in some open hemisphere with the same number of sides and with side length c , and the sum of the areas of the regular polygons assigned is just the area of S^2 . Then the maximum of the side-length in σ is at least c , and in case of equality each face is a regular polygon.*

Proof. We may assume that each edge in the tiling is of length at most c . Then Proposition 1 follows from the following fact which is the consequence of the isoperimetric inequality on the sphere (see [12]): For each tile the area is at most the area of the regular polygons assigned, and equality holds if and only if the tile and the regular polygon assigned are congruent. $\square$

We denote the side length of a spherical regular triangle of area $4\pi/n$ by c_n .

Assume that the planes H_1 and H_2 avoid the interior of the unit ball, and the normal vectors pointing towards H_1 and H_2 enclose an angle α , $0 < \alpha < \pi$. Then

$$d(o, H_1 \cap H_2) \geq \frac{1}{\cos(\alpha/2)}. \quad (1)$$

This can be shown by pushing the plane that lies farther from the origin towards the origin until H_1 and H_2 are at the same distance from o .

Lemma 1. *Let P be a polytope with at most 10 faces containing the unit ball. Then there exists an edge-line whose distance from the origin is greater than $\sqrt{\frac{3}{2}}$.*

Proof. Denote by $F_1, \dots, F_n$, $n \leq 10$, the facets of P , and by u_i the exterior unit normal to F_i .

Let σ be the spherical tiling whose tiles are cut out by the exterior normal cones at the vertices of P . Then $u_1, \dots, u_n$ are the vertices of σ . Finally, consider a triangulation

τ of σ , and hence if u_i and u_j are connected by an edge in τ then F_i and F_j intersect. Euler's theorem yields that τ has at most 16 tiles.

Note that

$$\frac{1}{\cos 35.7^\circ} = 1.2314 > \sqrt{\frac{3}{2}} = 1.2247.$$

Therefore, by (1), it is sufficient to prove that there exists an edge of τ whose length is at least 71.4° . In particular, we may assume that each edge is shorter than 90° .

Assume that there exists a vertex u_i with valence 3. Then the triangles containing u_i lie in the open hemisphere centred at u_i because each edge of τ is shorter than 90° . Uniting the three triangles at u_i results in a triangulation τ_1 with at most 14 tiles. Now Proposition 1 yields that there exists an edge of τ_1 (and hence of τ) with length at least $c_{14} = 73.3^\circ > 71.4^\circ$.

If there exists no vertex with valence 3 then Euler's theorem and $n \leq 10$ yield that there exists a vertex u_j with valence 4. Uniting the four triangles at u_j results in tiling τ_2 with one quadrilateral and at most twelve triangles.

Define c in a way that the area of the square with side length c plus twelve times the area of the regular triangle with side length c is 4π . Then $c = 72.15^\circ > 71.4^\circ$, and the lemma follows by Proposition 1. $\square$

Remark 1. The optimal lower bound in Lemma 1 is $\sqrt{3 - \sqrt{2}} = 1.2592$. If $n \leq 9$ then this can be proved using the ideas above. If $n = 10$ then similar arguments as the ones used in Theorem 6 in [5] yields the bound.

3. About the Circumradius of a Tetrahedron

The main purpose of this section is to prove Corollary 1 below, which is crucial for the discussions in the next section. On the way to verifying Corollary 1, we establish Theorem 2 about the circumradius of a tetrahedron with small dihedral angles, which can be useful in some other applications.

We start with an auxiliary statement. Assume that S is the surface of a ball centred at the origin in $\mathbb{R}^3$. A subset X of S is called spherically convex if it is contained in an open hemisphere, and for any two $x, y \in X$, the shorter arc of the great-circle through x and y is contained in X . In addition, a cap D on S with centre $p \in S$ and angular radius $\alpha < (\pi/2)$ is the set of points x on S with $\angle(x, p) \leq \alpha$. Observe that a cap is spherically convex.

