Conical Equipartitions of Mass Distributions*

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Abstract. A *conical dissection* of R^d is a decomposition of the space into polyhedral cones. An example of a conical dissection is a fan associated to the faces of a convex polytope. Motivated by some recent questions and results about (simultaneous) conical partitions of measures by Kaneko and Kano, Bárány and Matoušek, and Bespamyatnikh et al. [2], [4], [19], we study related partition problems in higher dimensions. In the case of a single measure, several conical partition results associated to a nondegenerated pointed simplex (Δ, a) in R^n are obtained with the aid of the Brouwer fixed point theorem. In the other direction, it is demonstrated that general “symmetrical” equipartition results [21] may be used to yield, by appropriate specialization, fairly general “asymmetric,” conical equipartitions for two or more mass distributions. Finally, the topological nature of these results is exemplified by their extension to the case of topological (projective) planes.

1. Introduction

1.1. *What are Equipartitions of Masses?*

Problems of equipartitions of sets of points and dissections of mass distributions have a long tradition in Combinatorial Geometry. Together with more general problems about extremal properties of mass distributions in R^d , they occupy one of the central positions in this field. An equipartition of a measurable set $M \subset R^n$, or more generally of the associated mass distribution (measure) μ_M defined on R^n , is a finite collection $\mathcal{K} = \{K_1, \dots, K_q\}$ of measurable sets forming a partition (dissection) of R^n such that $\mu_M(K_i) = (1/m)\mu_M(R^d)$ for each $i = 1, \dots, q$. A more general partition problem is formulated for a collection of measures $\mu_1, \dots, \mu_p$ and a matrix $A = (\alpha_{ij})$ of non-

* This research was supported by the Ministry for Science and Technology of Serbia, Grants 04M01 and 04M03.

negative numbers. Then $\mathcal{K} = \{K_1, \dots, K_q\}$ is called an A -partition if the inequalities $\mu_i(K_j) \geq \alpha_{ij}\mu_i(R^n)$ are satisfied for all $i = 1, \dots, p$ and $j = 1, \dots, q$. Interesting examples of these “asymmetrical” partitions can be found in [2]. Finally we say that a problem belongs to *extremal combinatorics of mass distributions* if the goal is to establish, for a given collection of measures $\mu_1, \dots, \mu_p$, the existence of a geometric object, or a configuration of objects so that a special associated class of inequalities is satisfied.

1.2. *What Is Known about Partitions of Masses?*

Some of the best known early equipartition/extremum results are the Borsuk–Ulam “ham sandwich theorem” (a consequence of the Borsuk theorem [6]), Rado’s “center point theorem” for general measures [17], Grünbaum’s “center point theorem” for convex bodies [12], etc. Survey articles [8], [14], [23]–[25] as well as the original papers [18], [21], and [27] cover different aspects of the equipartition problem and give a good picture of some of the more recent developments in this area. For example, Ramos in [18] covers the question of equipartitions of masses by hyperplanes, Vrećica and Živaljević [21] deal with equipartitions by regular wedge-like cones, and the “center transversal theorem” proved in [27] reveals a hidden connection between the center point theorem and the ham sandwich theorem, etc. On the other hand, some of the key “divide and conquer” paradigms for designing algorithms in Computational Geometry rely on the existence of “balanced” partitions for different classes of geometric objects. For example, Grünbaum raised in [12] a question whether, for a given measure μ in R^d , there always exist d hyperplanes such that each of the corresponding open orthants contains at most a fraction $1/2^d$ of the mass $\mu(R^d)$. This problem originally received new attention in Computational Geometry when it was connected with the problem of designing efficient algorithms for half-space range queries [22]. However, this approach was later superseded by other, more efficient, techniques, see [13].

Also, dissections or partitions of geometric objects of a different kind may involve indirectly equipartitions or other extremal properties of mass distribution. For example Rado’s “center point theorem” provides in [15] a key step in a proof of the existence of small separators for planar graphs.

1.3. *Tools*

The methods of Algebraic Topology, in particular the results about group actions and equivariant maps, have proven to be very powerful in this context, see [14] and [23]–[25]. However, one of the drawbacks is that equivariant methods typically lead to results about highly symmetric equipartitions. For example, the problem of partitioning mass distributions by convex cones, say by convex fans in R^n , seems at least at first sight to be amenable to direct topological (equivariant) methods only if these fans admit a nontrivial group of symmetries, see [21].

1.4. *This Paper*

In this paper we focus our attention on problems about “asymmetric,” convex partitions and show how some of these results can be deduced from other, more symmetrical, partition results. An immediate motivation for this paper came from the recent results of Bárány and Matoušek [1], [2]. For example, in response to a question by Kano, they established a result about an “asymmetric” partition for two planar sets, see Theorem 3.3. Our basic observation was that this type of result is closely related, actually in some sense reducible, to “symmetric” partition results established in our paper [21]. We were led to the conclusion that this type of reduction may be useful in other cases too, and that this approach should generate new results and simpler or at least more conceptual proofs of known facts.

One of our objectives was to test which of the asymmetrical, conical partition results admits generalizations to higher dimensions. In the second section we demonstrate that, at least in the case of a single measure, there always exist special conical partitions, called *radial* and *extended radial* convex (conical) partitions, associated to a pointed simplex (Δ, a) .

In the third section we prove a generalization of the equipartition result of Bárány and Matoušek and formulate some related conjectures. In the final section we show that this equipartition result admits a different type of generalization by showing that it holds for a fairly general class of topological (projective) planes. As a corollary we deduce that the result of Bárány and Matoušek remains true in the case of a hyperbolic, noneuclidean plane.

2. Radial Dissections of a Single Measure

2.1. *Admissible Measures*

As usual [21], [26], it is convenient to work with the class $\mathcal{A}$ of “admissible” probability Borel measures, Definition 2.1. The class $\mathcal{A}$ seems to be the most natural class of measures for equipartition problems because it contains both discrete and continuous mass distributions and, in principle, each result which holds for discrete and/or continuous measures can be extended to measures in $\mathcal{A}$.

