Erratum to: Signature Theory in Holographic Algorithms

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1. Theorem 6.2 of [4] contains an error. Specifically the condition in Theorem 6.2 is stated as both necessary and sufficient for a signature $G \in \{0, 1\}^n$ to be realizable as a matchgate signature under all basis transformations $\begin{bmatrix} 1 & x \\ 1 & -x \end{bmatrix}$. The condition is indeed sufficient, and the proof in Theorem 6.2 for this is valid; the error is the claim that it is also necessary.

On lines 16–17 of page 810, it claims that there must exist some i and $i + 1$, “both in X or both out of X , and one is in D and the other is out of D .” The reason given is that $D \neq \emptyset, [n], X, X^c$. This is incorrect. It is possible that $D \neq \emptyset, [n], X, X^c$, and yet all changes in membership from i to $i + 1$ in D are also changes in membership in X . This does not contradict $D \neq \emptyset, [n], X, X^c$ because there can be changes in membership in X that are not changes in membership in D . Here is a counterexample: $n = 8$, $X = \{2, 4, 6, 8\}$, $S = \{1, 2, 4, 5\}$, $S' = \{1, 4, 5, 6\}$, $D = S \oplus S' = \{2, 6\}$. Because of this error, we cannot prove the orthogonality of the coefficient vectors in (41) and thus we cannot deduce $G^S G^{S \oplus X} = 0$ in (41).

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As the condition in Theorem 6.2 is still sufficient, Corollary 6.1 is still valid. All other parts of the paper [4] are correct, and to our knowledge, no other subsequent results of ours and others depend on this part of Theorem 6.2.

2. As pointed out (correctly) on page 806 in [4] that for a signature $G \in \{0, 1\}^n$, a necessary and sufficient condition for $\begin{bmatrix} 1 & x \\ 1 & -x \end{bmatrix}^{\otimes n} G$ to be a standard matchgate signature for all $x \neq 0$ is that

- $G^S = 0$, for all $|S| \neq n/2$, and
- $H_2^{\otimes n} G$ is a standard matchgate signature, where $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

3. In the following, we give a characterization of the realizability of G under H_2 .

Let Γ be a matchgate of arity n satisfying $\Gamma^{11\dots 1} \neq 0$. We may normalize it to $\Gamma^{11\dots 1} = 1$. Define an $n \times n$ skew-symmetric matrix B where its (i, j) entry, for $1 \leq i < j \leq n$, is $\Gamma^{11\dots 1 \oplus e_i \oplus e_j}$, the signature value of Γ on the bit pattern that has two 1's at the i -th and j -th bit positions and 0 elsewhere. The theory of matchgate signatures ([3], see also [1, 2]) implies that for any $T \subseteq [n]$, $\Gamma^T = \text{Pf}(B[T])$, where Γ^T is the signature value for the bit pattern T denoted by its characteristic sequence, Pf denotes Pfaffian, $B[T]$ is the principal minor of B with all rows and columns in T removed.

Consider the transformation of Γ by the Hadamard matrix H_2 which is orthogonal up to a scalar $\frac{1}{\sqrt{2}}$. Let $G = H_2^{\otimes n} \Gamma$, then for all $S \subseteq [n]$,

$$G^S = \sum_{T \subseteq [n]} (-1)^{|S \cap T|} \Gamma^T.$$

If B is any $n \times n$ skew-symmetric matrix, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are n indeterminants, then Valiant [5] defined the Pfaffian Sum $\text{PfS}(B)$ to be a polynomial $\sum_{T \subseteq [n]} \left(\prod_{i \in T} \lambda_i \right) \text{Pf}(B[T])$.

Now for any $S \subseteq [n]$ we define a sequence of values

$$\lambda_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{otherwise} \end{cases}$$

then for these values of λ_i , and the B defined from Γ ,

$$\text{PfS}(B) = \sum_{T \subseteq [n]} (-1)^{|S \cap T|} \text{Pf}(B[T]) = \sum_{T \subseteq [n]} (-1)^{|S \cap T|} \Gamma^T = G^S. \quad (1)$$