Proposition 2. *Let S be the surface of a ball centred at the origin, consider a cap D on S with centre $p \in S$ and some $q \neq \pm p$. Assume that as a point x moves along the boundary of D , the angle $\angle(x, q)$ attains a local maximum at the position $x = x_0$. If $x_0 \neq -q$ then the shorter arc of the great-circle connecting q and x_0 contains either p or $-p$.*

Proof. Observe that the length of the shorter great-circle arc connecting $x, y \in S$, $x \neq y$, is proportional to $\angle(x, y)$. Denote the tangent vector at x_0 to the boundary arc

of D by l , and the tangent line at x_0 to the great-circle connecting x_0 and q by l' . Then the maximality of $\angle(x, q)$ yields that l and l' are orthogonal. Therefore p is contained in the great-circle through x_0 and q , which proves the proposition. $\square$

We consider the interior angle at an edge as the dihedral angle at the edge.

The Jung theorem (see [1]) yields that for any four unit vectors whose convex hull contains the origin, there exist two vectors whose angle is at least $\arccos(-1/3)$. Therefore the smallest among the dihedral angles of a tetrahedron is at most $\arccos \frac{1}{3}$, and equality holds if and only the tetrahedron is regular.

Let the tetrahedron $\tilde{T} = \text{conv}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\}$ satisfy that $\tilde{v}_1\tilde{v}_3 = \tilde{v}_2\tilde{v}_4 = 2a$ for some $1 \leq a < \sqrt{2}$, and the remaining edges have length 2. Observe that the faces of $\tilde{T}$ are congruent. Denote the dihedral angle of $\tilde{T}$ at $\tilde{v}_1\tilde{v}_2$ by α , the circumradius of the triangle $\tilde{v}_1\tilde{v}_2\tilde{v}_3$ by $\tilde{r}_\alpha$, and the circumradius of $\tilde{T}$ by $\tilde{R}_\alpha$. Then $0 < \alpha \leq \arccos \frac{1}{3}$, and

$$\cos \alpha = \frac{a^2}{4 - a^2}, \quad \tilde{r}_\alpha^2 = \frac{4}{4 - a^2} = 1 + \cos \alpha, \quad \tilde{R}_\alpha^2 = \frac{2 + a^2}{2} = \frac{1 + 3 \cos \alpha}{1 + \cos \alpha}.$$

In particular, both $\tilde{R}_\alpha$ and $\tilde{r}_\alpha$ increase as α decreases, and $\tilde{r}_\alpha < \tilde{R}_\alpha < \sqrt{2}$. We use the notation $\tilde{T}_\alpha$ when we wish to emphasize the dependence of $\tilde{T}$ on α .

Theorem 2. *Let T be a tetrahedron with edges of length at least 2, and with minimal dihedral angle $\alpha \leq \arccos \frac{1}{3}$. Then the circumradius R of T is at least $\tilde{R}_\alpha$, and $R = \tilde{R}_\alpha$ if and only if $T = \tilde{T}_\alpha$.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices of the tetrahedron T , and assume that the dihedral angle at the edge v_1v_2 is α .

If $\alpha = \arccos \frac{1}{3}$ then T and $\tilde{T}_\alpha$ are regular tetrahedra, and hence $R \geq \tilde{R}_\alpha$. Hence assume $\alpha < \arccos \frac{1}{3}$. Denote the centre of the ball circumscribed around T by o' , and the plane $v_1v_2v_3$ and $v_1v_2v_4$ by H_3 and H_4 , respectively.

First consider the case when either H_3 or H_4 separates o' and T , say H_3 does. Here we allow $o' \in H_3$. Translate T so that the origin is the circumcentre of the triangle $v_1v_2v_3$. If r_3 denotes the circumradius of the triangle $v_1v_2v_3$ then the ball with radius r_3 centred at the origin contains T . It is well known (see [14]) that among the five vectors $\pm v_4, v_1, v_2, v_3$, there exist two vectors whose angle is at most $\pi/2$. This yields an edge of T with length at most $\sqrt{2}r_3$, and hence $r_3 \geq \sqrt{2}$. Since $R \geq r_3$ and $\tilde{R}_\alpha < \sqrt{2}$, Theorem 2 follows in this case.

Therefore assume that o' lies in the open quadrant of $\mathbb{R}^3$ determined by H_3 and H_4 that contains $\text{int } T$. Translate T so that o' coincides with the origin o .