Definition 2.1. A probability measure μ defined on R^n is *admissible* if it is a weak limit of a sequence μ_n of probability measures absolutely continuous with respect to the Lebesgue measure. Recall [5] that a (probability) Borel measure μ , defined on locally compact space X , is a weak limit ($\mu_n \Rightarrow \mu$) of a sequence of measures μ_n if for each regularly closed set $F = \text{cl}(\text{int}(F))$,

$$\limsup \mu_n(F) \leq \mu(F). \quad (1)$$

Equivalently, $\mu_n \Rightarrow \mu$ if for each bounded, continuous map $f: X \rightarrow R$,

$$\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu.$$

The admissibility assumption permits us to assume in all arguments that the measures in question are continuous and then pass to the limit. For example, if the objective is to establish an inequality of the form $\nu(M) \geq \alpha$, then it is clearly sufficient to prove $\nu_n(M) \geq \alpha$ where $\nu_n \Rightarrow \nu$ is a sequence of continuous measures weakly converging to ν and then rely on the inequality (1).

The assumption that a measure ν is continuous, i.e., that it has a representation of the form $d\nu = f \, dm$ for some Lebesgue integrable function $f: R^n \rightarrow R$, ensures the continuity of all associated *test maps*. The following definition of a μ -equipartition is sufficiently general for the needs of this paper.

2.2. Radial and Extended Radial Dissections

Let μ be a Borel measure defined on R^n and let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a vector of nonnegative numbers such that $\alpha_1 + \dots + \alpha_m = 1$. In the spirit of Section 1.1, we say that a dissection $R^n = \bigcup_{i=1}^m F_i$ is an α -partition of the measure μ if $\mu(F_i) \geq \alpha_i \mu(R^n)$ for each i , $i = 1, \dots, m$. An α -partition is said to be *strict* if $\mu(F_i \cap F_j) = 0$ for $i \neq j$ in which case

$$\mu(F_i) = \alpha_i \mu(R^n), \quad i = 1, \dots, m.$$

Definition 2.2 (Radial Dissections). A pointed simplex is a pair (Δ, a) where $\Delta = \text{conv}\{a_0, \dots, a_n\}$ is a nondegenerated simplex in R^n and $a \in \text{int}(\Delta)$ is a point in its interior. Let $R_i = \text{cone}(a, \Delta_i) := \bigcup_{\lambda \geq 0} (a + \lambda(\Delta_i - a))$ be the cone with the vertex at the point a where $\Delta_i = \text{conv}\{a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$ is a facet of Δ . The dissection of R^n into $(n+1)$ convex cones $R_0, R_1, \dots, R_n$ is called a radial (Δ, a) -dissection and denoted by $\mathcal{R}_{(\Delta, a)}$.

Definition 2.3 (Extended Radial Dissections). Let (Δ, a) be a pointed simplex as before and let $\mathcal{R}_{(\Delta, a)}$ be the associated radial (Δ, a) -dissection of R^n defined in the previous definition. The extended radial (Δ, a) -dissection $\mathcal{E}_{(\Delta, a)}$, associated to the pointed simplex (Δ, a) , is defined by

$$\mathcal{E}_{(\Delta, a)} = \{E_0, E_1, \dots, E_n, E_{n+1}\},$$

where $E_i := R_i \setminus \text{int}(\Delta)$ for $i \leq n$ and $E_{n+1} := \Delta$. More generally, any extended radial dissection of R^n of the form $\mathcal{E}^L := \mathcal{E}_{(L(\Delta), L(a))} = \{E_0^L, E_1^L, \dots, E_n^L, E_{n+1}^L\}$, where $L: R^n \rightarrow R^n$ is a positive similarity transformation $x \mapsto \beta x + b$ ($\beta > 0, b \in R^n$), is called an *extended radial* dissection of type (Δ, a) .

Theorem 2.4. Let ν be an admissible probability measure on R^n (Definition 2.1) and let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ be a nonnegative vector such that $\alpha_0 + \alpha_1 + \dots + \alpha_n = 1$. Let (Δ, a) be a pointed, nondegenerated simplex $\Delta = \text{conv}\{a_0, a_1, \dots, a_n\}$, $a \in \text{int}(\Delta)$. Then there exists a vector $v \in R^n$ such that the radial dissection $\mathcal{R}_{(\Delta+v, a+v)} = \{R_0 + v, R_1 + v, \dots, R_n + v\}$ is an α -partition of ν . In other words, if $R_i = \text{cone}(a, \Delta_i)$, where $\Delta_i = \text{conv}\{a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$, then

$$\nu(R_i + v) \geq \alpha_i \cdot \nu(R^n) = \alpha_i.$$

Proof. As remarked at the beginning of Section 2, it is sufficient to establish the result in the case of a continuous measure ν , i.e., a measure of the form $d\nu = f \, dm$ for some Lebesgue measurable function f . It is natural to approach the proof of the theorem following the usual *configuration space/test map* procedure for applying topological methods in combinatorics, see [23]–[25] for a systematic exposition of these ideas. In our case the configuration space $X_{(\Delta,a)} = \{\mathcal{R}_\nu\}_{\nu \in R^n}$ of all radial dissections $\mathcal{R}_\nu := \mathcal{R}_{(\Delta+\nu,a+\nu)} = \{R_0 + \nu, R_1 + \nu, \dots, R_n + \nu\}$ is homeomorphic to R^n . The test map $f: X_{(\Delta,a)} \rightarrow R^{n+1}$ is defined by $f(\mathcal{R}_\nu) = (\nu(R_0 + \nu), \nu(R_1 + \nu), \dots, \nu(R_n + \nu))$. Note that the continuity of this map is precisely the reason why we had to assume that the measure ν is continuous. The problem is now reduced to the proof that $\alpha \in \text{Image}(f)$. In the usual *configuration space/test map* procedure [23, p. 115], the inner symmetries of the problem are supposed to show up at this stage turning f into an equivariant map with respect to an appropriate group. Unfortunately, in our case there are no symmetries in the problem whatsoever and the *configuration space/test map* procedure cannot be applied in its usual manner!

