Improved Bounds on the Planar Branchwidth with Respect to the Largest Grid Minor Size

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Abstract For graph G , let $\text{bw}(G)$ denote the branchwidth of G and $\text{gm}(G)$ the largest integer g such that G contains a $g \times g$ grid as a minor. We show that $\text{bw}(G) \leq 3\text{gm}(G)$ for every planar graph G . This is an improvement over the bound $\text{bw}(G) \leq 4\text{gm}(G)$ due to Robertson, Seymour and Thomas. Our proof is constructive and implies quadratic time constant-factor approximation algorithms for planar graphs for both problems of finding a largest grid minor and of finding an optimal branch-decomposition: $(3 + \epsilon)$ -approximation for the former and $(2 + \epsilon)$ -approximation for the latter, where ϵ is an arbitrary positive constant. We also study the tightness of the above bound. We show that for any constant $c < 2$, the bound of $\text{bw}(G) \leq c \text{gm}(G) + o(\text{gm}(G))$ does not hold in general for a planar graph G .

Keywords Graph · Graph minor · Grid · Branchwidth · Branch-decomposition · Planar graph · Hypergraph

1 Introduction

The notion of branchwidth of a graph is introduced by Robertson and Seymour [18] in their graph minor theory. It is similar and has close relationship to the more celebrated notion of treewidth [17]. In particular, the linear bound of $\text{bw}(G) \leq \text{tw}(G) + 1 \leq \frac{3}{2}\text{bw}(G)$ holds for every graph G with more than one edge, where $\text{bw}(G)$ denotes the branchwidth of G and $\text{tw}(G)$ the treewidth of G . Both treewidth and branchwidth

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play essential roles in the graph minor theory. Informally, a *branch-decomposition* of G is a system of vertex cuts of G represented as edges of a tree whose leaves are the edges of G . The width of a branch-decomposition is the maximum cardinality of the vertex cuts in the system and the *branchwidth* of G is the minimum width of all possible branch-decompositions of G . See Sect. 2 for more formal definitions.

Grid minors also play an important role in the graph minor theory. Let G and H be graphs. H is a minor of G if H can be obtained from some subgraph of G through a possibly empty sequence of edge contractions. A $k \times h$ grid is a graph on vertex set $\{(i, j) \mid 0 \leq i < k, 0 \leq j < h, i, j : \text{integer}\}$, such that vertices (i, j) and (i', j') are adjacent with each other if and only if $|i - i'| + |j - j'| = 1$. We denote by $\text{gm}(G)$ the largest integer g such that G contains a $g \times g$ grid as a minor. Since the branchwidth of a $g \times g$ grid is g and $\text{bw}(G) \geq \text{bw}(H)$ holds if H is a minor of G , $\text{gm}(G)$ is a lower bound on $\text{bw}(G)$.

Robertson and Seymour [17] showed that there is a function f such that, for every graph G , $\text{tw}(G) \leq f(\text{gm}(G))$. This function f , which was enormous in this result, was improved by Robertson, Seymour, and Thomas [20] to $f(k) = 20^{2(k+1)^5}$. In fact, this bound is shown via a bound on the branchwidth, namely $\text{bw}(G) \leq 20^{2\text{gm}(G)(\text{gm}(G)+1)^4}$. In the same paper, these three authors also show a much better bound for planar graphs: every planar G satisfies $\text{bw}(G) \leq 4\text{gm}(G)$.¹

This linear bound of the form

$$\text{bw}(G) \leq c \text{gm}(G) + o(\text{gm}(G)) \quad (1)$$

has been extended to bounded genus graphs [7] and to graphs excluding a fixed minor [6], providing a combinatorial basis of the bidimensionality theory [5, 6, 8, 10]. Informally, a graph parameter $f(G)$ is *bidimensional*, if the value of f for $g \times g$ grid grows as g increases and $f(H) \leq f(G)$ if H is a minor of G . The bidimensionality theory provides a uniform framework for designing subexponential fixed parameter algorithms for computing bidimensional graph parameters. To decide if $f(G) > k$, we first determine or estimate $\text{bw}(G)$. If $\text{bw}(G)$ is sufficiently large then, owing to the linear bound, we may conclude that $f(G) > k$, since G contains a large grid minor. Otherwise, the small-width branch-decomposition of G is used in a dynamic programming algorithm to exactly compute $f(G)$, which typically runs in time polynomial in the size of G but exponential in $\text{bw}(G)$. Combined with the techniques of kernelization (see [11] for introduction), this approach often leads to subexponential fixed parameter algorithms.

Since the coefficient c in the linear bound (1) appears in the exponent of the running time of those algorithms derived from the bidimensionality framework, reducing this coefficient is of extreme importance. The derivations of the bound (1) for bounded genus graphs [7] and for fixed-minor free graphs [6] both depend on the bound for planar graphs and the coefficient c for these classes of graphs, as given by

¹ Their statement of the bound is that every planar graph with a tangle of order $4g - 3$ has a $g \times g$ grid minor. Equivalently, since G does not have a $g \times g$ grid minor with $g = \text{gm}(G) + 1$, every planar graph does not have a tangle of order $4(\text{gm}(G) + 1) - 3 = 4\text{gm}(G) + 1$, which implies that G admits a branch-decomposition of width $4\text{gm}(G)$.

those derivations, depends linearly on the coefficient for planar graphs. Therefore, it is natural to ask for the best possible coefficient in the bound (1) for planar graphs.

Our main results in this paper are the following. We improve the coefficient 4 given by [20] to 3.

Theorem 1.1 *For every planar graph G , we have $\text{bw}(G) \leq 3\text{gm}(G)$.*

We also show that the coefficient cannot be smaller than 2.

Theorem 1.2 *There is a family of planar graphs $\{G_h\}$, $h = 2, 3, \dots$, such that $\text{gm}(G_h) = h$ and $\text{bw}(G_h) = 2h$. Therefore, the coefficient c in the linear bound $\text{bw}(G) \leq c\text{gm}(G) + o(\text{gm}(G))$ for planar graphs must satisfy $c \geq 2$.*

The family $\{G_h\}$ in Theorem 1.2 is explicitly defined as follows. Let $k \geq 3$ and $h \geq 1$ be integers. A $k \times h$ cylinder, denoted by $C_{k,h}$, is a graph on vertex set $\{(i, j) \mid 0 \leq i < k, 0 \leq j < h, i, j : \text{integer}\}$, such that vertices (i, j) and (i', j') are adjacent if and only if $i' \equiv (i \pm 1) \pmod k$ and $j' = j$ or $i' = i$ and $|j - j'| = 1$. In other words, $C_{k,h}$ is the Cartesian product of a cycle on k vertices and a path on h vertices.

We show that $\text{bw}(C_{k,h}) = \min\{k, 2h\}$ and:

Theorem 1.3 *For every integer $h \geq 2$, we have $\text{gm}(C_{2h,h}) = h$.*

With $G_h = C_{2h,h}$, Theorem 1.2 follows from this theorem.

The proof of Theorem 1.1 uses an extension of some known upper bounds on the branchwidth of planar graphs and hypergraphs. These upper bounds are based on the “radius” of planar graphs [16], which roughly corresponds to the outerplanarity [1], and are first observed for the treewidth of planar graphs [2, 16] and later for the branchwidth of planar graphs [4, 22] and of planar hypergraphs [22]. Although our results are on planar graphs, our proof of Theorem 1.1 involves hypergraphs and requires a non-trivial extension of the bound of [22], which is embodied by Theorem 3.1 proved in Sect. 3. This extension may be of an independent interest.

The bound of the form (1) on the branchwidth of planar graphs implies similar bounds on the treewidth through the linear relation mentioned above. In particular, the bound $\text{bw}(G) \leq 4\text{gm}(G)$ of [20] implies $\text{tw}(G) \leq 6\text{gm}(G)$. Thomas [23] and Grigoriev [12] independently improve the constant in this bound to 5. Theorem 1.1 gives a better constant 4.5. Grigoriev *et al.* [13] study the tightness of these bounds for treewidth and conjecture that the best constant in this bound is 2. Theorem 1.3 verifies one side of their conjecture that the constant in the bound cannot be smaller than 2, since $\text{tw}(C_{2h,h}) = 2h = 2\text{gm}(C_{2h,h})$.

In algorithms derived in the bidimensionality framework mentioned above, we need to compute a large grid minor of the given graph, if we want not only to decide whether the parameter value exceeds k but also to obtain the evidence of a positive answer. It is not known whether a largest $(\text{gm}(G) \times \text{gm}(G))$ grid minor of a planar graph can be found in polynomial time. The linear bound of [20] implies a 4-approximation algorithm for this largest grid minor problem on planar graphs. Bodlaender, Grigoriev and Koster give an $O(n^2 \log n)$ time algorithm with the same approximation ratio for

this problem [3], where n is the number of vertices of G . Our proof for Theorem 1.1 is constructive and implies an $O(n^2)$ time $(3 + \epsilon)$ -approximation algorithm, where ϵ is an arbitrary positive constant.

Theorem 1.1 is a consequence of a slightly more general result.

Theorem 1.4 *Let G be a planar graph and k, h be integers with $k \geq 3$ and $h \geq 1$. Then G has either a minor isomorphic to a $k \times h$ cylinder or a branch-decomposition of width at most $k + 2h - 3$.*

Setting $h = k$ in this theorem gives Theorem 1.1 since a $k \times h$ grid is a subgraph of a $k \times h$ cylinder. Another interesting case is when $h = \lceil k/2 \rceil$. As we show in Sect. 6, the branchwidth of $C_{k, \lceil k/2 \rceil}$ is k . This motivates us to define another lower bound on the branchwidth: $\text{cm}(G)$ is the largest k such that G contains $C_{k, \lceil k/2 \rceil}$ as a minor. Then, our theorem implies that $\text{bw}(G) \leq 2\text{cm}(G)$ for planar graph G . Thus, $\text{cm}(G)$ is a better lower bound on $\text{bw}(G)$ than $\text{gm}(G)$ in the sense that it is provably tight within a factor of two for planar graphs.