If $R < \tilde{R}_\alpha$ then $(\tilde{R}_\alpha/R)T$ is a tetrahedron satisfying the conditions of Theorem 2 and whose circumradius is $\tilde{R}_\alpha$. Therefore Theorem 2 is equivalent to the statement

$$\text{if } R = \tilde{R}_\alpha \text{ then } T = \tilde{T}_\alpha. \quad (2)$$

Let T be a tetrahedron satisfying the conditions of Theorem 2, and assume that $R = \tilde{R}_\alpha$. Denote by S the boundary of the ball with radius $\tilde{R}_\alpha$ centred at o . Place T so that $v_1, v_2, \tilde{v}_1, \tilde{v}_2$ are contained in the same great semi-circle of S and the lines v_1, v_2 and $\tilde{v}_1, \tilde{v}_2$ are parallel.

Observe that if $d(x, y) = 2$ holds for $x, y \in S$ then $\angle(x, y) = \tilde{\beta}$ where

$$\cos \tilde{\beta} = \frac{a^2 - 2}{a^2 + 2}.$$

In particular, $\tilde{\beta}$ is obtuse.

We define certain regions on S . Denote by K the set of points $x \in S$ such that the Euclidean distance to x both from v_1 and from v_2 is at least 2. Since $\beta = \angle(v_1, v_2) \geq \tilde{\beta}$ is obtuse, K is the intersection of the caps centred at $-v_1$ and at $-v_2$ with angular radius $180^\circ - \tilde{\beta}$. We deduce that K is spherically convex and $-v_1, -v_2 \notin K$.

Denote by L the intersection of S and the quadrant determined by H_3 and H_4 that contains T , and set $Q = L \cap K$. Then either none of the two vertices of K is in L , and hence Q is bounded by four circular arcs, or one of the vertices of K lies in L , and hence Q is bounded by three circular arcs (Q is not spherically convex in either of the cases).

We turn to $\tilde{T}$. Then the analogous regions $\tilde{K}$, $\tilde{L}$ and $\tilde{Q}$ can be defined, and we denote the vertices and angles by a tilde over the corresponding symbol. Then $\tilde{Q}$ is bounded by four circular arcs. Observe that $\tilde{v}_3, \tilde{v}_4$ are vertices of $\tilde{Q}$, and denote the other two vertices by $\tilde{v}'_3, \tilde{v}'_4$. Now the tetrahedron with vertices $\tilde{v}_1, \tilde{v}_2, \tilde{v}'_3, \tilde{v}'_4$ is congruent to $\tilde{T}$, and $\angle(\tilde{v}_3, \tilde{v}_4) = \angle(\tilde{v}'_3, \tilde{v}'_4) = \tilde{\beta}$.

Denote by H'_3 (H'_4) the plane through $\tilde{v}_1, \tilde{v}_2$ parallel to H_3 (H_4), and by L' the part of S in the quadrant determined by H'_3 and H'_4 and containing T . Set $Q' = L' \cap \tilde{K}$. If $d(v_1, v_2) = 2$ then $Q = Q'$, and if $d(v_1, v_2) > 2$ then $K \subset \text{int } \tilde{K}$ and $Q \subset \text{int } Q'$.

We claim that if $x, y \in Q'$ then

$$\begin{aligned} \angle(x, y) &\leq \tilde{\beta}, & \text{with equality if and only if} \\ Q' &= \tilde{Q}, & \text{and } \{x, y\} = \{\tilde{v}_3, \tilde{v}_4\} \text{ or } \{x, y\} = \{\tilde{v}'_3, \tilde{v}'_4\}. \end{aligned} \quad (3)$$

Observe that $v_3, v_4 \in Q'$ and $\angle(v_3, v_4) \geq \beta$, therefore the claim yields (2).

It may happen that H'_3 or H'_4 avoids $\tilde{K}$. In this case we can increase Q' by rotating H'_3 and H'_4 and preserving the angle, hence assume that each of H'_3 and H'_4 intersects $\tilde{K}$.

Next let $p_3 \in S$ ($p_4 \in S$) be the centre of the spherical circle $H'_3 \cap S$ ($H'_4 \cap S$). We verify that

$$-p_3 \text{ } (-p_4) \text{ is strictly separated by } H'_4 \text{ } (H'_3) \text{ from } Q'. \quad (4)$$

Note that if $\alpha = \arccos \frac{1}{3}$ then $-p_3$ and $-p_4$ are vertices of $\tilde{K}$, and $-p_3 \in H'_4$ ($-p_4 \in H'_3$). Therefore it is essential to assume that $\alpha < \arccos \frac{1}{3}$.