However, we can still complete the proof by a degree theoretic argument which is nothing more than the essence of the usual homological proof of the Brouwer fixed point theorem. Assume that ν has a bounded support, i.e., that $\nu(B) = 1$ for some bounded set $B \subset R^n$. We also assume that $\Delta := \text{conv}\{a_0, a_1, \dots, a_n\}$ is a simplex with the base point at the origin ($a = 0$) and that $B \subset \text{int}(\Delta)$. Hence, the radial dissection $\mathcal{R}_{(\Delta,a)} = \{R_0, R_1, \dots, R_n\}$ corresponds to the fan associated to the simplex Δ where $R_i = \text{cone}(\Delta_i)$ and Δ_i is the facet of Δ opposite the vertex a_i . Let $\partial\Delta$ be the boundary of Δ and let $\hat{f}: \partial\Delta \rightarrow R^n$ be a modified test map defined by

$$\hat{f}(x) = \sum_{i=0}^n \nu(R_i + x) a_i.$$

If $x \in \Delta_i$, then $(R_i + x) \cap \text{int}(\Delta) = \emptyset$ which implies that $\hat{f}(x) \in \Delta_i$. This shows that $\hat{f}$ is a self-map of $\partial\Delta$ and, since Δ_i is convex, $\hat{f}$ is linearly homotopic to the identity map $\text{Id}: \partial\Delta \rightarrow \partial\Delta$. Hence, $\deg(\hat{f}) = +1$. From here we deduce that $\Delta \subseteq \text{Image}(\hat{f})$, otherwise $\hat{f}: \partial\Delta \rightarrow \partial\Delta$ would be homotopic to a constant map and have degree zero. In particular, $\alpha_0 a_0 + \dots + \alpha_n a_n \in \text{Image}(\hat{f})$ and the result follows. $\square$

Remark 2.5. Theorem 2.4 can be seen as an “asymmetrical” version and a generalization of Proposition 1.2 from [21]. The proof of that proposition is based on Kakutani’s fixed point theorem and it is not difficult to see that this proof can be easily modified to yield a proof of Theorem 2.4. One of the reasons we chose instead a degree theoretic argument is that, in light of the parametrized index theorem from [10], it opens a possibility of proving vector bundle analogs and extensions of these results in the spirit of the general “*Combinatorial geometry on vector bundles*” program formulated in [26].

Theorem 2.6. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n, \alpha_{n+1})$ be a positive vector such that $\alpha_0 + \dots + \alpha_{n+1} = 1$. Let μ be an admissible probability measure on R^n (Definition 2.1) such that for each $u \in R^n$, $\mu(\{u\}) < \alpha_{n+1}$. Let (Δ, a) be a nondegenerated, pointed simplex $\Delta = \text{conv}\{a_0, \dots, a_n\}$ with $a \in \text{int}(\Delta)$. Then there exists an extended radial dissection $\mathcal{E}^L = \{E_0^L, E_1^L, \dots, E_n^L, E_{n+1}^L\}$ (Definition 2.3) which is an α -partition of μ . In other

words, there exists a similarity transformation $L: R^n \rightarrow R^n$, $x \mapsto \beta x + b$, ($\beta > 0$, $b \in R^n$), such that

$$\mu(E_i^L) \geq \alpha_i \cdot \mu(R^n) = \alpha_i,$$

where $E_i^L = L(E_i)$ and $\mathcal{E}_{(\Delta, a)} = \{E_0, E_1, \dots, E_{n+1}\}$ is the extended, radial (Δ, a) -dissection from Definition 2.3.

Proof. We suppose that R^n is isometrically embedded in R^{n+1} , for example, R^n can be identified with the horizontal coordinate hyperplane $R^n := \text{span}\{e_1, e_2, \dots, e_n\}$, where $\{e_1, e_2, \dots, e_{n+1}\}$ is the standard basis of R^{n+1} . Let $\hat{\mu}$ be the measure on R^{n+1} which is concentrated on R^n where it agrees with μ . More precisely, for each Borel set $B \subset R^{n+1}$, let $\hat{\mu}(B) := \mu(B \cap R^n)$. Let $(\hat{\Delta}, \hat{a})$ be the pointed simplex in R^{n+1} defined by $\hat{\Delta} = \text{conv}(\Delta \cup \{a + e_{n+1}\})$, $\hat{a} := a + \frac{1}{2}e_{n+1}$. Let $\hat{\Delta}_i$ for $i = 0, 1, \dots, n$ be the face of $\hat{\Delta}$ opposite the vertex $a_i \in \Delta \subset \hat{\Delta}$ and for $i = n+1$, let $\hat{\Delta}_{n+1} := \Delta$ be the face of $\hat{\Delta}$ opposite the vertex $a + e_{n+1} \in \hat{\Delta}$. Then by Theorem 2.4 applied on the measure $\hat{\mu}$ there exists a vector $v \in R^{n+1}$ such that the radial dissection

$$\mathcal{R}_{(\hat{\Delta}+v, \hat{a}+v)} = \{R_0 + v, R_1 + v, \dots, R_{n+1} + v\},$$

where $R_i = \text{cone}(\hat{a}, \hat{\Delta}_i)$, is an α -equipartition of $\hat{\mu}$. In other words,

$$\mu((R_i + v) \cap R^n) = \hat{\mu}(R_i + v) \geq \alpha_i$$

for all $i = 0, 1, \dots, n+1$. Note that $\hat{a} + v \notin R^n$. Indeed, otherwise we would have $\alpha_{n+1} \leq \hat{\mu}(R_{n+1} + v) = \mu((R_{n+1} + v) \cap R^n) = \mu(\{\hat{a} + v\})$ which would contradict the assumption that $\mu(\{u\}) < \alpha_{n+1}$ for each $u \in R^n$. Since $\hat{a} + v \notin R^n$, it is easy to see that $\mathcal{E} := \{(R_i + v) \cap R^n\}_{i=0}^{n+1}$ is an extended, radial dissection of type (Δ, a) which forms an α -partition of μ . This completes the proof of the theorem. $\square$

Corollary 2.7. *Let A be a Lebesgue measurable set in R^n . More generally, let μ be a continuous mass distribution on R^n associated to a nonnegative, integrable function $f: R^n \rightarrow R$, i.e., $\mu(B) = \int_B f \, dm$ for each measurable $B \subset R^n$. Let (Δ, a) be a nondegenerated, pointed simplex $\Delta = \text{conv}\{a_0, a_1, \dots, a_n\}$, $a \in \text{int}(\Delta)$. Then there exists a (positive) similarity transformation $L: R^n \rightarrow R^n$, $L(x) = \beta x + b$ ($\beta > 0$, $b \in R^n$) such that the extended radial dissection $\mathcal{E}_{(L(\Delta), L(a))} = \{E_0^L, E_1^L, \dots, E_{n+1}^L\}$ of R^n , associated to the pointed simplex $(L(\Delta), L(a))$, is a strict μ -equipartition in the sense that*

$$\mu(E_0^L) = \mu(E_1^L) = \dots = \mu(E_{n+1}^L) = 1/(n+2)\mu(R^n).$$

Corollary 2.8. *Let S be a finite set in R^n which has at least two elements, $|S| \geq 2$. Let μ_S be the normalized counting measure associated to S defined by $\mu_S(B) = |B \cap S|/|S|$. Then for each nondegenerated pointed simplex (Δ, a) , there exists a positive similarity transformation $L: R^n \rightarrow R^n$, $L(x) = \beta x + b$ ($\beta > 0$, $b \in R^n$), such that the extended radial dissection $\mathcal{E}_{(L(\Delta), L(a))} = \{E_0^L, E_1^L, \dots, E_{n+1}^L\}$, associated to the pointed simplex $(L(\Delta), L(a))$, is a μ_S -equipartition of R^n .*