In a related paper [15], based on the construction in Theorem 1.4, we develop efficient approximation algorithms for the largest grid minor and the optimal branch-decomposition of planar graphs. These algorithms give a trade-off between the running time and the approximation ratio. At one end, they run in $O(n^2)$ time and give a $(3 + \epsilon)$ -approximation for the largest grid-minor and a $(2 + \epsilon)$ -approximation for the optimal branch-decomposition. At the other end, they run in $O(n^{1+\epsilon})$ time and give constant-factor approximations where the constant is, roughly speaking, inversely proportional to ϵ . Although this approximation algorithm for branch-decomposition is less important than that for grid-minor construction, as a polynomial time algorithm for exact optimal decomposition is known for planar graphs [21], (see also [14] for an improvement on the running time), it is still significantly faster than the exact algorithm of [14], which runs in $O(n^3)$ time.

The organization of this paper is as follows. In the next section, we give some preliminaries to what follows. In Sect. 3, we prove Theorem 3.1, the radius-based upper bound. In Sect. 4, based on this bound, we prove Theorem 1.4. In Sect. 5, we show that the upper bound of Theorem 3.1 is tight. In Sect. 6, we determine the branchwidth of a cylinder and prove Theorem 1.3.

2 Preliminaries

Although our results are on graphs, we need to work with hypergraphs, in the inductive proof of Theorem 1.4. To obtain a branch-decomposition of given graph G , we find an appropriate separation of G , construct the branch-decompositions of the two separated parts, and then combine them into a branch-decomposition of G . As will be clear, in each of these “parts” we need to regard the other part as a hyperedge in order for this approach to work.

A *hypergraph* Γ is a pair of $V(\Gamma)$, the vertex set of Γ , and $E(\Gamma)$, the edge set of Γ , together with an incidence relation between $V(\Gamma)$ and $E(\Gamma)$. For each $e \in E(\Gamma)$, we denote by $V_\Gamma(e)$ the set of vertices in $V(\Gamma)$ that are incident with e . For $A \subseteq E(\Gamma)$, we

denote by $V_\Gamma(A)$ the set of vertices incident with edges in A : $V_\Gamma(A) = \bigcup_{e \in A} V_\Gamma(e)$. We denote by $\partial_\Gamma(A) = V_\Gamma(A) \cap V_\Gamma(E(\Gamma) \setminus A)$ the set of vertices incident with edges in A and with edges in $E(\Gamma) \setminus A$. The *order* of an edge e in Γ is the number $|V_\Gamma(e)|$ of vertices incident with e . A hypergraph Γ is a *graph* if every edge of Γ has order 2. Note that our definition of a graph allows parallel edges but not self-loops. A hypergraph Δ is a *subhypergraph* of a hypergraph Γ if $V(\Delta) \subseteq V(\Gamma)$, $E(\Delta) \subseteq E(\Gamma)$, and the incidence relation of Δ is a restriction of that of Γ on $V(\Delta) \times E(\Delta)$. Thus, for each $e \in E(\Delta)$, $V_\Delta(e) = V_\Gamma(e) \cap V(\Delta)$.

A *walk* on graph G is a sequence $v_0, e_1, v_1, \dots, e_k, v_k$, which alternates between vertices and edges such that vertex v_i is incident with edge e_i for $1 \leq i \leq k$ and with e_{i+1} for $0 \leq i < k$. We say the above walk is from v_0 to v_k . For each walk ω on G , we denote by $V(\omega)$ and $E(\omega)$ the set of vertices and the set of edges, respectively, appearing in ω . The *length* of a walk is the number of edge occurrences in the walk. We call a walk a *path* if it does not contain duplicate occurrences of a vertex in the sequence. A walk is *closed* if it starts and ends with the same vertex. A closed walk is a *cycle* if no vertex occurs in the walk more than once, except for the first/last vertex. If ω_1 is a walk to some vertex v and ω_2 is a walk from v , then we denote by $\omega_1 + \omega_2$ the concatenated walk, ω_1 followed by ω_2 . It may happen that $\omega_1 + \omega_2$ is a closed walk.

A *separation*, in this paper, of hypergraph Γ is an ordered pair (A, B) of subsets of edges of Γ , such that $A \cup B = E(\Gamma)$ and $A \cap B = \emptyset$. The *order* of separation (A, B) is the cardinality of $V_\Gamma(A) \cap V_\Gamma(B) = \partial_\Gamma(A) = \partial_\Gamma(B)$, the set of vertices shared by both sides of the separation.

A *branch-decomposition* of hypergraph Γ is a pair (T, ϕ) , where T is a tree each internal node of which has degree 3 and ϕ is a bijection from the set of leaves of T to $E(\Gamma)$. Consider an edge a of T and let L_1 and L_2 denote the sets of leaves of T in the two respective subtrees of T obtained by removing a . We say that the separation $(\phi(L_1), \phi(L_2))$ is associated with this edge a of T and also that this separation belongs to this branch-decomposition. We define the width of the branch-decomposition (T, ϕ) to be the largest order of the separations associated with edges of T or 0 if $|E(\Gamma)| \leq 1$ and hence T does not have any edge. The *branchwidth* of Γ , denoted by $\text{bw}(\Gamma)$, is the minimum integer w such that there is a branch-decomposition of Γ with width w . In the rest of this paper, we identify a branch-decomposition (T, ϕ) with the tree T , leaving the bijection implicit and regarding each leaf of T as an edge of Γ .

Let Γ be a hypergraph and (A, B) a separation of Γ . We denote by $\Gamma|A$ the hypergraph with vertex set $V_\Gamma(B)$ and edge set $B \cup \{e_A\}$, where e_A is an edge not in $E(\Gamma)$, such that each edge $e \in E(\Gamma|A) \setminus \{e_A\}$ is incident with $v \in V(\Gamma|A)$ if and only if e is incident with v in Γ and e_A is incident with each vertex in $\partial_\Gamma(A)$. Thus in $\Gamma|A$, all edges in A are replaced by a single edge e_A and each incidence of a vertex in $V_\Gamma(B)$ with an edge in A is kept as an incidence with e_A . The following lemma is straightforward from the definition of branch-decompositions and is implicit in the literature [18, 21]. It is the use of this lemma that makes us work on hypergraphs even though our main results are on graphs.

Lemma 2.1 *Let Γ be a hypergraph and (A, B) a separation of Γ . Let T_A be a branch-decomposition of $\Gamma|A$ of width w_A and T_B a branch-decomposition of $\Gamma|B$*

of width w_B . Let e_A and e_B be the edges replacing A and B , respectively, in $\Gamma|A$ and $\Gamma|B$. Let tree T be the result of concatenating T_A and T_B by identifying the leaf e_A of T_A and the leaf e_B of T_B and then ignoring the identified node to join the two incident edges into one. Then, T is a branch-decomposition of Γ of width $\max\{w_A, w_B\}$.

Corollary 2.1 Let Γ be a hypergraph and $\mathcal{A} = \{A_1, \dots, A_m\}$ a non-empty collection of mutually disjoint subsets of $E(\Gamma)$. For each $1 \leq i \leq m$, let $B_i = E(\Gamma) \setminus A_i$ and T_i a branch-decomposition of $\Gamma|B_i$ and let w_i be the width of this branch-decomposition. Let $\Gamma_0 = \Gamma$ and, for each $1 \leq i \leq m$, let $\Gamma_i = \Gamma_{i-1}|A_i$. Let T_0 be a branch-decomposition of Γ_m and let w_0 be the width of this branch-decomposition. For each $1 \leq i \leq m$, let e_i be the edge in T_0 replacing A_i and e'_i the edge in T_i replacing B_i . Construct tree T from $T_0, T_1, \dots, T_m$ by identifying, for $1 \leq i \leq m$, leaf e_i of T_0 with leaf e'_i of T_i and then ignoring this node to merge the two incident edges into one. Then, T is a branch-decomposition of Γ and the width of T is $\max_{0 \leq i \leq m} w_i$.

Let Σ be a fixed sphere. A *plane graph* is the standard representation of graphs on Σ : a vertex is a point on Σ distinct from any other vertex, an edge is a curve segment on Σ between two vertices such that its interior does not contain any intersection with other edges. A graph is *planar* if it is isomorphic, as a graph, to some plane graph.

We view walks on a plane graph as a curve on Σ that threads the edges in the walk. Formally, a *curve* on Σ is a continuous function $\alpha : [0, 1] \rightarrow \Sigma$. We say a curve α is *simple* if $\alpha(s) \neq \alpha(t)$ for every pair s, t with $0 \leq s < t < 1$. We say a curve α is *closed* if $\alpha(0) = \alpha(1)$; otherwise it is *non-closed*. We refer to simple non-closed curves as *segments*. For each curve α on the sphere, we denote by $\text{rev}(\alpha)$ its reversal: $\text{rev}(\alpha)(t) = \alpha(1 - t)$ for $0 \leq t \leq 1$. A *disc* on Σ is a set of points of Σ that is homeomorphic to the unit closed disc $\{(x, y) \mid x^2 + y^2 \leq 1\}$ on the plane. For a disc d on the sphere, the *interior* $\text{int}(d)$ is the open region whose closure is d . The *boundary* of d , denoted by $\text{bd}(d)$ is $d \setminus \text{int}(d)$. We also denote by $\vec{\text{bd}}(d)$ the simple closed curve whose image is $\text{bd}(d)$ and goes clockwise around d . For each simple closed curve α , we denote by $R(\alpha)$ the closed disc d such that $\vec{\text{bd}}(d) = \alpha$.

We say that a closed curve α on Σ is *non-crossing* if it has an infinitesimal perturbation that is simple. Let α be a non-crossing closed curve and α' an infinitesimal perturbation of α that is simple. When we perturb α' back to α , $\text{int}(R(\alpha'))$ transforms into a set of open discs. We denote by $\text{discs}(\alpha)$ the set of the closures of these open discs. We denote by $R(\alpha)$ the union $\alpha \cup \bigcup_{d \in \text{discs}(\alpha)} d$ of α , as a point set, with those discs in $\text{discs}(\alpha)$. Note that this notation is consistent with the special case where α is simple. We say that non-crossing closed curve α is *tree-shaped* if $\Sigma \setminus R(\alpha)$ is connected.