Denote by g the great-circle cut out by the perpendicular bisector of the segment $\tilde{v}_1 \tilde{v}_2$, and denote by h_3 (h_4) the intersection of $g \cap \tilde{K}$ and H'_3 (H'_4). Then (4) follows by verifying that the points $-p_3$ and $-p_4$ of g lie outside of the shorter arc $h_3 h_4$. Observe that the line passing through o and $-p_3$ is perpendicular to H'_3 . Now it is easy to see that if (4) holds and one rotates H'_3 and H'_4 (preserving the angle between them) towards the position where they are symmetric with respect to $\text{lin}\{\tilde{v}_1, \tilde{v}_2\}$ then (4) continues to hold. Therefore it is sufficient to consider the case when h_4 is a vertex of $\tilde{K}$, and to show that $-p_3$ lies outside of the shorter arc $h_3 h_4$ in this case.

Hence assume that h_4 is a vertex of $\tilde{K}$, and set $m = \frac{1}{2}(\tilde{v}_1 + \tilde{v}_2)$. We have that

$$\cos \angle(m - h_4, o - h_4) = \sqrt{\frac{8}{3(2 + a^2)}}.$$

Now, elementary calculations and $a > 1$ yields that

$$\cos \angle(m - h_4, o - h_4) > \sin \alpha,$$

and hence $\angle(m - h_4, o - h_4) < 90^\circ - \alpha$. Therefore $-p_3$ lies outside of the shorter arc h_3h_4 , which yields (4).

Let $x, y \in Q'$ be such that $\angle(x, y)$ is a maximum, and hence x and y are contained in the union of the four arcs bounding Q' . Since $-\tilde{v}_1, -\tilde{v}_2$ are not contained in the spherically convex set $\tilde{K}$ and $Q' \subset \tilde{K}$, we deduce by Proposition 2 that x and y are contained in the arcs $H'_3 \cap \tilde{K}$ and $H'_4 \cap \tilde{K}$. Now if either $H'_3 \cap \tilde{K}$ or $H'_4 \cap \tilde{K}$ contains both x and y then they should be vertices. Finally observe that the shorter arc of the great-circle connecting x and y is a subset of Q' because o lies on the same side of H'_3 or H'_4 as L' . Since the points $-p_3$ and $-p_4$ lie outside Q' , Proposition 2 yields that neither x nor y can lie in the interior of a bounding arc. We conclude that both x and y are vertices of Q' .

Denote the endpoints of $H'_3 \cap \tilde{K}$ and $H'_4 \cap \tilde{K}$ by q'_3 and q_3 , and by q_4 and q'_4 , respectively, in a way that q'_3 and q_4 are neighbours. Possibly, $q'_3 = q_3$ or $q_4 = q'_4$. In addition, denote the two points of intersection of the boundary of $\tilde{K}$ and $\text{lin}\{\tilde{v}_1, \tilde{v}_2\}$ by s_1 and s_2 .

Since Q' is symmetric with respect to g , we deduce that $\angle(q'_4, q_3) \leq \angle(q_3, q_4)$. It also follows that each of $\angle(q'_3, q_3)$ and $\angle(q_4, q'_4)$ is at most $\angle(s_1, s_2) = 360^\circ - 3\tilde{\beta} < \tilde{\beta}$. Therefore (3) follows if $\angle(q_3, q_4) < \tilde{\beta}$ unless $Q' = \tilde{Q}$, which in turn is equivalent to

$$d(q_3, q_4) \leq 2, \quad \text{with equality if and only if } Q' = \tilde{Q}. \quad (5)$$

Denote the circumcentres of the triangles $\tilde{v}_1\tilde{v}_2q_3$ and $\tilde{v}_1\tilde{v}_2q_4$ by o_3 and by o_4 , respectively. The points o, o_3, m, o_4 belong to the perpendicular bisector of the segment $\tilde{v}_1\tilde{v}_2$. The angle of the quadrilateral oo_3mo_4 at o is α , while the angle at each of o_3 and o_4 is 90° . In particular, o_3 and o_4 lie on the circle with diameter om . Since