Fig. 1

Proof. If $|S| \geq n + 3$, then $\mu_S(\{u\}) \leq 1/(n + 3)$ for each $u \in R^n$ and the result follows from Theorem 2.6. We show that the result is still true if $2 \leq |S| \leq n + 2$. Let $u, v \in S$ be two distinct points and let $u - v = \sum_{j=0}^n \lambda_j a_j$, $\sum_{j=0}^n \lambda_j = 1$ be the barycentric representation of $u - v$ with respect to the simplex Δ . Then $u - v$ also has a representation $u - v = \sum_{j=0}^n \alpha_j a_j$ where $\alpha_j := \lambda_j - 1/(n + 1)$, so in particular $\sum_{j=0}^n \alpha_j = 0$. Let $I_+ = \{j \mid \alpha_j > 0\}$, $I_- = \{j \mid \alpha_j < 0\}$ and let $F_+ = \text{conv}\{a_j \mid j \in I_+\}$ and $F_- = \text{conv}\{a_j \mid j \in I_-\}$ be the corresponding faces of Δ . Let $\alpha := \sum_{j \in I_+} \alpha_j = -\sum_{j \in I_-} \alpha_j$, and $u_1 \in F_+$, $v_1 \in F_-$ be the vectors defined by $u_1 = (1/\alpha) \sum_{j \in I_+} \alpha_j a_j$, $v_1 = (1/\alpha) \sum_{k \in I_-} (-\alpha_k) a_k$. Since $u_1 - v_1 = (1/\alpha)(u - v)$, we observe that there exists a positive similarity transformation $L: R^n \rightarrow R^n$, $L(x) = (1/\alpha)x + b$, for some $b \in R^n$, such that $L(u_1) = u$ and $L(v_1) = v$. Let $\mathcal{E}_{(L(\Delta), L(a))} = \{E_0^L, E_1^L, \dots, E_{n+1}^L\}$ be the extended radial dissection of R^n associated to the pointed simplex $(L(\Delta), L(a))$. Each E_i^L contains a facet of $L(\Delta)$. Since u and v belong to disjoint faces $L(F_+)$ and $L(F_-)$ of $L(\Delta)$, each facet of $L(\Delta)$ contains at least one of these two points. From here we deduce that for each $i = 0, 1, \dots, n + 1$, $\mu_S(E_i^L) \geq \mu_S(\{u\}) = \mu_S(\{v\}) = 1/|S| \geq 1/(n + 2)$ which completes the proof. $\square$

As a “variation on a theme” we prove one more result about radial equipartitions for single measures in R^n . The proof relies on the following form of the Brouwer fixed point theorem.

Theorem 2.9. *Let K be a compact, convex body in R^n . Suppose that $f: K \rightarrow R^n$ is a continuous map such that for each $x \in \partial K$, the image $f(x)$ belongs to the supporting cone $\text{cone}_x(K)$ of K at point x ,*

$$f(x) \in \text{cone}_x(K) := \bigcup_{\lambda \geq 0} (x + \lambda(K - x)). \quad (2)$$

Then $f(x) = x$ for some $x \in K$.

Proof. Let $B \subset R^n$ be a ball such that $K \cup f(K) \subset B$. Let $v: B \rightarrow K$ be the “nearest point map,” i.e., $y = v(x)$ is the unique point $y \in K$ such that $d(x, y) = d(x, K)$. It is elementary to check that v is well defined and continuous. Let $\varphi := f \circ v: B \rightarrow B$. By the usual form of the Brouwer fixed point theorem, $\varphi(x) = x$ for some $x \in B$. If

$x \notin K$, then $y = v(x) \in \partial K$. Let H be the supporting hyperplane of K at y which is orthogonal to the line segment $[x, y]$ and let H^+ be the associated closed halfspace such that $K \subset H^+$. Then by (2), $\varphi(x) = f(v(x)) \in \text{cone}_{v(x)} K \subset H^+$, which implies that $x \neq \varphi(x)$. Hence, $x \in K$, $v(x) = x$, and $f(x) = f(v(x)) = \varphi(x) = x$. $\square$

Theorem 2.10. *Let $\Delta = \text{conv}\{a_0, a_1, \dots, a_n\}$ be a nondegenerated simplex in R^n . Let $\Delta_i = \text{conv}\{a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$ be the facet of Δ opposite to the vertex a_i , $i = 0, \dots, n$. Let μ be an admissible probability measure on R^n such that $\text{supp}(\mu) \subseteq \text{int}(\Delta)$, i.e., such that*

$$\mu(\Delta') = \mu(R^n) = 1 \quad \text{for some simplex } \Delta' \subset \text{int}(\Delta) \quad (3)$$

and let $\alpha = (\alpha_0, \dots, \alpha_n)$ be a positive vector such that $\alpha_0 + \dots + \alpha_n = 1$. Then there exists $x \in \text{int}(\Delta)$ such that the associated dissection $\hat{\mathcal{R}}_{(\Delta, x)} = \{R_{0,x}, R_{1,x}, \dots, R_{n,x}\}$, where $R_{i,x} := \text{cone}(x, \Delta_i)$, is an α -partition of μ .

Proof. We apply Theorem 2.9 on the map $f: \Delta \rightarrow R^n$ defined by

$$f(x) := (\lambda_0(x) + \alpha_0 - \mu(R_{0,x}))a_0 + \dots + (\lambda_n(x) + \alpha_n - \mu(R_{n,x}))a_n, \quad (4)$$

where $\{\lambda_i(x)\}_{i=0}^n$ are the barycentric coordinates of a point $x \in \Delta$. Strictly speaking the cone $R_{i,x} = \text{cone}(x, \Delta_i)$ is not well defined if $x \in \partial\Delta$ but this does not cause a problem since condition (3) guarantees that $\mu(R_{i,x})$ is well defined. In order to ensure the continuity of $\mu(R_{i,x})$ as a function of $x \in \Delta$ and so the continuity of f , we assume that μ is a continuous measure. As before, a limit argument allows us to extend the result to a general admissible measure. It is obvious that $x \in \Delta$ is a fixed point of f if and only if $\mathcal{R}_{(\Delta, x)}$ is a radial α -partition of R^n . So it suffices to show that f satisfies condition (2) of Theorem 2.9. Let $x \in \partial\Delta$. The supporting cone of Δ at the point x has the following simple description:

$$\text{cone}_x(\Delta) = \{y \in R^n \mid (\forall j \in I_0) \lambda_j(y) \geq 0\}, \quad (5)$$

where $I_0 := \{j \mid \lambda_j(x) = 0\}$. Since for each $j \in I_0$, $\lambda_j(x) = \mu(R_{j,x}) = 0$, we have $\lambda_j(f(x)) = \alpha_j \geq 0$ which shows that $f(x)$ satisfies the condition defining $\text{cone}_x(\Delta)$ in (5). Hence, condition (2) is also satisfied and the proof of the theorem is complete. $\square$