For hypergraphs, we use slightly non-standard embeddings: vertices, edges, and faces are all represented by discs. A *tiling* is a finite set D of discs on Σ such that $\bigcup_{d \in D} d = \Sigma$ and, for each distinct pair $d_1, d_2 \in D$, $d_1 \cap d_2$, is either empty or a single closed segment. In the latter case, we say that the two discs d_1 and d_2 are *incident* with each other. A tiling D is *tripartite*, if D has a tripartition (D_1, D_2, D_3) such that, for each $i = 1, 2, 3$ and for each distinct discs $d_1, d_2 \in D_i$, $d_1 \cap d_2 = \emptyset$. In this situation, we say that (D_1, D_2, D_3) is a tripartite tiling. See Fig. 1.

Fig. 1 (a) A tripartite tiling with 4 dark, 3 light, and 3 white disks. A white disk in the sphere is drawn as an infinite area in the plane. (b) Not a tiling since a light and a dark disc intersect each other in two segments

A *plane hypergraph*, in this paper, is a pair $\Gamma = (V(\Gamma), E(\Gamma))$, such that $(V(\Gamma), E(\Gamma), F(\Gamma))$ is a tripartite tiling for some set $F(\Gamma)$ of discs. Note that, given a plane hypergraph Γ , the set $F(\Gamma)$ such that $(V(\Gamma), E(\Gamma), F(\Gamma))$ is a tripartite tiling is unique. We call each disc in $F(\Gamma)$ a *face* of the plane hypergraph. A plane hypergraph is a hypergraph with the natural incidence relation: vertex $v \in V(\Gamma)$ and edge $e \in E(\Gamma)$ are incident with each other if $v \cap e \neq \emptyset$. Although not all hypergraphs that are legitimately considered planar can be embedded in the sphere as a plane hypergraph in our definition (for example, if we try to embed a hypergraph that is disconnected or has a cut vertex, there would be a face that is not a closed disc), the restricted definition suffices for our purposes.

A plane hypergraph is *trivial* if it has exactly one vertex and exactly one edge; otherwise it is *non-trivial*.

Proposition 2.1 *Each edge of a non-trivial plane hypergraph has order at least two.*

Proof Suppose a plane hypergraph Γ has an edge e of order one and let $V_\Gamma(e) = \{v\}$. Then, since $\text{bd}(e) \cap \text{bd}(v)$ is a single segment (as $(V(\Gamma), E(\Gamma), F(\Gamma))$ is a tripartite tiling), $\text{bd}(e) \setminus \text{bd}(v)$ belongs to the boundary $\text{bd}(r)$ of some single face r . Since $\text{bd}(v) \cap \text{bd}(r)$ is a single segment with endpoints on $\text{bd}(e)$, $\{\{e\}, \{v\}, \{r\}\}$ is a tiling and hence Γ is trivial. $\square$

Let Γ be a plane hypergraph. For each disc $d \in V(\Gamma) \cup E(\Gamma) \cup F(\Gamma)$, we denote by $V_\Gamma(d)$, $E_\Gamma(d)$, and $F_\Gamma(d)$ the set of discs in $V(\Gamma)$, $E(\Gamma)$, and $F(\Gamma)$, respectively, that are incident with d . We extend this notation to sets of discs: $V_\Gamma(D) = \bigcup_{d \in D} V_\Gamma(d)$, and so on.

We define the radial graph [19] of a plane hypergraph, that represents the incidences between vertices and faces. Let Γ be a plane hypergraph and G a plane graph. We say that G is the *radial graph* of Γ , if there is a bijection $\phi : V(\Gamma) \cup F(\Gamma) \rightarrow V(G)$ such that:

1. $\phi(x) \in \text{int}(x)$ for each $x \in V(\Gamma) \cup F(\Gamma)$, and
2. for each $v \in V(\Gamma)$ and $r \in F(\Gamma)$, there is an edge between $\phi(v)$ and $\phi(r)$ in $\mathcal{R}_\Gamma$ if and only if v and r are incident with each other in Γ , and
3. for each $v \in V(\Gamma)$ and $r \in F(\Gamma)$ incident with each other, the edge of $\mathcal{R}_\Gamma$ between $\phi(v)$ and $\phi(r)$ is contained in $\text{int}(v) \cup \text{int}(r) \cup (\text{bd}(v) \cap \text{bd}(r))$ and intersects $\text{bd}(v) \cap \text{bd}(r)$ at exactly one point.

Fig. 2 A plane hypergraph Γ , its radial graph, and a simple Γ -noose v : (a) Edges, vertices, and faces are drawn in dark gray, light gray, and white respectively. The solid and dashed segments are the edges of $\mathcal{R}_\Gamma$. Γ -noose v consists of dashed edges and is oriented clockwise. (b) The disc $R(v)$ is shown in dark gray, as a new edge. This entire picture of (b) shows the hypergraph $\Gamma|_v$

Although the radial graph of Γ is not unique, it is unique up to homeomorphism. We assume a fixed representative radial graph for each plane hypergraph Γ and denote it by $\mathcal{R}_\Gamma$. We often view $\mathcal{R}_\Gamma$ as a graph on $V(\Gamma) \cup F(\Gamma)$ embedded as plane graph $\mathcal{R}_\Gamma$ in the sphere. In particular, we not only view a walk ω in $\mathcal{R}_\Gamma$ as a curve on the sphere, we also view it as a walk through $V(\Gamma) \cup F(\Gamma)$, saying that $x \in V(\Gamma) \cup F(\Gamma)$ is on ω if the corresponding vertex $\phi(x)$ is on ω . We also say that the walk is from x to y , where $x, y \in V(\Gamma) \cup F(\Gamma)$, if it is, more precisely, from $\phi(x)$ to $\phi(y)$. For each walk ω on $\mathcal{R}_\Gamma$, we denote by $\text{length}_\Gamma(\omega)$ the length of this walk, that is, the number of edge occurrences it contains.

Let Γ be a plane hypergraph. For $x, y \in V(\Gamma) \cup F(\Gamma)$, the Γ -distance between x and y , denoted by $\text{dist}_\Gamma(x, y)$, is the length of the shortest path in $\mathcal{R}_\Gamma$ connecting x and y . We extend this notion of distance to sets of vertices/faces: for $X, Y \subseteq V(\Gamma) \cup F(\Gamma)$, $\text{dist}_\Gamma(X, Y) = \min_{x \in X, y \in Y} \text{dist}_\Gamma(x, y)$. We also write $\text{dist}_\Gamma(x, Y)$ for $\text{dist}_\Gamma(\{x\}, Y)$ and $\text{dist}_\Gamma(X, y)$ for $\text{dist}_\Gamma(X, \{y\})$.

Let Γ be a plane hypergraph. For our purposes, a Γ -noose is an oriented cycle of the radial graph of $\mathcal{R}_\Gamma$ (see Fig. 2).

Let μ be an oriented path or cycle of $\mathcal{R}_\Gamma$. For $x, y \in V(\Gamma) \cup F(\Gamma)$ on μ , we denote by $\mu[x, y]$ the oriented subpath of μ from x to y . In particular, $\mu[x, x]$ is a subpath of length zero consisting of the single vertex of $\mathcal{R}_\Gamma$ corresponding to x . If μ is an oriented path and y precedes x on μ then $\mu[x, y]$ is undefined.

Let v be a Γ -noose. We say that a separation (A, B) of Γ is induced by v , if every edge in A is contained in $R(v)$ and every edge in B is contained in $R(\text{rev}(v))$. Note that if separation (A, B) is induced by a Γ -noose v , then $V_\Gamma(A) \cap V_\Gamma(B)$ is the set of vertices of Γ on v . Therefore, the order of the separation induced by v is $\text{length}_\Gamma(v)/2$. Let e be an arbitrary edge of Γ and v a closed walk of $\mathcal{R}_\Gamma$ threading the vertices and faces incident with e clockwise around e . Because of our definition of

plane hypergraphs based on tiling, each face or vertex incident with e appears exactly once on v and therefore v is a Γ -noose that induces separation $(\{e\}, E(\Gamma) \setminus \{e\})$.

A branch-decomposition of Γ is called a *sphere-cut decomposition* [9] if each separation that belongs to this decomposition is induced by a Γ -noose.

Let Γ be a plane hypergraph and v a Γ -noose. We denote by $\Gamma|v$ a plane hypergraph with vertex set $\{v \setminus \text{int}(R(v)) \mid v \in V(\Gamma), v \not\subseteq R(v)\}$ and edge set $\{e \in E(\Gamma) \mid e \cap R(v) = \emptyset\} \cup \{R(v)\}$. Note that, in $\Gamma|v$, we replace all the edges of Γ contained in $R(v)$ by one edge $R(v)$ (see Fig. 2(b)). Thus, $\Gamma|v$ is isomorphic to $\Gamma|A$ as a hypergraph, where A is the set of edges of Γ contained in $R(v)$.

We need to slightly generalize the notion of Γ -nooses and related concepts. We call a closed walk v of $\mathcal{R}_\Gamma$ a *pseudo- Γ -noose* if it is non-crossing and tree-shaped as a closed curve on Σ . Let v be a pseudo- Γ -noose. We denote by $\Gamma|v$ the plane hypergraph obtained from Γ by making each disc $d \in \text{discs}(v)$ into a single edge. More formally, $\Gamma|v$ has vertex set $V(\Gamma|v) = \{v \setminus \bigcup_{d \in \text{discs}(v)} \text{int}(d) \mid v \in V(\Gamma), v \not\subseteq \bigcup_{d \in \text{discs}(v)} d\}$ and edge set $E(\Gamma|v) = \{e \in E(\Gamma) \mid e \cap \bigcup_{d \in \text{discs}(v)} d = \emptyset\} \cup \text{discs}(v)$.