$$d(o, m)^2 = \frac{2 \cos \alpha}{1 + \cos \alpha},$$

we deduce that

$$d(o_3, o_4)^2 = 2 \cos \alpha (1 - \cos \alpha).$$

Now after some elementary calculations, we obtain

$$d(q_3, q_4)^2 = 36 - 32 \cdot \frac{1 + 2 \cos \alpha - 2 \cos^2 \alpha + 3 \cdot d_3 d_4 \cdot \cos \alpha}{1 + 2 \cos \alpha - 2 \cos^2 \alpha + 2 \cdot d_3 d_4 \cdot \cos \alpha + d_3^2 d_4^2},$$

where $d_3 = d(o_3, m)$ and $d_4 = d(o_4, m)$. Using the function

$$f(t) = \frac{3pt + q}{t^2 + 2pt + q}$$

with $p = \cos \alpha$, $q = 1 + 2 \cos \alpha - 2 \cos^2 \alpha$ and $t = d_3 d_4$, we get

$$d(q_3, q_4)^2 = 36 - 32 \cdot f(t). \quad (6)$$

Next we bound the possible values of t . Observe that $\frac{1}{2}d_3d_4 \sin \alpha$ is the area of the triangle o_3o_4m , and the length of the side o_3o_4 depends on α only. If H'_3 and H'_4 are rotated away from the symmetric position (when $Q' = \tilde{Q}$) then the secant o_3o_4 of the circle oo_3o_4m turns towards m , and hence t decreases as the area of the triangle o_3o_4m decreases. In particular,

$$t \text{ is maximal if and only if } Q' = \tilde{Q}. \quad (7)$$

In order to minimize t , rotate H'_3 and H'_4 while preserving the angle between them in a way that H'_4 contains o . If α is relatively large then a vertex of $\tilde{K}$ gets strictly between H'_3 and H'_4 , but calculating the corresponding t at this position still provides a lower bound. Therefore

$$t \geq \frac{2 \cos^2 \alpha}{1 + \cos \alpha}.$$

Since $\cos \alpha > \frac{1}{3}$, we deduce that

$$t > \frac{\cos \alpha}{2} = \frac{p}{2}. \quad (8)$$

Now

$$f'(t) = \frac{-3pt^2 - 2qt + pq}{(t^2 + 2pt + q)^2}.$$

The quadratic polynomial in the numerator of $f'(t)$ is positive if $t = 0$, and negative if $t = p/2$. Therefore (8) yields that $f'(t)$ is negative for the possible values of t . In turn, (6) and (7) yield that $d(q_3, q_4)$ is maximal if and only if $Q' = \tilde{Q}$. We conclude (5), and hence the proof of Theorem 2 is complete. $\square$

Next, we consider the analogue of Theorem 2 for the circumradii of the faces of the tetrahedron.

Corollary 1. *Let $T = \text{conv}\{v_1, v_2, v_3, v_4\}$ be a tetrahedron with edges of length at least 2, and let $\alpha \leq \arccos \frac{1}{3}$ be the dihedral angle at the edge v_1v_2 . Denote the circumradius of the face $v_1v_2v_3$ ($v_1v_2v_4$) by r_3 (r_4). Then*

$$\max\{r_3, r_4\} \geq \tilde{r}_\alpha,$$

and equality holds if and only if $T = \tilde{T}$.

Proof. The first half of the proof is virtually the same as that of Theorem 2. We may assume again that $\alpha < \arccos \frac{1}{3}$.

Denote by o the centre of the ball circumscribed around T , and by H_3 and H_4 the planes $v_1v_2v_3$ and $v_1v_2v_4$, respectively.

If either H_3 or H_4 separates o from T , say H_3 does, then the same argument as for Theorem 2 shows that $r_3 \geq \sqrt{2}$. Since $\tilde{r}_\alpha < \sqrt{2}$, the corollary follows in this case. Therefore assume that o lies in the open quadrant of $\mathbb{R}^3$ determined by H_3 and H_4 that contains $\text{int } T$.