3. Common Radial Dissection for Several Measures

The motivating problem of Kano mentioned in the Introduction reads as follows. Given $3b$ blue and $3r$ red points in the plane, is it possible to find a point and three halflines starting from this point so that each of the three sectors contains exactly b blue and r red points? At the conference Geometric Combinatorics, Kotor '98, Bárány presented a proof of a result, based on his joint work with Matoušek, giving essentially a positive answer to Kano's question.

Of course the result of Bárány and Matoušek is naturally formulated for other classes of measures. As in the first section, the class of admissible measures (Definition 2.1)

Fig. 2

seems to be the most convenient class of measures for all equipartition problems. In this generality Kano's question is seen as the problem of the existence of a common radial equipartition for two or more measures in R^n .

As emphasized in the Introduction we view the result of Bárány and Matoušek (Theorem 3.3), and other related results of this type, as offsprings of "symmetric" equipartition results proved or conjectured in [21]. In order to formulate these statements, we need a preliminary definition where the radial dissections, defined in Definition 2.2, are extended to the case when the pointed simplex (Δ, a) and the ambient space R^n are not of the same dimension.

Definition 3.1. Let (Δ, a) be a pointed, nondegenerated simplex in R^n where $\Delta = \text{conv}\{a_0, \dots, a_m\}$, $a \in \text{rel int}(\Delta)$, and $\dim(\Delta) = m \leq n$. As before, let Δ_i be the face of Δ opposite the vertex a_i , let $p := \text{aff}(\Delta)$ be the affine hull of Δ , and let $p^\perp$ be the linear subspace of R^n orthogonal to p . Let R_i be the wedge-like cone defined by $R_i := p^\perp + \text{cone}(a, \Delta_i)$. Given an m -dimensional simplex $\Delta \subseteq R^n$, the associated radial dissection $\mathcal{R}_{(\Delta, a)} := \{R_i\}_{i=0}^m$ will be simply called the radial (Δ, a) -dissection. If (Δ, a) is a regular simplex and $a = 1/(m+1)(a_0 + \dots + a_m)$ its baricenter, the dissection $\mathcal{R}_\Delta = \mathcal{R}_{(\Delta, a)}$ is referred to as the *regular*, radial dissection associated to Δ . Note that for $m = 1$ we obtain a hyperplane dissection of R^n .

Conjecture $B(n, k)$ [21]. Let $\mu_0, \mu_1, \dots, \mu_k$ be a family of admissible, probability measures, $0 \leq k \leq n-1$, where admissible means that all measures belong to the class $\mathcal{A}$ defined in Definition 2.1. Then there exists an $(n-k)$ -dimensional, regular simplex Δ such that the associated regular, radial dissection $\mathcal{R}(\Delta) = \{R_i\}_{i=0}^{n-k}$ of R^n is a μ_i -equipartition for each $i = 0, \dots, k$. In other words, $\mathcal{R}_\Delta$ has the property that for every $i, 0 \leq i \leq k$, and $s, 0 \leq s \leq n-k$, $\mu_i(R_s) \geq 1/(n-k+1)$. Specially, if all measures are continuous, then $\mu_i(R_s) = \mu_i(R_t)$ for all $i, 0 \leq i \leq k$, and $s, t, 0 \leq s, t \leq n-k$.

This conjecture was reduced in [21] to a purely topological conjecture $C(n, k)$ stating that a naturally defined vector bundle does not admit a nowhere zero continuous cross-section. In that paper we were able to establish $C(n, k)$ and the conjecture $B(n, k)$ in the

case $k = n - 2$ which led to the following result, see Theorem 2.2 and Corollary 2.3 of [21].

Theorem 3.2. *The statement $B(n, n - 2)$ is true. In other words, if $\mu_1, \dots, \mu_{n-1}$ is a family of admissible, probability measures defined in R^n , then there exists a two-dimensional regular simplex Δ such that the associated dissection $\mathcal{R}_\Delta := \{R_0, R_1, R_2\}$ of R^n is a μ_i -equipartition of R^n for each $i = 1, \dots, n - 1$. Equivalently, $\mathcal{R}_\Delta$ has the following property:*

$$\forall s = 0, 1, 2 \quad \text{and} \quad \forall i = 1, \dots, n - 1, \quad \mu_i(R_s) \geq \frac{1}{3}.$$

Theorem 3.3 [2]. *Let μ_1 and μ_2 be admissible probability measures in R^2 . Then there exist three sectors S_0, S_1, S_2 in R^2 determined either by three halflines starting from a common endpoint or with two parallel lines such that for each $j \in \{1, 2\}$ and each $k \in \{0, 1, 2\}$, $\mu_j(S_k) \geq \frac{1}{3}$.*

Proof. As in the proof of Theorem 2.6, we assume that R^2 is isometrically embedded in R^3 , say as a horizontal plane P . We also assume that both μ_1 and μ_2 are continuous measures on R^2 with compact support, i.e., measures of the form $d\mu_i = f_i dm$ where $f_i: R^2 \rightarrow R$ are Lebesgue measurable function which are zero away from a fixed compact set B . Let $\hat{\mu}_1$ and $\hat{\mu}_2$ be two measures defined on R^3 which are concentrated on $R^2 = P$ where they agree with μ_1 and μ_2 , respectively. In other words, for each measurable $B \subset R^3$, $\hat{\mu}_i(B) := \mu_i(B \cap R^2)$. It is clear that measures $\hat{\mu}_i$ are admissible in R^3 but they are certainly not continuous. So it will be convenient to introduce two sequences of measures $\hat{\mu}_1^k$ and $\hat{\mu}_2^k$ on R^3 obtained as the product of measures μ_1 and μ_2 respectively with uniformly distributed measures on the interval $[-1/k, +1/k]$ contained in the line which is the orthogonal complement of P in R^3 .