A $k \times h$ cylinder, denoted by $C_{k,h}$, where $k \geq 3$ and $h > 0$ are integers, is the Cartesian product of a cycle on k vertices and a path on h vertices. Its vertex set is $V(C_{k,h}) = \{(i, j) \mid 0 \leq i < k, 0 \leq j < h\}$ and its vertices (i, j) and (i', j') are adjacent to each other if and only if $i = i'$ and $(j' - j) \equiv \pm 1 \pmod{k}$ or $j = j'$ and $(i' - i) = \pm 1$. For $0 \leq j < h$, we call the cycle of $C_{k,h}$ induced by the vertex set $\{(i, j) \mid 0 \leq i < k\}$ the j -th row, or row j , of $C_{k,h}$. For $0 \leq i < k$, we call the path of $C_{k,h}$ induced by the vertex set $\{(i, j) \mid 0 \leq j < h\}$ the i -th column, or column i , of $C_{k,h}$. A $k \times h$ grid, denoted by $G_{k,h}$, where $k, h > 0$ are integers, is similarly defined as a Cartesian product of a path on k vertices and a path on h vertices. The rows and columns of a grid are defined similarly.

3 A Radius-based Bound on the Branchwidth of Plane Hypergraphs

The goal of this section is to prove an upper bound on the branchwidth of a plane hypergraph based on the radius of its radial graph, which serves as the base case of the inductive proof of Theorem 1.4. This type of radius-based upper bounds are first observed for the treewidth of planar graphs [2, 16], and later for the branchwidth [4, 22]. Tamaki [22] generalizes the bound to plane hypergraphs and gives a linear time algorithm for constructing a branch-decomposition achieving the bound.

We prove and use the following upper bound on the branchwidth of plane hypergraphs which extends the bound in [22]. While the previous bound is with respect to the radius of the radial graph with the center corresponding to a face of the plane hypergraph, our bound is, roughly speaking, with respect to the radius with the center being an edge of the plane hypergraph.

Theorem 3.1 *Let $k \geq 2$ and $d \geq 1$ be integers. Let Γ be a plane hypergraph of maximum edge order at most k . Suppose there is an edge $e_0 \in E(\Gamma)$ such that $\text{dist}_\Gamma(v, V_\Gamma(e_0)) \leq d$ for every vertex $v \in V(\Gamma)$. Then, we have $\text{bw}(\Gamma) \leq k + d$.*

We need some lemmas, all of which involve the notion of d -compact nooses defined below. For a plane hypergraph Γ , a connected subset R of Σ , and elements

$x, y \in V(\Gamma) \cup F(\Gamma)$ such that the vertices of $\mathcal{R}_\Gamma$ corresponding to x and y are contained in R , we denote by $\text{dist}_{\Gamma, R}(x, y)$ the length of the shortest path of $\mathcal{R}_\Gamma$ from x to y contained in R ; $\text{dist}_{\Gamma, R}(x, y)$ is undefined if such a path does not exist. This notation is extended to $\text{dist}_{\Gamma, R}(x, Y)$, $\text{dist}_{\Gamma, R}(X, y)$, and $\text{dist}_{\Gamma, R}(X, Y)$, where X and Y are subsets of $V(\Gamma) \cup F(\Gamma)$.

Let d be a positive integer, Γ a plane hypergraph, ν a pseudo- Γ -noose, and X either a set of vertices or a set of faces of Γ . We say that ν is d -compact for Γ with center X , if the following conditions are satisfied.

1. The vertices or the faces in X appear consecutively on ν , that is, there is a subwalk ν' of ν such that X is the set of vertices on ν' or the set of faces on ν' .
2. For every vertex or face y of Γ that intersects $R(\nu)$, we have $\text{dist}_{\Gamma, R(\nu)}(y, X) \leq d$.

Lemma 3.1 *Let ν be a pseudo- Γ -noose and suppose ν is d -compact with center X for some d and X . Then, each Γ -noose μ such that $R(\mu) \in \text{discs}(\nu)$ is d -compact for some Y , where $Y \subseteq X$ or $|Y| = 1$.*

Proof Let T be a breadth-first forest, with roots in X , of the subgraph of $\mathcal{R}_\Gamma$ contained in $R(\nu)$. Since ν is d -compact with center X , the depth of this forest is at most d . Let T_μ be the subgraph of T consisting of the vertices and edges in $R(\mu)$. Let Y be the set of elements of X that are intersected by $R(\mu)$. Since ν is tree-shaped, if Y is non-empty then T_μ is a sub-forest of T with roots in Y and therefore μ is d -compact with center Y . Otherwise, T_μ is a subtree of some tree of T . We redefine Y to be the singleton set consisting of the root of this subtree. Then, μ is d -compact with center Y . $\square$

Lemma 3.2 *Let k, d be positive integers. Let Γ be a plane hypergraph with maximum edge order k . Suppose there is a Γ -noose ν that is $(d + 1)$ -compact for Γ with some center X with $|X| \leq k/2 + 1$ and has length $\text{length}_\Gamma(\nu) \leq 2d + 2k$, then $\Gamma|\text{rev}(\nu)$ has a branch-decomposition of width at most $d + k$.*

Proof Let k, d, Γ, ν , and X be as in the statement of the lemma. We prove by induction on the number of edges of $\Gamma|\text{rev}(\nu)$ that $\Gamma|\text{rev}(\nu)$ has a branch-decomposition of width at most $d + k$.

If $|E(\Gamma|\text{rev}(\nu))| \leq 2$, then the statement is trivial. So, assume that $|E(\Gamma|\text{rev}(\nu))| \geq 3$.

Let V_0 denote the set of vertices of Γ that have corresponding vertices in $\Gamma|\text{rev}(\nu)$: $V_0 = \{v \in V(\Gamma) \mid v \cap R(\nu) \neq \emptyset\}$. Similarly define F_0 to be the set of faces of Γ that have corresponding faces in $\Gamma|\text{rev}(\nu)$.

We construct a breadth-first forest T in $\mathcal{R}_\Gamma$ contained in $R(\nu)$ and spans the vertices of $\mathcal{R}_\Gamma$ corresponding to vertices and faces in $V_0 \cup F_0$, where the roots are the vertices of $\mathcal{R}_\Gamma$ corresponding to the elements of X . We view this forest as a forest on $V_0 \cup F_0$, referring to the vertices in X as the roots of T . For each $y \in (V_0 \cup F_0) \setminus X$, let $\pi(y)$ denote the parent of y in T . For each $y \in V_0 \cup F_0$, let $\rho(y)$ denote the root of the tree in T to which y belongs and let δ_y denote the oriented path in T from y to $\rho(y)$. Since T is breadth-first, the length of δ_y equals $\text{dist}_{\Gamma, R(\nu)}(y, \rho(y)) = \text{dist}_{\Gamma, R(\nu)}(y, X)$.

Fig. 3 Separating paths in Case 1: Drawing conventions are the same as in the previous figure. The Γ -noose ν is shown by a dashed curve and is oriented clockwise. The edges of the breadth-first forest T are shown by *solid lines*. Where an edge of T is drawn along ν , it must be considered as identical to the corresponding segment of ν . For each face or vertex x , the corresponding vertex in $\mathcal{R}_\Gamma$ is shown by a *small black circle or square*, the latter being the case when $x \in X$. The paths δ_y and $\delta_{y'}$ in $\mathcal{R}_\Gamma$, for face y and vertex y' , are both separating. The first point at which δ_y intersects ν after y is in face $z \neq \rho(y)$. The first point at which $\delta_{y'}$ intersects ν after y' is in the root $\rho(y')$ of y'

Let y be a vertex or face on ν but not in X . We say that δ_y is *separating*, if neither δ_y nor its reversal is a subpath of ν .

(Case 1) There is some vertex or face y such that δ_y is separating (see Fig. 3). Let y_0 be a vertex or face on ν but not in X such that δ_{y_0} is separating and is the shortest subject to this condition. Then, $\pi(y_0)$ is not on ν ; otherwise, $\delta_{\pi(y_0)}$ would be separating and shorter. Let z be the first vertex or face on δ_{y_0} , after y_0 , that is on ν . Then, since $\delta_z = \delta_{y_0}[z, \rho(y_0)]$ is not separating, it is either a subpath of ν or $\text{rev}(\nu)$. Thus, the subpath $\delta_{y_0}[y_0, z]$ divides $R(\nu)$ into two closed discs R_1 and R_2 . Let v_i , for $i = 1, 2$, be the Γ -noose such that $R(v_i) = R_i$. We define X_i for $i = 1, 2$ as follows. If no element of X is on v_i then $X_i = \{z\}$; otherwise, X_i is the set of vertices and faces of X on v_i .

We claim that v_i is $(d + 1)$ -compact for Γ with center X_i for $i = 1, 2$. To see this, simply observe that, if $y \in V(\Gamma) \cup F(\Gamma)$ intersects R_i , $i = 1$ or 2 , either δ_y is contained in R_i or z is on δ_y and $\delta_y[y, z]$ is contained in R_i .

Fix $i = 1$ or 2 . We have confirmed above that v_i is $(d + 1)$ -compact with center X_i . Moreover, we have $\text{length}_\Gamma(v_i) \leq \text{length}_\Gamma(\nu)$, because v_i is obtained from ν by replacing a subpath of ν with a chordal path $\delta_{y_0}[y_0, z]$ not longer than the replaced subpath.

Observe furthermore that $R(v_{3-i})$ contains at least one edge. This is because $\pi(y_0)$ is not on ν and, for each vertex or face on ν that is adjacent to y_0 on ν , there must be an edge to make it distinct from $\pi(y_0)$. Therefore, we may apply the induction hypothesis to v_i and obtain a branch-decomposition of $\Gamma|\text{rev}(v_i)$ with width at most $d + k$.

The hypergraph $((\Gamma|\text{rev}(v)|v_1)|v_2)$, consisting of three edges, has a unique branch-decomposition of width at most $d + k$. Therefore, applying Corollary 2.1, we obtain from these decompositions a branch-decomposition of $\Gamma|\text{rev}(v)$ of width at most $d + k$.

(Case 2) There is no vertex or face y on v such that δ_y is separating. Thus, for every vertex or face $y \in V(\Gamma) \cup F(\Gamma)$ that is on v , δ_y is a subpath of v or $\text{rev}(v)$. Let x_1 and x_2 be the two elements of X such that all the elements of X are on $v[x_2, x_1]$. If X is all the vertices on v or all the faces on v then such a pair of x_1 and x_2 is not unique but an arbitrary choice suffices.