The second half of the proof goes along a different line; here we actually apply Theorem 2. Denote the centres of the circumscribed circles of the triangles $v_1 v_2 v_3$ and $v_1 v_2 v_4$ by o_3 and o_4 , respectively. In addition, set $m = \frac{1}{2}(v_1 + v_2)$ and $\omega = \angle(o - m, o_4 - m)$. We may assume that $d(o, o_4) \leq d(o, o_3)$, and hence $\omega \leq \alpha/2$. Set

$$x = d(v_1, m), \quad y = d(o_4, m), \quad z = d(o, o_4),$$

and denote by $\tilde{y}$ and $\tilde{z}$ the corresponding quantities for $\tilde{T}_\alpha$. Then

$$r_4^2 = x^2 + y^2, \quad R^2 = x^2 + y^2 + z^2 \quad \text{and} \quad \tilde{r}_\alpha^2 = 1 + \tilde{y}^2, \quad \tilde{R}_\alpha^2 = 1 + \tilde{y}^2 + \tilde{z}^2.$$

Assume that $r_4 \leq \tilde{r}_\alpha$, and hence $x^2 + y^2 \leq \tilde{r}_\alpha^2$. Since $x \geq 1$ holds by $d(v_1, v_2) \geq 2$, we deduce that $y^2 \leq \tilde{r}_\alpha^2 - 1 = \tilde{y}^2$. Thus

$$z \cdot \operatorname{ctg} \frac{\alpha}{2} \leq z \cdot \operatorname{ctg} \omega = y \leq \tilde{y} = \tilde{z} \cdot \operatorname{ctg} \frac{\alpha}{2},$$

and hence $z \leq \tilde{z}$. Finally, the inequality $R \geq \tilde{R}_\alpha$ of Theorem 2 yields that

$$r_4^2 = x^2 + y^2 = R^2 - z^2 \geq \tilde{R}_\alpha^2 - \tilde{z}^2 = \tilde{r}_\alpha^2.$$

We conclude that $r_4 \geq \tilde{r}_\alpha$, which proves the inequality of Corollary 1. If equality holds then $R = \tilde{R}_\alpha$, hence $T = \tilde{T}_\alpha$. $\square$

4. Basic Facts About Edge Closeness

In this section we prove some properties of packings with small edge closeness.

Lemma 2. *Assume that in a packing of unit balls, there exists a vertex at which at least five Dirichlet–Voronoi cells meet. Then the edge closeness of the packing is at least $\sqrt{2}$.*

Proof. Assume that the Dirichlet–Voronoi cells corresponding to the centres $o_1, \dots, o_5$ meet at the common vertex v .

Note that if two sides of a triangle have length at least 2, and the angle they enclose is at least 90° then the circumradius of the triangle is at least $\sqrt{2}$. Therefore it is sufficient to prove that there exists a triangle whose vertices are among $o_1, \dots, o_5$, and one of the angles is at least 90° .

By Radon's theorem (see [1]), we may assume that the segment $o_1 o_2$ and the triangle $o_3 o_4 o_5$ have a common point m . Now the points $o_1, \dots, o_5$ are contained in the boundary S of a supporting ball whose centre is v . If o_1 and o_2 are opposite points of S then $\angle(o_1 - o_3, o_2 - o_3)$ is the right angle, and we are done.

Assume that o_1 and o_2 are not opposite points. Denote by H the plane containing o_1 and o_2 , and orthogonal to the plane $vo_1 o_2$. Since the convex hull of $\{o_3, o_4, o_5\}$ intersects the segment $o_1 o_2$, there exists an o_i , $3 \leq i \leq 5$, such that o_i is separated from v by H (possibly $o_i \in H$). Then $\angle(o_1 - o_i, o_2 - o_i)$ is at least 90° , which completes the proof of Lemma 2. $\square$

Note that the triangle with sides 2, 2 and $4/\sqrt{3}$ is acute, and the circumradius is $\sqrt{\frac{3}{2}}$. This yields

Proposition 3. *If two sides of a triangle are at least 2, and the third side is at least $4/\sqrt{3}$, then the circumradius of the triangle is at least $\sqrt{\frac{3}{2}}$. Equality holds if and only if two sides are of length 2, and the third side is of length $4/\sqrt{3}$.*

Let T^* be a Delone tile of the space centred cubic lattice as in Theorem 1. Two opposite edges of T^* have length $4/\sqrt{3}$, the other four edges have length 2, and the circumradius of the faces is $\sqrt{\frac{3}{2}}$.