By applying the statement $B(3, 1)$, Theorem 3.2, on measures $\hat{\mu}_1^k$ and $\hat{\mu}_2^k$, we observe that there exists a two-dimensional dissection $\mathcal{R}^k = \{R_0^k, R_1^k, R_2^k\}$ of R^3 , associated to a regular 2-simplex $\Delta_k = \text{conv}\{a_0^k, a_1^k, a_2^k\}$, such that $\hat{\mu}_i^k(R_s) \geq \frac{1}{3}$ for each $s, 0 \leq s \leq 2$, and $i, 1 \leq i \leq 2$. Actually, the continuity of measures $\hat{\mu}_1^k$ and $\hat{\mu}_2^k$ guarantees the equalities $\hat{\mu}_i(R_s) = \frac{1}{3}$. A compactness argument allows us to assume that the sequence of two simplices Δ_k converges to a 2-simplex Δ and let $\mathcal{R} = \{R_0, R_1, R_2\}$ be the associated regular, radial dissection of R^3 . This dissection is obviously a $\hat{\mu}_i$ -equipartition of R^3 for $i \in \{1, 2\}$.

Let l be the line $l = R_0 \cap R_1 \cap R_2$. Obviously, by the definition of $\hat{\mu}_i$, the intersection of the dissection $\mathcal{R}$ with P forms a, possibly degenerated, equipartition of $R^2 = P$ with respect to both measures μ_1 and μ_2 .

Depending on the mutual position of l and p we have the following three cases:

1. $l \cap P = \{p\}$. In this case one easily verifies that $R_0 \cap P, R_1 \cap P$, and $R_2 \cap P$ are the three sectors of P formed by three halflines starting from the point p , having the same measures with the respect to both μ_1 and μ_2 .
2. $l \parallel P$ and $l \not\subset P$. In this case it can be also easily seen that $R_0 \cap P, R_1 \cap P$, and $R_2 \cap P$ are three sectors in P formed by two parallel lines having the same measure with the respect to both measures μ_1 and μ_2 .

3. $l \subseteq P$. By the definition of $\hat{\mu}_i$, we see that one of the halfplanes forming a dissection should be contained in P . Otherwise the wedge-like cone R_{j_0} bounded by two halfplanes on the same side of P would have the property $R_{j_0} \cap P = l$. This would lead to a contradiction. Indeed, both measures μ_1 and μ_2 are assumed to be continuous on P , hence $\mu_1(l) = \mu_2(l) = 0$. From here we deduce $\mu_1(R_{j_0} \cap P) = \mu_1(l) = 0$ which is in contradiction with $\mu_i(R_{j_0}) \geq \frac{1}{3}$.

Now we turn our attention to the fact that the dissection $\mathcal{R} = \{R_0, R_1, R_2\}$ arose as a limit of dissections $\mathcal{R}^k = \{R_0^k, R_1^k, R_2^k\}$. Since the measures $\hat{\mu}_i^k$ are continuous, every wedge-like cone of the dissection $\mathcal{R}^k$ contains *exactly* one-third of both measures μ_i^k in R^3 . Let R_{i_0} be the wedge-like cone in the dissection $\mathcal{R}$ formed by two halfplanes which do not belong to P . By construction of measures $\hat{\mu}_i^k$, we deduce that $\hat{\mu}_1(R_{i_0}) = \hat{\mu}_2(R_{i_0}) = \frac{1}{3}$. Let P^- and P^+ be the two halfplanes in P determined by l where $P^- \subset R_{i_0}$. So, $\mu_1(P^-) = \mu_2(P^-) = \frac{1}{3}$ and $\mu_1(P^+) = \mu_2(P^+) = \frac{2}{3}$. Now we apply the “ham sandwich theorem” to the part of measures μ_1 and μ_2 contained in the halfplane P^+ . This way we obtain another line l' in P which divides P^+ in two parts each having one-third of both measures μ_1 and μ_2 . If $l \parallel l'$, these two parallel lines form the dissection of P with the required properties. If $l \cap l' = \{q\}$, three halflines starting from q (both halflines of l and the halfline of l' contained in P^+) form the dissection with the required properties.

We have established the result for continuous measures μ_1 and μ_2 which have compact supports. An approximation argument leads to the result for general admissible measures. $\square$

Using exactly the same idea and the statement $B(n, n - 2)$, we obtain the following result.

Theorem 3.4. *Let $\mu_1, \dots, \mu_n$ be admissible probability measures in R^n . Then there are three sectors S_0, S_1, S_2 in R^n determined either with an $(n - 2)$ -dimensional flat F and three half-hyperplanes bounded by F , or with two parallel hyperplanes such that for $i \in \{1, \dots, n\}$ and $s \in \{0, 1, 2\}$ we have $\mu_i(S_s) \geq \frac{1}{3}$.*

The proof of this statement is essentially the same as the proof of Theorem 3.3 so we omit the details.

4. Equipartitions by Pseudolines

In this section we provide further evidence that Theorem 3.3 is a result of topological nature. This is demonstrated by showing that the result remains true in the case of fairly general *topological planes*. Our proof in its key steps follows the original argument of Bárány and Matoušek from [2] and the main additional ingredient is the observation that the space of all flags in a topological projective plane is homeomorphic to the tangent circle bundle on P^2 . We warn the reader that we are not able to present a simple proof of this result for the case of general topological projective planes and this is the

main reason why we restrict our attention to a restricted class of planes described in Proposition 4.3.

Recall that a topological (projective) plane $Q = (P^2, \mathcal{L})$ is the usual projective plane P^2 together with a collection $\mathcal{L}$ of *pseudolines* which obey the basic incidence axioms of geometry.

In this paper we assume that a pseudoline is a simple closed curve which is *piecewise smooth*, i.e., it consists of a finite number of smooth curves so that at each point the left and right tangent vectors never point in the same direction. In other words, cusps of the form $y^2 = x^3$ are not allowed. A more general definition would require that pseudolines are only Jordan curves.

Assuming this definition of a pseudoline, a usual definition, see [11], of a topological plane Q is the following.

Definition 4.1. A pair $Q = (P^2, \mathcal{L})$ is a topological plane if the following conditions are satisfied:

1. P^2 is the usual projective plane and $\mathcal{L} = \mathcal{L}(Q)$ is a topological space consisting of simple, piecewise smooth closed curves (pseudolines) in P^2 playing the role of lines in Q ;
2. for any two distinct points $p, q \in P^2$, there is a unique pseudoline $c(p, q) \in \mathcal{L}(Q)$ containing both p and q ;
3. every two distinct pseudolines $l, l' \in \mathcal{L}(Q)$ intersect in exactly one point where they intersect transversally;
4. $c(p, q)$ varies continuously as a function of p and q ;
5. $l \cap l'$ varies continuously as a function of l and l' .