For each $y \in V(\Gamma) \cup F(\Gamma)$ on $v[x_1, x_2]$, since δ_y is a subpath of v or $\text{rev}(v)$ and ends in an element of X , either $\rho(y) = x_1$ or $\rho(y) = x_2$. Therefore, there are $y_1, y_2 \in V(\Gamma) \cup F(\Gamma)$ that appear consecutively on $v[x_1, x_2]$ in this order, such that $\delta_{y_1} = \text{rev}(v)[y_1, x_1]$ and $\delta_{y_2} = v[y_2, x_2]$. Since one of y_1 and y_2 is a vertex and the other is a face, it is impossible for both to be at distance $d + 1$ from the roots in T . Therefore, without loss of generality, we may assume that

$$\text{length}_\Gamma(\delta_{y_1}) \leq d + 1 \quad (2)$$

and

$$\text{length}_\Gamma(\delta_{y_2}) \leq d \quad (3)$$

Let f denote the unique edge of Γ contained in $R(v)$ that is incident with both y_1 and y_2 . Let α be the Γ -noose that induces separation $(E(\Gamma) \setminus \{f\}, \{f\})$. Note that the length of α is twice the order of edge f and is at most $2k$.

Let y_3 be a vertex or face of Γ on α , such that

$$\text{length}_\Gamma(\alpha[y_2, y_3]) = \text{length}_\Gamma(\alpha[y_3, y_2]) \leq k. \quad (4)$$

This implies that

$$\text{length}_\Gamma(\alpha[y_1, y_3]) \leq k - 1. \quad (5)$$

Let z be the first point on δ_{y_3} that is also on v . Since $\delta_z = \delta_{y_3}[z, \rho(y_3)]$ is not separating, it must be either a subpath of v or $\text{rev}(v)$ (see Fig. 4).

Let $z_1 = z$ if z is on $v[\rho(y_3), y_1]$ and $z_1 = \rho(y_3)$ otherwise. Let $z_2 = z$ if z is on $v[y_2, \rho(y_3)]$ and $z_2 = \rho(y_3)$ otherwise. Define closed walks v_1 and v_2 of $\mathcal{R}_\Gamma$ by

$$v_1 = v[z_1, y_1] + \alpha[y_1, y_3] + \delta_{y_3}[y_3, z_1],$$

and

$$v_2 = v[y_2, z_2] + \text{rev}\delta_{y_3}[y_3, z_2] + \alpha[y_3, y_2].$$

Then, v_i for $i = 1, 2$ is a pseudo- Γ -noose. Note that the discs in $\text{discs}(v_1)$, those in $\text{discs}(v_2)$, $R(\text{rev}(v))$, and $R(\text{rev}(\alpha))$ have mutually disjoint interiors and together cover the entire sphere Σ .

For $i = 1, 2$, let X_i be the set of elements of X on v_i if there are such elements; otherwise let $X_i = \{z_i\}$. Then, for $i = 1, 2$, v_i is d -compact for Γ with center X_i , since for each vertex or face y that intersects $R(v_i)$, δ_y stays in $R(v_i)$ until it reaches $\rho(y)$ or z_i .

Fig. 4 Case 2, no separating paths from elements on v :

Drawing conventions are the same as in the previous figures. The deviations of the drawings of the pseudo- Γ -nooses v_1 and v_2 from the corresponding segments of v and from the corresponding edges of T are for the sake of visibility

We first bound the length of v_i for each $i = 1, 2$. Using (2) and (5), we have

$$\begin{aligned} \text{length}_{\Gamma}(v_1) &\leq \text{length}_{\Gamma}(v[z_1, y_1]) + \text{length}_{\Gamma}(\alpha[y_1, y_3]) + \text{length}_{\Gamma}(\delta_{y_3}[y_3, z_1]) \\ &\leq d + 1 + 2(|X_1| - 1) + k - 1 + d + 1 \\ &\leq 2d + 2k + 1, \end{aligned}$$

where we have used $|X_1| \leq |X| \leq k/2 + 1$. Since the length of a noose is even, the bound reduces to $2d + 2k$. Similarly, using (3) and (4), we have

$$\begin{aligned} \text{length}_{\Gamma}(v_2) &\leq \text{length}_{\Gamma}(v[y_2, z_2]) + \text{length}_{\Gamma}(\alpha[y_3, y_2]) + \text{length}_{\Gamma}(\delta_{y_3}[y_3, z_2]) \\ &\leq d + 2(|X_2| - 1) + k + d + 1 \\ &\leq 2d + 2k + 1, \end{aligned}$$

and hence $\text{length}_{\Gamma}(v_2) \leq 2d + 2k$.

For each disc $g \in \text{discs}(v_1) \cup \text{discs}(v_2)$, let μ_g denote the Γ -noose such that $R(\mu_g) = g$. Since v_i for $i = 1, 2$ is $(d + 1)$ -compact for Γ with center X_i , $|X_i| \leq |X|$, μ_g is $(d + 1)$ -compact for Γ with some center X_g , $|X_g| \leq |X| \leq k/2 + 1$, by Lemma 3.1. Therefore, we may apply the induction hypothesis to μ_g and obtain a branch-decomposition T_g of $\Gamma|_{\text{rev}(\mu_g)}$ of width at most $d + k$.

Let $\Gamma_0 = ((\Gamma|_{\text{rev}(v)})|_{v_1})|_{v_2}$ be the plane hypergraph obtained from $\Gamma|_{\text{rev}(v)}$ by replacing the edges contained in disc g by a single edge g , for every $g \in \text{discs}(v_1) \cup \text{discs}(v_2)$. Consider separations (A_i, B_i) , $1 \leq i \leq 6$, where

1. $A_1 = \{R(\text{rev}(v))\}$, $B_1 = \text{discs}(v_1) \cup \text{discs}(v_2) \cup \{f\}$,
2. $A_2 = \text{discs}(v_1)$, $B_2 = \{R(\text{rev}(v))\} \cup \text{discs}(v_2) \cup \{f\}$,
3. $A_3 = \text{discs}(v_2)$, $B_3 = \{R(\text{rev}(v))\} \cup \text{discs}(v_1) \cup \{f\}$,
4. $A_4 = \{f\}$, $B_4 = \{R(\text{rev}(v))\} \cup \text{discs}(v_1) \cup \text{discs}(v_2)$,
5. $A_5 = \text{discs}(v_1) \cup \{f\}$, $B_5 = \{R(\text{rev}(v))\} \cup \text{discs}(v_2)$, and
6. $A_6 = \text{discs}(v_2) \cup \{f\}$, $B_6 = \{R(\text{rev}(v))\} \cup \text{discs}(v_1)$.

Separation (A_1, B_1) is induced by $\text{rev}(v)$ and hence its order is at most $d + k$. Separation (A_2, B_2) is induced by the pseudo- Γ -noose v_1 and hence its order is at most $d + k$. Separation (A_3, B_3) is induced by the pseudo- Γ -noose v_2 and hence its order is at most $d + k$. The order of separation (A_4, B_4) is at most $|V_\Gamma(f)| \leq k$.

Let v'_1 and v'_2 denote the pseudo- Γ -nooses that induce separations (A_5, B_5) and (A_6, B_6) , respectively. By an analysis similar to the one to bound the length of v_1 and v_2 , we have $\text{length}_\Gamma(v'_i) \leq \text{length}_\Gamma(v_i) + 2$, for $i = 1, 2$. The bound of $\text{length}_\Gamma(v_i) \leq 2d + 2k$ is tight for at most one of $i = 1, 2$, because we cannot have $|X_1| = k/2 + 1$ and $|X_2| = k/2 + 1$ at the same time. Therefore, we have $\text{length}_\Gamma(v'_i) \leq 2d + 2k$ for either $i = 1$ or 2 and either separation (A_5, B_5) or separation (A_6, B_6) has order at most $d + k$. Assume without loss of generality that (A_5, B_5) has order at most $d + k$.

Let T_0 be a branch-decomposition of Γ_0 that consists of separations (A_i, B_i) , $1 \leq i \leq 5$, together with other separations induced by pseudo- Γ -nooses that are subwalks of v_1 or v_2 . Each of those other separations has order at most $d + k$, since $\text{length}_\Gamma(v_i) \leq 2(d + k)$ for $i = 1, 2$, and therefore the width of T_0 is at most $d + k$.

Applying Corollary 2.1 to branch-decompositions T_0 and T_g , $g \in \text{discs}(v_1) \cup \text{discs}(v_2)$, we obtain a branch-decomposition of $\Gamma|\text{rev}(v)$ of width at most $d + k$. $\square$

For the next lemma, we need some additional structure in the radial graph. Let Γ be a plane hypergraph, v a vertex of Γ and r a face of Γ . Because $V(\Gamma) \cup E(\Gamma) \cup F(\Gamma)$ is a tiling by the definition of plane hypergraphs, $v \cap r$ is either empty or a single closed segment. In the latter case, we call the segment the *b-arc* demarcating v and r . Note that the endpoints of a b-arc are on the boundary of edges of Γ . The graph whose vertices are the edges of Γ and whose edges are b-arcs is essentially the *medial graph* of Γ introduced in [21].

Lemma 3.3 *Let d, k be positive integers, with $k \geq 2$. Let Γ be a non-trivial plane hypergraph, with maximum edge order k . Suppose there is an edge $e_0 \in E(\Gamma)$ such that $\text{dist}_\Gamma(v, V_\Gamma(e_0)) \leq d$ for every vertex $v \in V(\Gamma)$. Then, there are t pseudo- Γ -nooses v_j , $1 \leq j \leq t$, where $t = 2$ or 3 , t sets X_j , $1 \leq j \leq t$, each of which is either a set of vertices or set of faces of Γ , and an edge $f \in E(\Gamma)$ such that the following conditions are satisfied.*

1. *The interiors of the discs in $\bigcup_{1 \leq j \leq t} \text{discs}(v_j)$ are mutually disjoint.*
2. *For $1 \leq j \leq t$, $e_0 \cap R(v_j) = \emptyset$ and $f \cap R(v_j) = \emptyset$.*
3. *$E(\Gamma) = \bigcup_{1 \leq j \leq t} (E_j) \cup \{e_0, f\}$, where E_j is the set of edges of Γ contained in $R(v_j)$.*
4. *For $1 \leq j \leq t$, the elements of X_j appear consecutively on v_j .*
5. *For $1 \leq j \leq t$, v_j is $(d + 1)$ -compact for Γ with center X_j .*
6. *For $1 \leq j \leq t$, $|X_j| \leq k/2 + 1$.*
7. *For $1 \leq j \leq t$, $\text{length}_\Gamma(v_j) \leq 2(d + k)$.*
8. *Let Γ_j , $0 \leq j \leq t$, be defined by $\Gamma_0 = \Gamma$ and $\Gamma_j = \Gamma_{j-1}|v_j$ for $0 < j \leq t$. Then, Γ_t , which has edges e_0 , f , and those in $\text{discs}(v_j)$, $1 \leq j \leq t$, has a branch-decomposition of width at most $d + k$.*

Proof Let β denote the Γ -noose that induces separation $(E(\Gamma) \setminus \{e_0\}, \{e_0\})$ of Γ .