Lemma 3. *We consider a packing of unit balls with edge closeness at most $\sqrt{\frac{3}{2}}$, and assume that the Dirichlet–Voronoi cells corresponding to the centres o_1 and o_2 share a face and $d(o_1, o_2) \geq 4/\sqrt{3}$. Then $d(o_1, o_2) = 4/\sqrt{3}$, the common face is a square with side length $\sqrt{\frac{2}{3}}$, and exactly four Delone tiles contain $o_1 o_2$, each of which is congruent to T^* . In particular, the edge closeness is $\sqrt{\frac{3}{2}}$.*

Proof. Since the edge closeness is at most $\sqrt{\frac{3}{2}}$, Lemma 2 yields that each Delone tile is a tetrahedron. Then the fact that $d(o_1, o_2) = 4/\sqrt{3}$ follows directly from Proposition 3.

Assume that in the Delone tiling, the n tetrahedra have $o_1 o_2$ as an edge, say $o_1 o_2 o_3 o_4, \dots, o_1 o_2 o_{n+1} o_{n+2}, o_1 o_2 o_{n+2} o_3$. According to Proposition 3, we have $d(o_1, o_i) = d(o_2, 4o_i) = 2, i = 3, \dots, n+2$. We have that

$$2 \leq d(o_i, o_{i+1}), d(o_{n+2}, o_3) \leq \frac{4}{\sqrt{3}}, \quad (9)$$

where the upper bound is a consequence of Proposition 3.

Denote the midpoint of the segment $o_1 o_2$ by m . Now $d(m, o_i) = \sqrt{\frac{8}{3}}$, therefore the lower bound in (9) yields that $n \leq 4$. On the other hand, we deduce by the upper bound in (9) that $n = 4$, and the quadrilateral $o_3 o_4 o_5 o_6$ is a square with side length $4/\sqrt{3}$. Therefore the common facet of the two Dirichlet–Voronoi cells is a square of side length $\sqrt{\frac{2}{3}}$, while the four Delone tiles containing $o_1 o_2$ are congruent to T^* . $\square$

Finally, we investigate the possibility that a face of some Dirichlet–Voronoi cell has at least six sides.

Lemma 4. *Consider a packing of unit balls with edge closeness at most $\sqrt{\frac{3}{2}}$, and assume that the Dirichlet–Voronoi cells corresponding to the centres o_1 and o_2 share a face F of at least six sides. Then $d(o_1, o_2) = 2$, the common face is a regular hexagon with side length $\sqrt{\frac{2}{3}}$, and exactly six Delone tiles contain $o_1 o_2$ each of which is congruent to T^* . In particular, the edge closeness is $\sqrt{\frac{3}{2}}$.*

Proof. Since the edge closeness is at most $\sqrt{\frac{3}{2}}$, Lemma 2 yields that each Delone tile is a tetrahedron. Suppose that v is the vertex of F where the sides enclose the maximal angle. If this angle is $180^\circ - \alpha$ then $\alpha \leq 60^\circ$. Denote by T the Delone tile corresponding to v , and observe that the dihedral angle of T at $o_1 o_2$ is α .

If $\alpha = 60^\circ$ then $a = 2/\sqrt{3}$, $\tilde{r}_\alpha = \sqrt{\frac{3}{2}}$, and $\tilde{T}_\alpha$ is congruent to T^* in terms of the notation of Corollary 1. Therefore if $\alpha < 60^\circ$ then Corollary 1 yields that the edge closeness is greater than $\sqrt{\frac{3}{2}}$, which is absurd. We deduce that $\alpha = 60^\circ$, which yields that F is a hexagon and each of its angles is 120° . Now we deduce by the equality case of Corollary 1 that each Delone tile at the vertices of F is congruent to T^* , which completes the proof of the lemma. $\square$

5. Proof of Theorem 1

We precede the proof with a quote of two statements, which are Lemmas 2 and 3 in [4]. Recall that a polytope in $\mathbb{R}^3$ is simple, if each vertex is contained in exactly three faces.

Lemma 5. *There exists no simple polytope in $\mathbb{R}^3$ which is bounded by one quadrilateral and ten pentagons.*

Lemma 6. *There exists no face-to-face tiling of 3-space by convex polyhedra such that each polyhedron is bounded by twelve pentagons, and at each vertex four polyhedra meet.*

We note that in the packing of unit balls determined by the space centred cubic lattice the edge closeness is $\sqrt{\frac{3}{2}}$. The Dirichlet–Voronoi cells are congruent to the truncated octahedron DV^* , which is bounded by eight regular hexagons and six squares, each of edge length $\sqrt{\frac{2}{3}}$.