It was shown in [11] that every finite arrangement $\mathcal{A}$ of pseudolines in P^2 can be extended to a topological plane. It is well known that there exist many examples of *nonstretchable* arrangements of pseudolines, i.e., arrangements of pseudolines nonisomorphic to any arrangement of straight lines. Hence topological planes arising from nonstretchable arrangements are certainly not isomorphic to the usual projective plane. Here is another example of a topological plane which is used later to show that Theorem 3.3 also remains true in the hyperbolic plane $\mathcal{H}$.

Example 4.2. Let Δ be the Poincaré disk model of the hyperbolic plane. In other words, Δ is the interior of the unit disk in the usual euclidean plane E^2 while the hyperbolic lines are modeled as the arcs of circles orthogonal to the boundary $S^1 = \partial(\Delta)$ of Δ . Let $l = k \cap \Delta$ be the hyperbolic line which is obtained from a circle k orthogonal to the boundary of Δ and let p and q be the points in $k \cap \partial(\Delta)$. We extend l to a pseudoline l' in the euclidean plane by extending l' outside of Δ as the straight line $s(p, q)$ determined by p and q . For all other pairs (p, q) of points in the plane for which the associated straight line $s(p, q)$ does not intersect Δ , we take $s(p, q)$ itself to be the pseudoline associated to points p, q . If we add, as usual, the line at infinity, we obtain a topological plane $Q_{\mathcal{H}}$ which is nonisomorphic to the usual projective plane. We note that such modifications of topological planes were used already by Hilbert in his work on the foundations of geometry.

An usual model of the projective plane P^2 is the euclidean unit sphere S^2 with identified antipodal points x and $-x$. In other words, $P^2 = S^2/Z_2$ where the cyclic group acts by the antipodal action. We find it more convenient to work directly with the sphere S^2 than with the quotient space P^2 , so we essentially reformulate Definition 4.1 in terms of centrally symmetric pseudocircles on the sphere emphasizing the properties which will be needed in the proof of Theorem 4.6. Alternatively, Proposition 4.3 can be seen as an alternative definition of a topological plane.

Proposition 4.3. *Let $(S^2, \mathcal{F})$ be a sphere together with a collection $\mathcal{F}$ of pseudocircles satisfying the following conditions:*

1. *Each pseudocircle $c \in \mathcal{F}$ is an oriented, simple, piecewise smooth, centrally symmetric curve in S^2 . Moreover, it is assumed that for each point $x \in c$, the curve c is the image of a piecewise smooth map $f_{c,x}: [0, 2\pi] \rightarrow S^2$, satisfying $f(t) = -f(t + \pi)$, such that the parametrization $f_{c,x}$ is a constant times the arc length of c .*
2. *For each $x \in S^2$, the space $\mathcal{F}_x := \{c \in \mathcal{F} \mid x \in c\}$ is homeomorphic to the circle S^1 . More precisely, if we denote by S_x^1 the collection of unit tangent vectors to S^2 at x , then the map $\varphi_x: \mathcal{F}_x \rightarrow S_x^1$ defined by $\varphi_x(c) = v_x/|v_x|$ is 1–1 and onto, where $x = f_{c,x}(t_0)$ and $v_x := df_{c,x}(\partial/\partial t|_{t=t_0})$.*
3. *Let $\mathcal{S} = \{(x, c) \in S^2 \times \mathcal{F} \mid x \in c\}$ and let $UT(S^2)$ be the manifold of all unit tangent vectors in S^2 . Both spaces $\mathcal{S}$ and $UT(S^2)$ are naturally fibered over S^2 . We assume that the map $\Phi: \mathcal{S} \rightarrow UT(S^2)$ defined by $\Phi(x, c) := \varphi_x(c)$ is a homeomorphism.*

Under these conditions, there exists a topological plane $Q = (P^2, \mathcal{L})$ associated to $(S^2, \mathcal{F})$ such that $P^2 = S^2/Z_2$ and $\mathcal{L} = \{c/Z_2 \mid c \in \mathcal{F}\}$.

Remark 4.4. As remarked above, we see Proposition 4.3 as an alternative definition of a topological projective plane essentially equivalent to Definition 4.1. Consequently condition 3 in this proposition is seen as an observation that for any topological plane in the sense of Definition 4.1, the space $\mathcal{S}$ of all flags in the plane is homeomorphic to the unit tangent bundle over the projective plane P^2 . This bundle is naturally visualized as a quotient of the unit tangent bundle over the sphere S^2 . Conditions 1 and 2 imply that there is a natural action of the circle group S^1 on $\mathcal{S}$ such that $\mathcal{F} \cong \mathcal{S}/S^1$. From here one deduces that the space $\mathcal{F}$ of all pseudolines is homeomorphic to the projective plane P^2 . This is known to hold for more general topological planes, see [11] and the references therein. So it is reasonable to conjecture that part 3 of Proposition 4.3, which is the key assumption for Theorem 4.6, holds for general topological planes where the pseudolines are not necessarily piecewise smooth.

Definition 4.5. Let $c \in \mathcal{F}$ be a pseudocircle with the orientation inherited from the parametrization f_c . For an ordered pair of points in c , $p = f_c(t_0)$ and $q = f_c(t_1)$ where $0 \leq t_0 < t_1 \leq 2\pi$, let $[p, q]_c$ be the image in c of the interval $[t_0, t_1]$. If $x \in c$, then $[x, -x]_c$ is called a (pseudo) semicircle associated to x and denoted by c_x^+ or simply by c^+ if x is clear from the context.