We construct a breadth-first spanning forest T of $\mathcal{R}_\Gamma$ where the roots are the vertices of $\mathcal{R}_\Gamma$ corresponding to vertices of Γ incident with e_0 . As we did in the proof of Lemma 3.2, we view this forest as a forest on $V(\Gamma) \cup F(\Gamma)$ with roots in $V_\Gamma(e_0)$. For each vertex or face $y \in V(\Gamma) \cup F(\Gamma)$, we denote by $\rho(y)$ the root of the tree in this forest y belongs to, and δ_y the oriented path in this forest from y to $\rho(y)$. If $y \notin V_\Gamma(e_0)$, we furthermore denote by $\pi(y)$ the parent of y in this forest. The choice of this breadth-first forest is arbitrary except for the following rule: for each face r incident with e_0 , $\pi(r)$ is the vertex on β immediately after r .

For b-arc a of Γ , we denote by $\text{edges}(a)$ the set of two edges of Γ to which the endpoints of a belong. Since Γ is non-trivial, we have $|\text{edges}(a)| = 2$ for each b-arc a of Γ .

Let T^* be the graph whose vertex set is $E(\Gamma)$ and whose edges are the b-arcs of Γ that are not intersected by edges of T . Let $T^\dagger$ denote the subgraph of T^* induced by $E(\Gamma) \setminus \{e_0\}$. Since Γ is non-trivial, the order of e_0 is at least two by Proposition 2.1 and $E(\Gamma) \setminus \{e_0\}$ is non-empty. We claim that $T^\dagger$ is a tree. Suppose for contradiction that $T^\dagger$ is not connected. Then, there is a cycle C of $\mathcal{R}_\Gamma$ each edge of which belongs either to T or to β . Since T is a forest, C must contain an edge that belongs to β . Let p be a maximal subpath of C that does not contain an edge of β . Then p is between some $x, y \subseteq V_\Gamma(e_0) \cup F_\Gamma(e_0)$. Since p is a path in T , x and y must belong to the same tree in T . But this is possible only if one of x and y is the root of the tree and the other is an immediate child of the root, a contradiction because then the edge between x and y together with p would form a cycle in T . Therefore, $T^\dagger$ is connected. Suppose for contradiction that $T^\dagger$ contains a cycle. Then, the b-arcs forming this cycle, together with the edges of Γ on this cycle, separate Σ into two regions both containing vertices or faces, a contradiction since there must be an edge of T connecting these two regions. Therefore, $T^\dagger$ does not have a cycle and hence is a tree.

For each b-arc a of $T^\dagger$ and each edge $e \in \text{edges}(a)$, let $A(a, e)$ denote the set of edges $e' \in E(\Gamma) \setminus \{e_0\}$ such that the path between e and e' in $T^\dagger$ contains a . Thus, if $\text{edges}(a) = \{e_1, e_2\}$ then $(A(a, e_1), A(a, e_2))$ is a bipartition of $E(\Gamma) \setminus \{e_0\}$, with $e_1 \in A(a, e_2)$ and $e_2 \in A(a, e_1)$.

We now orient each b-arc of T^* . Each b-arc with one end on $\text{bd}(e_0)$ is oriented away from e_0 . Other b-arcs, that is, those in $T^\dagger$, are oriented as follows. For each $A \subseteq E(\Gamma) \setminus \{e_0\}$, let $\text{weight}(A)$ denote the number of b-arcs between A and e_0 that are in T^* . Note that $\text{weight}(E(\Gamma) \setminus \{e_0\})$ is the order of e_0 , as each vertex of Γ incident with e_0 contributes exactly one b-arc in T^* incident with e_0 . Let $\text{edges}(a) = \{e_1, e_2\}$. We orient a from e_1 to e_2 if $\text{weight}(A(a, e_1)) > \text{weight}(A(a, e_2))$ or $\text{weight}(A(a, e_1)) = \text{weight}(A(a, e_2))$ and $|A(a, e_1)| > |A(a, e_2)|$, from e_2 to e_1 if $\text{weight}(A(a, e_2)) > \text{weight}(A(a, e_1))$ or $\text{weight}(A(a, e_2)) = \text{weight}(A(a, e_1))$ and $|A(a, e_2)| > |A(a, e_1)|$, and arbitrarily if $\text{weight}(A(a, e_1)) = \text{weight}(A(a, e_2))$ and $|A(a, e_1)| = |A(a, e_2)|$. Since $T^\dagger$ is a tree, the resulting directed graph has a sink f .

We say that a face r incident with f is *bad* if $\text{dist}_\Gamma(r, V_\Gamma(e_0)) = d + 1$ and $\pi(r)$ is not incident with f . If there is a bad face r , we eliminate it by redefining $\pi(r)$ to be one of the two vertices incident with r and f , modifying T and hence T^* and $T^\dagger$, provided that f remains a sink of the orientated version of T^* . Let v_1 and v_2 be two vertices incident with both r and f . Let a_i , for $i = 1, 2$, the b-arc demarcating r and v_i . Then, f ceases to be a sink after the above modification only if

Fig. 5 Forest T and tree T^* : The edges of the breadth-first forest T are shown with *thin solid lines*. The oriented b-arcs of T^* are shown with *thick arrows*: those in T^* are *black* and others are *gray*. The b-arcs of T^* around the sink f and the vertices/faces incident with them are shown with *symbols* to denote them

$\text{weight}(a_1, f) + \text{weight}(a_2, f) \geq |V_\Gamma(e_0)|/2$. Since this weight condition may hold for at most one face incident with f , if there are more than one bad face, then we can eliminate at least one of them. Thus, we assume in the following that there is at most one bad face.

Let α be the Γ -noose that induces separation $(E(\Gamma) \setminus \{f\}, \{f\})$. Let $a_1, \dots, a_m$ be the b-arcs of Γ incident with f that are in T^* , listed in the counterclockwise order around f . For convenience, we denote a_m also by a_0 . We claim that $m \geq 2$. If f is the only edge of Γ other than e_0 , this is the case since the order of e_0 is at least two. Otherwise, f has an incoming b-arc in T^* , say a , and because of the orientation of this b-arc, we have that $\text{weight}(A(a, e)) > 0$, where e is the other end of a , and hence f has at least one more b-arc of T^* incident with it.

For each $0 \leq i \leq m$, let x_i^+ and x_i^- denote the vertex and the face (not respectively) that a_i demarcates, with x_i^+ on the left and x_i^- on the right as we look from f toward the other end of a_i (see Fig. 5).

For each pair of distinct indices i, j , $1 \leq i, j \leq m$, we define a pseudo- Γ -noose μ_{ij} as follows. Fix i, j , $1 \leq i, j \leq m$, $i \neq j$. If $\rho(x_i^-) \neq \rho(x_{j-1}^+)$ then we let

$$\mu_{ij} = \delta_{x_{j-1}^+} + \beta[\rho(x_{j-1}^+), \rho(x_i^-)] + \text{rev}(\delta_{x_i^-}) + \alpha[x_i^-, x_{j-1}^+]. \quad (6)$$

Note that μ_{ij} may not be a Γ -noose because $\beta[\rho(x_{j-1}^+), \rho(x_i^-)]$ and $\alpha[x_i^-, x_{j-1}^+]$ may share vertices. If $\rho(x_i^-) = \rho(x_{j-1}^+)$ then we let z_{ij} be the first vertex or face that

Fig. 6 Pseudo- Γ -nooses μ_{ij} : For the same instance as in the previous figure, only $\mu_{51} = \mu_{01}$ and μ_{34} are shown. The pseudo- Γ -nooses in this example are Γ -nooses

is on both $\delta_{x_i^-}$ and $\delta_{x_{j-1}^+}$ and let

$$\mu_{ij} = \delta_{x_{j-1}^+}[x_{j-1}^+, z_{ij}] + \text{rev}(\delta_{x_i^-}[z_{ij}, x_i^-]) + \alpha[x_i^-, x_{j-1}^+]. \quad (7)$$

Note that the orientation of μ_{ij} specified here, in either of the above two cases, is such that f lies outside of $R(\mu_{ij})$ (see Fig. 6). If e_0 also lies outside of $R(\mu_{ij})$ then we say that the ordered pair (i, j) is *legal*; otherwise it is *illegal*.

Let X'_{ij} be the set of vertices on μ_{ij} that are incident with e_0 . If $X'_{ij} \neq \emptyset$ then we let $X_{ij} = X'_{ij}$; otherwise, if pair (i, j) is legal then we let $X_{ij} = \{z_{ij}\}$; otherwise we let $X_{ij} = V_\Gamma(e_0) \cup F_\Gamma(e_0)$. The last case is for technical convenience utilized later.

We claim that if μ_{ij} is legal then it is $(d+1)$ -compact with center X_{ij} . To see this, first observe that, for every vertex v of Γ that intersects $R(\mu_{ij})$, the path in T from v to $\rho(v)$ stays in $R(\mu_{ij})$ until it reaches either z_{ij} (when μ_{ij} is defined by (7)) or a vertex or face incident with e_0 . Moreover, if μ_{ij} is defined by (6) then we have $\rho(v) \in X_{ij}$. If μ_{ij} is defined by (7) then we have either $\rho(v) \in X_{ij}$ or the face that is on δ_v immediately before $\rho(v)$ is incident with another root of T in X_{ij} . Therefore, we have $\text{dist}_{\Gamma|\text{rev}(\mu_{ij})}(v, X_{ij}) \leq d$ for every vertex v and $\text{dist}_{\Gamma|\text{rev}(\mu_{ij})}(r, X_{ij}) \leq d+1$ for every face r .