In order to prove Theorem 1, we consider a packing of unit balls whose edge closeness is at most $\sqrt{\frac{3}{2}}$. What we prove below is that each Dirichlet–Voronoi cell of the packing is congruent to DV^* , and that the packing is determined by the space centred cubic lattice.

We deduce by Lemma 2 that

$$\begin{aligned} &\text{at every vertex exactly four Dirichlet–Voronoi cells meet, and hence} \\ &\text{each Dirichlet–Voronoi cell is a simple polytope.} \end{aligned} \tag{10}$$

We claim that there exists a face F of some Dirichlet–Voronoi cell which has at least six sides. First, $1.23 > \sqrt{\frac{3}{2}}$ and Lemma 1 yield that each Dirichlet–Voronoi cell has at least eleven faces.

Assume that there exists a Dirichlet–Voronoi cell with eleven faces. If each face has at most five sides then the Euler theorem implies that the cell is bounded by one quadrilateral and ten pentagons. This contradicts Lemma 5, and hence the claim follows in this case.

If there exists a Dirichlet–Voronoi cell with at least thirteen faces then the claim is a consequence of the Euler theorem. Finally, assume that each Dirichlet–Voronoi cell has twelve faces. Now the Euler theorem yields that a simple polytope with twelve faces either has a face with at least six sides, or each face is a pentagon. Therefore Lemma 6 and statement (10) complete the proof of the claim.

By virtue of Lemma 4 we have just verified that the minimal edge closeness is $\sqrt{\frac{3}{2}}$. However, we know much more. Assume that F corresponds to the centres o_1 and o_2 . Lemma 4 yields that $d(o_1, o_2) = 2$, F is a regular hexagon with side length $\sqrt{\frac{2}{3}}$, and the six Delone tiles containing $o_1 o_2$, are congruent to T^* . Then among the six faces of DV_1 sharing an edge with F , three pairwise disjoint ones are at a distance of $2/\sqrt{3}$ from o_1 .

Therefore we deduce by Lemma 3 that these three faces, say F_1 , F_3 and F_5 , are squares with side length $\sqrt{\frac{2}{3}}$, and the Delone tiles corresponding to F_1 , F_3 and F_5 are congruent to T^* .

Let F_2 be the facet of DV_1 sharing common edges with F_1 , F and F_3 , and assume that these edges are $v_1 v_2$, $v_2 v_3$, $v_3 v_4$, respectively. We know that the angles of F_2 at v_1 , v_2 , v_3 , v_4 are 120° each, and the edges $v_1 v_2$, $v_2 v_3$, $v_3 v_4$ have length $\sqrt{\frac{2}{3}}$ each.

Denote by F_{21} (by F_{23}) the face of DV_1 neighbouring F_2 and F_1 (F_2 and F_3) and different from F . If F_2 were a pentagon then its fifth vertex v_5 would be the common vertex of F_2 , F_{21} and F_{23} , and hence the intersection of F_{21} and F_{23} would be an edge. The distance from the affine hull of this edge to o_1 would be $\sqrt{3} > \sqrt{\frac{3}{2}}$, which is impossible. Therefore F_2 has at least six sides. We deduce by Lemma 4 that F_2 is a regular hexagon, and each of the six Delone tiles corresponding to F_2 is congruent to T^* . A similar statement holds for the last two faces F_4 and F_6 of DV_1 neighbouring F . We deduce, applying the same argument to F_2 , F_4 and F_6 , and finally to their neighbours, that DV_1 is congruent to DV^* .

Let DV_3 (DV_4) be the other Dirichlet cell having the hexagon F_2 (the square F_1) as a face. Then DV_2 and DV_3 are congruent to DV^* . Checking the dihedral angles of DV^* shows that the common face of DV_2 and DV_3 is a square, and the common face of DV_3 and DV_4 is a hexagon. In particular, DV_4 is congruent to DV^* as well. Therefore applying the argument above to the hexagonal faces of the known Dirichlet–Voronoi cells, we obtain a tiling of $\mathbb{R}^3$ by DV^* . Since this tiling coincides with the one determined by the space centred cubic lattice, the proof of Theorem 1 is complete.

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