By the Pfaffian Sum Theorem of [5] this can be expressed as a single Pfaffian. Suppose n is even, then define an $n \times n$ skew-symmetric matrix $\Lambda^{(n)}(S)$ with its (i, j) entry $(-1)^{j-i-1} \lambda_i \lambda_j = (-1)^{j-i-1+x_S(i)+x_S(j)}$ for $1 \leq i < j \leq n$. This matrix has the form

$$\begin{bmatrix} (-1)^{x_S(1)} & 0 & \dots & 0 \\ 0 & (-1)^{x_S(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (-1)^{x_S(n)} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & \dots & 1 \\ -1 & 0 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & -1 & \dots & 0 \end{bmatrix} \begin{bmatrix} (-1)^{x_S(1)} & 0 & \dots & 0 \\ 0 & (-1)^{x_S(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (-1)^{x_S(n)} \end{bmatrix} \quad (2)$$

This is the skew-symmetric matrix with the strict alternating pattern of ± 1 starting each row with $+1$ in the upper triangular half, *modified* by -1 at every (i, j) with $|\{i, j\} \oplus S| = 1$. Valiant's Pfaffian Sum Theorem in [5] states that $G^S = \text{PfS}(B) = \text{Pf}(B + \Lambda^{(n)}(S))$.

If n is odd, then $G = H_2^{\otimes n} \Gamma$ is also expressed as a Pfaffian Sum. We define B^+ to be the $(n+1) \times (n+1)$ matrix of which the first n rows and columns equal B , and the $(n+1)$ st row and column are all 0. Extend $\Lambda^{(n)}(S)$ to $\Lambda^{(n+1)}(S)$ with $\lambda_{n+1} = 1$. Then $G^S = \text{PfS}(B) = \text{Pf}(B^+ + \Lambda^{(n+1)}(S))$.

Finally, suppose $\Gamma^{11\dots 1} = 0$. If Γ is identically 0, then it can be represented by the Pfaffian of an all zero matrix. Suppose Γ is not identically 0, and $\Gamma^{T_0} = \lambda \neq 0$, for some $T_0 \subset [n]$. We can define a modified matchgate $\tilde{\Gamma}$ extending the i -th external node x_i by an edge (x_i, x'_i) of weight one, making x'_i the new external node, for every $i \notin T_0$. Also add one isolated edge with weight $1/\lambda$. Then the signature of $\tilde{\Gamma}$ satisfies $\Gamma^T = \lambda \tilde{\Gamma}^{T \oplus T_0^c}$, and $\tilde{\Gamma}^{11\dots 1} = \Gamma^{T_0}/\lambda = 1$. Now we can apply the construction above to matchgate $\tilde{\Gamma}$. More specifically, define $\tilde{B}$ from $\tilde{\Gamma}$, and $\Lambda^{(n)}(S)$ as before. Let $G = H_2^{\otimes n} \Gamma$, then for n even,

$$\begin{aligned} G^S &= \sum_{T \subseteq [n]} (-1)^{|S \cap T|} \Gamma^T \\ &= \lambda \sum_{T \subseteq [n]} (-1)^{|S \cap T|} \tilde{\Gamma}^{T \oplus T_0^c} \\ &= \lambda \sum_{T' \subseteq [n]} (-1)^{|S \cap T'|} (-1)^{|S \cap T_0^c|} \tilde{\Gamma}^{T'} \\ &= \lambda (-1)^{|S \cap T_0^c|} \text{Pf}(\tilde{B} + \Lambda^{(n)}(S)). \end{aligned}$$

For n odd, the same result follows using $\tilde{B}^+$ and $\Lambda^{(n+1)}(S)$ with $\lambda_{n+1} = 1$.

We summarize the discussion as follows:

Theorem 0.1 *A signature G is realizable as a matchgate signature under the Hadamard transformation H_2 iff it can be parameterized by an $n \times n$ skew-symmetric matrix as follows. If n is even, then there exist $\lambda \in \mathbb{C}$, $T_0 \subseteq [n]$, and $n \times n$ skew-symmetric matrix B , such that for all $S \subseteq [n]$,*

$$G^S = \lambda (-1)^{|S \cap T_0^c|} \text{Pf}(B + \Lambda^{(n)}(S)),$$

where $\Lambda^{(n)}(S)$ is defined in (2). If n is odd, then it is suitably modified as described above

$$G^S = \lambda (-1)^{|S \cap T_0^c|} \text{Pf}(B^+ + \Lambda^{(n+1)}(S)).$$

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