To bound the length of μ_{ij} , let l_{ij} denote the number of vertices of Γ on $\mu_{ij}[x_i^-, x_{j-1}^+]$. From the assumption in the lemma, $\text{length}_\Gamma(\delta_{x_i^-}) \leq d$ if x_i^- is a vertex and $\text{length}_\Gamma(\delta_{x_i^-}) \leq d+1$ if x_i^- is a face. In either case, at most $d/2$ vertices of Γ are on $\delta_{x_i^-}$, excluding $\rho(x_i^-)$. Similarly, at most $d/2$ vertices of Γ are on $\delta_{x_{j-1}^+}$, excluding $\rho(x_{j-1}^+)$. Therefore, the number of vertices of Γ on $\mu_{ij}[x_{j-1}^+, x_i^-]$ excluding those in

$\{x_i^-, x_{j-1}^+\} \cup X_{ij}$ is at most d . Therefore, we have

$$\text{length}_\Gamma(\mu_{ij}) \leq 2(d + l_{ij} + |X_{ij}|) \quad (8)$$

We say that an ordered pair (i, j) , $1 \leq i, j \leq m$, is *good* if $l_{ij} \leq (k-1)/2$ and $|X_{ij}| \leq |V_\Gamma(e_0)|/2 + 1$. Note that from the technical definition of X_{ij} for illegal pair (i, j) , an illegal pair can never be good.

We say that a t -tuple of indices $(i_1, i_2, \dots, i_t = i_0)$ with $1 \leq i_1 < i_2 < \dots < i_t \leq m$ is *cyclically good*, if, for each $1 \leq j \leq t$, pair (i_{j-1}, i_j) is good. We claim that there is a cyclically good t -tuple with $t = 2$ or 3 . To show this, we first show that the m -tuple $(1, 2, \dots, m)$ is cyclically good. Fix $0 \leq i < m$. We show that pair $(i, i+1)$ is good. If $\rho(x_i^-) = \rho(x_i^+)$ and hence $X_{i(i+1)} = \{z_{i(i+1)}\}$ then certainly $|X_{i(i+1)}| \leq |V_\Gamma(e_0)|/2 + 1$. Otherwise, each vertex of $X_{i(i+1)}$ has a b-arc in T^* on its boundary that directly connects an edge in $A(a_i, f)$ and e_0 . Therefore, we have $|X_{i(i+1)}| \leq \text{weight}(A(a_i, f))$. From the definition of the orientation of the b-arcs in T^+ and the choice of f as a sink, we have $\text{weight}(A(a_i, f)) \leq |V_\Gamma(e_0)|/2$. Therefore, we have $|X_{i(i+1)}| \leq |V_\Gamma(e_0)|/2$. We also have $l_{i(i+1)} = 0$, since x_i^- and x_i^+ are adjacent around f and, though one of them is a vertex, it is not counted in $l_{i(i+1)}$. Therefore, pair $(i, i+1)$ is good for every $0 \leq i < m$ and hence $(1, 2, \dots, m)$ is cyclically good.

We now show that, if there is a cyclically good t -tuple and $t > 3$, then there is a cyclically good $(t-1)$ -tuple. Together with the existence of a cyclically good m -tuple we proved above, this shows that there is a cyclically good t -tuple with $t \leq 3$.

Let $(i_1, i_2, \dots, i_t = i_0)$ be a cyclically good t -tuple, with $t > 3$.

Since $\text{int}(R(\mu_{i_0 i_2}))$ and $\text{int}(R(\mu_{i_2 i_4}))$ are disjoint, $X_{i_0 i_2}$ and $X_{i_2 i_4}$ may have at most two elements in common. Therefore, we have $|X_{i_0 i_2}| + |X_{i_2 i_4}| \leq |V_\Gamma(e_0)| + 2$, and either $|X_{i_0 i_2}| \leq |V_\Gamma(e_0)|/2 + 1$ or $|X_{i_2 i_4}| \leq |V_\Gamma(e_0)|/2 + 1$. Similarly, we have either $|X_{i_1 i_3}| \leq |V_\Gamma(e_0)|/2 + 1$ or $|X_{i_3 i_5}| \leq |V_\Gamma(e_0)|/2 + 1$ if $t \geq 5$ and either $|X_{i_1 i_3}| \leq |V_\Gamma(e_0)|/2 + 1$ or $|X_{i_3 i_1}| \leq |V_\Gamma(e_0)|/2 + 1$ if $t = 4$. With possible renumbering, we need to consider only two cases: (Case 1) $t \geq 5$, $|X_{i_0 i_2}| \leq |V_\Gamma(e_0)|/2 + 1$, and $|X_{i_3 i_5}| \leq |V_\Gamma(e_0)|/2 + 1$; (Case 2) $|X_{i_0 i_2}| \leq |V_\Gamma(e_0)|/2 + 1$, and $|X_{i_1 i_3}| \leq |V_\Gamma(e_0)|/2 + 1$.

(Case 1) First observe that $l_{i_0 i_2} + l_{i_3 i_5} \leq |V_\Gamma(f)| - 1 \leq k - 1$, since at least one vertex of $V_\Gamma(f)$ is between $x_{i_2-1}^+$ inclusive and $x_{i_3}^-$, and is not counted in $l_{i_0 i_2} + l_{i_3 i_5}$. Therefore, we have either $l_{i_0 i_2} \leq (k-1)/2$ or $l_{i_3 i_5} \leq (k-1)/2$ and hence either pair (i_0, i_2) or pair (i_3, i_5) is good. It follows that either $(t-1)$ -tuple $(i_2, i_3, \dots, i_t)$ or $(t-1)$ tuple $(i_1, i_2, i_3, i_5, \dots, i_t)$ is cyclically good.

(Case 2) We claim that, for each i with $i_1 \leq i < i_2$, either $l_{i_0(i+1)} \leq (k-1)/2$ or $l_{i i_3} \leq (k-1)/2$. Since $l_{i_0 i_1} \leq (k-1)/2$ and $l_{i_2 i_3} \leq (k-1)/2$, this claim implies that if we let i^* be the largest i in the range $i_1 \leq i \leq i_2$ such that $l_{i_0 i} \leq (k-1)/2$ then $l_{i_0 i^*} \leq (k-1)/2$ and $l_{i^* i_3} \leq (k-1)/2$. Therefore, both pairs (i_0, i^*) and (i^*, i_3) are good and hence $(i^*, i_3, \dots, i_t)$ is a cyclically good $(t-1)$ -tuple.

To verify the claim, let i be arbitrary with $i_1 \leq i < i_2$. No vertices in $V_\Gamma(f)$ are counted both in $l_{i_0(i+1)}$ and in $l_{i i_3}$, since if x_i^- is a vertex then it is not counted in $l_{i i_3}$ and if x_i^+ is a vertex then it is not counted in $l_{i_0(i+1)}$. Moreover, from $x_{i_3-1}^+$ to $x_{i_0}^-$ in counterclockwise order around f , there must be at least one vertex since $t \geq 4$ and hence $i_3 \neq i_0$, which is not counted either in $l_{i_0(i+1)}$ or in $l_{i i_3}$. Therefore we have

$l_{i_0(i+1)} + l_{ii_3} \leq k - 1$, and hence either $l_{i_0(i+1)} \leq (k - 1)/2$ or $l_{ii_3} \leq (k - 1)/2$. This verifies the claim.

We have verified that there is a cyclically good t -tuple with $t \leq 3$. Let $(i_1, \dots, i_t = i_0)$ be a cyclically good t -tuple with $t \leq 3$. It is clear that t cannot be 1, so $t = 2$ or 3. We set $v_j = \mu_{i_{j-1}i_j}$ and $X_j = X_{i_{j-1}i_j}$ for $1 \leq j \leq t$.

We now verify that the conditions in the lemma are satisfied.

1. Let $g_1 \in \text{discs}(v_j)$ and $g_2 \in \text{discs}(v_{j'})$ be two distinct discs, where $1 \leq j, j' \leq t$. If $j = j'$ then $\text{int}(g_1) \cap \text{int}(g_2) = \emptyset$ by the definition of pseudo- Γ -nooses. If $j \neq j'$, then $\text{int}(g_1) \cap \text{int}(g_2) = \emptyset$ because the subtree of $T^\dagger$ enclosed by v_j is disjoint from the subtree enclosed by $v_{j'}$.
2. For $1 \leq j \leq t$, we have $f \cap R(v_j) = \emptyset$ because of the definition of v_j and $e_0 \cap R(v_j) = \emptyset$ because the pair i_{j-1}, i_j is chosen to be legal.
3. $E(\Gamma) = \bigcup_{1 \leq j \leq t} (E_j) \cup \{e_0, f\}$, where E_j is the set of edges of Γ contained in $R(v_j)$, because $\bigcup_{1 \leq i \leq m} A(a_i, f) = E(\Gamma) \setminus \{e_0, f\}$ and, for each $1 \leq i \leq m$, $A(a_i, f)$ is contained in $R(v_j)$ for some $1 \leq j \leq t$.
4. For $1 \leq j \leq t$, elements of X_j appear consecutively on v_j , by the definition of X_j .
5. For $1 \leq j \leq t$, v_j is $(d + 1)$ -compact for Γ with center X_j . This is a consequence of the more general statement that μ_{ij} is $(d + 1)$ -compact for X_{ij} for every legal pair of indices $1 \leq i, j \leq m$, which we have already verified.
6. For $1 \leq j \leq t$, we have $|X_j| \leq |V_\Gamma(e_0)|/2 + 1 \leq k/2 + 1$ because $|X_{i_{j-1}i_j}| \leq |V_\Gamma(e_0)|/2 + 1$ from the definition of a good pair.
7. For $1 \leq j \leq t$, $\text{length}_\Gamma(v_j) \leq 2(d + k)$. This follows from the fact that the pair (i_{j-1}, i_j) is good: we have $l_{i_{j-1}i_j} \leq (k - 1)/2$, $|X_{i_{j-1}i_j}| \leq k/2 + 1$, and hence $l_{i_{j-1}i_j} + |X_{i_{j-1}i_j}| \leq k + 1/2$. But since this is the number of vertices and possibly faces, it must be integral and we may remove $1/2$ from the bound. Our required bound follows from inequality (8).