Given an admissible measure μ on S^2 , a collection $\{c_1^+, c_2^+, c_3^+\}$ of semicircles associated to $x \in S^2$ is called an equipartition of μ if the sphere S^2 is divided by these semicircle into three closed regions D_1, D_2, D_3 such that

$$\mu(D_i) \geq 1/3\mu(S^2), \quad \text{for each } i = 1, 2, 3.$$

Theorem 4.6. *Let $(S^2, \mathcal{F})$ be a sphere with a collection $\mathcal{F}$ of pseudocircles satisfying conditions of Proposition 4.3. Let μ_1 and μ_2 be admissible measures defined on S^2 . Then there exists a point $x \in S^2$ and three pseudocircles c_1, c_2, c_3 in $\mathcal{F}$ containing x , such that the associated semicircles $\{c_1^+, c_2^+, c_3^+\}$ form an equipartition for both measures μ_1 and μ_2 .*

Proof. Without loss of generality we can assume that both measures μ_1 and μ_2 are continuous. We can also assume that μ_1 is “strictly positive” in the sense that $\mu_1(U) > 0$ for any nonempty open set $U \subset S^2$. Indeed, the limit argument used in Section 2 allows us to assume that both measures are continuous. For the second part we can replace μ_1 by the measure $\mu_1^\varepsilon = \mu_1 + \varepsilon \cdot \nu$, where ν is the invariant Lebesgue measure on S^2 , and appeal to the limit argument again.

By assumption, the space $\mathcal{S}$ of all pairs $(x, c) \in S^2 \times \mathcal{F}$, $x \in c$, is homeomorphic to the unit tangent vector bundle $UT(S^2)$ on the sphere S^2 . Let $\Psi: UT(S^2) \rightarrow \mathcal{S}$ be the homeomorphism inverse to Φ . Given $(x, a) \in UT(S^2)$, let $(x, c_{x,a}) := \Psi(x, a) \in \mathcal{S}$ where by assumption the function $c_{(\cdot, \cdot)}: UT(S^2) \rightarrow \mathcal{F}$ is continuous. If the “base point” x is clear from the context, we abbreviate $c_{x,a}$ by c_a .

By definition let $(x; a_1, a_2, a_3) \in \mathcal{E}$ if a_1, a_2, a_3 is an ordered triple of unit vectors (oriented counterclockwise and orthogonal to x) such that the ordered triple $\{c_{a_1}^+, c_{a_2}^+, c_{a_3}^+\}$ of pseudosemicircles associated to x, a_1, a_2, a_3 forms an equipartition for the measure μ_1 . In other words, $\mathcal{E}$ is the space of all equipartitions for the measure μ_1 alone. By the positivity assumption on μ_1 , the point $(x; a_1, a_2, a_3)$ is determined by x and any of the vectors a_1, a_2, a_3 which implies that $\mathcal{E}$ itself is also homeomorphic to $UT(S^2)$. Moreover, there is a continuous action of the cyclic group Z_3 on $\mathcal{E} \cong UT(S^2)$ which cyclically permutes vectors a_1, a_2, a_3 .

The theorem will be proved if we show that for some $x \in S^2$ and $a_1 \perp x$, the ordered triple $\{c_{a_1}^+, c_{a_2}^+, c_{a_3}^+\}$, associated to the point $(x; a_1, a_2, a_3) \in \mathcal{E}$, is an equipartition for the measure μ_2 . Let D_1, D_2, D_3 be the ordered triple of closed regions on the sphere separated by the semicircles $\{c_{a_1}^+, c_{a_2}^+, c_{a_3}^+\}$. We define a test function $T: UT(S^2) \rightarrow R^3$ by $T(x, a_1) := (\mu_2(D_1), \mu_2(D_2), \mu_2(D_3))$. Clearly, the function T is Z_3 -equivariant with respect to the obvious cyclic permutation action on R^3 . If $\Delta := \{(t, t, t) \mid t \in R\}$ is the diagonal in R^3 , then it is sufficient to show that $\text{Im}(T) \cap \Delta \neq \emptyset$. Assuming the opposite, there arises a Z_3 -equivariant map from $UT(S^2)$ to $R^3 \setminus \Delta$. If we observe that $R^3 \setminus \Delta$ is Z_3 -equivariantly homotopic to the unit circle in the plane $\Delta^\perp \subset R^3$ perpendicular to Δ , then there exists a Z_3 -equivariant map $F: UT(S^2) \rightarrow S^1$.

It is well known that such a map does not exist. Indeed, the question of the existence of a G -equivariant map of the form $F: P \rightarrow V \setminus \{0\}$ is known to be closely related to the question of the existence of a nonzero continuous section of the associated vector bundle $V \rightarrow P \times_G V \rightarrow P/G$, see [7] or Proposition 3.1 in [21]. It turns out that the desired statement is in our case equivalent to the statement $C(3, 1)$, which was already

mentioned in Section 3 and which is a special case of the statement $C(n, n-2)$ proved in [21]. $\square$

For the reader's convenience, we reproduce a nice direct elementary proof that a Z_3 -equivariant map $F: UT(S^2) \rightarrow S^1$ does not exist as given in [2]. We start with the observation that there does exist a Z_3 -equivariant map $e: S^1 \rightarrow UT(S^2)$. Indeed, $UT(S^2)$ is easily recognized as the space $V_{3,2}$ of all positive, orthonormal frames in R^3 , or equivalently as the group $SO(3)$ of all rotations in R^3 . Let e_1, e_2, e_3 be the usual orthonormal frame in R^3 . We denote by R_α , $0 \leq \alpha \leq 2\pi$, the rotation in $SO(3)$ which keeps e_1 fixed and rotates the y - z plane through the angle α . It is easily seen that the Z_3 -action on $SO(3)$, induced from the identification $UT(S^2) \cong SO(3)$, has the form $g \mapsto g \cdot R_{2\pi/3}$. Let $e: S^1 \rightarrow UT(S^2)$ be defined by $e(\alpha) = R_\alpha$. Then $g := F \circ e: S^1 \rightarrow S^1$ is also a Z_3 -equivariant map. It is an easy exercise to show that $\deg(g)$ must be congruent to 1 modulo 3, see [3] or [7] for more general results of this type. Hence, $g \circ g$ also has a nonzero degree, i.e., it represents a nontrivial map from S^1 to S^1 . This is a contradiction since $2e$ represents a trivial element in $\pi_1(SO(3)) \cong Z_2$ which means that $g \circ g$, representing the same element in $\pi_1(S^1)$ as $2g$, must also be trivial.

Corollary 4.7. *Let μ_1 and μ_2 be admissible probability measures defined in a hyperbolic plane in H^2 . Then there exist three sectors S_0, S_1, S_2 in H^2 , determined either by three halflines starting from a common endpoint or with two (hyper)parallel lines such that for each $j \in \{1, 2\}$ and each $k \in \{0, 1, 2\}$, $\mu_j(S_k) \geq \frac{1}{3}$.*

Proof. It suffices to apply Theorem 4.6 on the topological plane Q_H described in the Example 4.2. $\square$

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Received December 2, 1999, and in revised form July 1, 2000. Online publication February 1, 2001.