It remains to verify the last condition that Γ_t has branchwidth at most $d + k$. We show this for $t = 3$; the case $t = 2$ is similar and simpler. For indices $j = 1, 2, 3$, we use a cyclic arithmetic where $j + 1 = 1$ if $j = 3$ and $j - 1 = 3$ if $j = 1$.

Recall that the edges of Γ_3 are e_0, f , and the discs in $\text{discs}(v_j)$, $j = 1, 2, 3$. For each subset I of $\{1, 2, 3\}$, let S_I denote the separation (A, B) of Γ_3 , with $A = \{f\} \cup \bigcup_{j \in I} \text{discs}(v_j)$ and $B = \{e_0\} \cup \bigcup_{j \notin I} \text{discs}(v_j)$. Our goal is to find two distinct indices $j_1, j_2 \in \{1, 2, 3\}$ such that two separations $S_{\{j_1\}}$ and $S_{\{j_1, j_2\}}$ have order at most $k + d$. These separations essentially determine a branch-decomposition of Γ_3 . This branch-decomposition is completed by adding a set of separations, for each $j = 1, 2, 3$, used to decompose $\text{discs}(v_j)$ further. Such a set of separations can be chosen so that each separation in the set is induced by a pseudo- Γ -noose that is a subwalk of v_j and hence has order at most $d + k$. Therefore, finding indices j_1 and j_2 as above will finish the proof.

For $j = 1, 2, 3$, let α_j denote the common subpath of α and v_j ; similarly, β_j is the common subpath of β and v_j . If v_j and β do not have any edge or vertex in common, we say that β_j is empty.

Suppose first that β_j is empty for some j , say $j = 1$. If furthermore β_2 is empty, then index pair (i_3, i_1) used to define β_3 would be illegal, so β_2 is not empty. Similarly, β_3 is not empty.

Fig. 7 α_j , β_j , and γ_j

Separation $S_{\{1\}} = (\{f\} \cup \text{discs}(v_1), \{e_0\} \cup \text{discs}(v_2) \cup \text{discs}(v_3))$ is induced by a Γ -noose consisting of $\text{rev}(\alpha_2 + \alpha_3)$ and a path in T of length at most $2d$. The length of this Γ -noose is at most $2d + 2k$ and therefore the order of separation $S_{\{1\}}$ is at most $d + k$. Separation $S_{\{1,2\}} = (\{f\} \cup \text{discs}(v_1) \cup \text{discs}(v_2), \{e_0\} \cup \text{discs}(v_3))$ is induced by a Γ -noose obtained from $\text{rev}(v_3)$ by replacing its subpath $\text{rev}(\beta_3)$ by β_2 . The length of this Γ -noose is at most $2d + 2k$, by a calculation similar to the one that bounded the length of v_3 . Therefore, the order of separation $S_{\{1,2\}}$ is at most $d + k$ and we are done in this case.

So suppose β_j is non-empty for $j = 1, 2, 3$. For $j = 1, 2, 3$, let v_j be decomposed as

$$v_j = \alpha_j + \gamma_j + \beta_j + \text{rev}(\gamma_{j-1}),$$

where γ_j is a path of T (see Fig. 7). Because of the radius condition, we have $\text{length}_\Gamma(\gamma_j) \leq d + 1$ for $j = 1, 2, 3$ and, because there is at most one bad face, we have $\text{length}_\Gamma(\gamma_j) = d + 1$ for at most one value of j .

First suppose $\text{length}_\Gamma(\alpha) \leq \text{length}_\Gamma(\beta)$. Then, for at least one $j \in \{1, 2, 3\}$, $\text{length}_\Gamma(\alpha_j) + \text{length}_\Gamma(\alpha_{j+1}) \leq \text{length}_\Gamma(\beta_j) + \text{length}_\Gamma(\beta_{j+1})$ holds. Without loss of generality, we assume $\text{length}_\Gamma(\alpha_2) + \text{length}_\Gamma(\alpha_3) \leq \text{length}_\Gamma(\beta_2) + \text{length}_\Gamma(\beta_3)$. Define a closed walk ω_1 in $\mathcal{R}_\Gamma$ by

$$\omega_1 = \text{rev}(\alpha_2 + \alpha_3) + \gamma_1 + \beta_1 + \text{rev}(\gamma_3).$$

Then ω_1 is a Γ -noose that induces separation $S_{\{1\}}$. Its length is bounded by

$$\begin{aligned} \text{length}_\Gamma(\omega_1) &= \text{length}_\Gamma(\alpha_2) + \text{length}_\Gamma(\alpha_3) + \text{length}_\Gamma(\beta_1) + \text{length}_\Gamma(\gamma_1) \\ &\quad + \text{length}_\Gamma(\gamma_3) \\ &\leq \text{length}_\Gamma(\beta_2) + \text{length}_\Gamma(\beta_3) + \text{length}_\Gamma(\beta_1) + 2d + 1 \\ &\leq 2k + 2d + 1. \end{aligned}$$

Therefore, the order of separation $S_{\{1\}}$ is at most $d + k$. Since $\text{length}_\Gamma(\alpha_2) + \text{length}_\Gamma(\alpha_3) \leq \text{length}_\Gamma(\beta_2) + \text{length}_\Gamma(\beta_3)$, we have either $\text{length}_\Gamma(\alpha_2) \leq \text{length}_\Gamma(\beta_2)$

or $\text{length}_\Gamma(\alpha_3) \leq \text{length}_\Gamma(\beta_3)$. We assume the latter is the case; the other case is similar. Define a closed walk ω_2 in $\mathcal{R}_\Gamma$ by

$$\omega_2 = \text{rev}(\alpha_3) + \gamma_2 + \beta_2 + \beta_1 + \text{rev}(\gamma_3).$$

Then ω_2 is a Γ -noose that induces separation $S_{\{1,2\}}$. Its length is bounded by

$$\begin{aligned} \text{length}_\Gamma(\omega_2) &= \text{length}_\Gamma(\alpha_3) + \text{length}_\Gamma(\beta_2) + \text{length}_\Gamma(\beta_1) + \text{length}_\Gamma(\gamma_2) \\ &\quad + \text{length}_\Gamma(\gamma_3) \\ &\leq \text{length}_\Gamma(\beta_3) + \text{length}_\Gamma(\beta_2) + \text{length}_\Gamma(\beta_1) + 2d + 1 \\ &\leq 2k + 2d + 1. \end{aligned}$$

Therefore the order of $S_{\{2,3\}}$ is at most $d + k$ and we are done in this case as well.

Finally suppose $\text{length}_\Gamma(\alpha) > \text{length}_\Gamma(\beta)$. Similarly as above, we find two distinct indices j_1 and j_2 such that $\text{length}_\Gamma(\beta_{j_1}) < \text{length}_\Gamma(\alpha_{j_1})$ and $\text{length}_\Gamma(\beta_{j_1}) + \text{length}_\Gamma(\beta_{j_2}) < \text{length}_\Gamma(\alpha_{j_1}) + \text{length}_\Gamma(\alpha_{j_2})$. We assume below that $j_1 = 1$ and $j_2 = 2$. Other cases are similar. Define Γ -nooses ω_1 and ω_2 as above. Then,

$$\begin{aligned} \text{length}_\Gamma(\omega_1) &= \text{length}_\Gamma(\alpha_2) + \text{length}_\Gamma(\alpha_3) + \text{length}_\Gamma(\beta_1) + \text{length}_\Gamma(\gamma_1) \\ &\quad + \text{length}_\Gamma(\gamma_3) \\ &< \text{length}_\Gamma(\alpha_2) + \text{length}_\Gamma(\alpha_3) + \text{length}_\Gamma(\alpha_1) + 2d + 1 \\ &< 2k + 2d + 1, \end{aligned}$$

and

$$\begin{aligned} \text{length}_\Gamma(\omega_2) &= \text{length}_\Gamma(\alpha_3) + \text{length}_\Gamma(\beta_2) + \text{length}_\Gamma(\beta_1) + \text{length}_\Gamma(\gamma_2) \\ &\quad + \text{length}_\Gamma(\gamma_3) \\ &< \text{length}_\Gamma(\alpha_3) + \text{length}_\Gamma(\alpha_2) + \text{length}_\Gamma(\alpha_1) + 2d + 1 \\ &< 2k + 2d + 1. \end{aligned}$$

Therefore, the orders of both separations $S_{\{1\}}$ and $S_{\{1,2\}}$ are at most $d + k$ and we are done. $\square$

We are now ready to prove Theorem 3.1.

Let $k \geq 2$ and $d \geq 1$ be integers. Let Γ be a plane hypergraph of maximum edge order at most k and suppose there is an edge $e_0 \in E(\Gamma)$ such that $\text{dist}_\Gamma(v, V_\Gamma(e_0)) \leq d$ for every vertex $v \in V(\Gamma)$. We construct a branch-decomposition of Γ with width at most $d + k$ as follows. We first apply Lemma 3.3 and obtain an edge f , t pseudo- Γ -nooses v_j , $1 \leq j \leq t$, where $t = 2$ or 3 , and t sets of vertices or faces X_j , $1 \leq j \leq t$, such that the conditions of Lemma 3.3 are satisfied. We construct Γ_j , $1 \leq j \leq t$, as in the condition in the lemma, and a branch-decomposition T_t of Γ_t with width at most $d + k$.

Let $D = \bigcup_{1 \leq j \leq t} \text{discs}(v_j)$. For each disc $g \in D$, let ω_g be the Γ -noose such that $R(\omega_g) = g$. Since v_j for $j = 1, 2, 3$ is $(d + 1)$ -compact for Γ with center X_j

with $|X_j| \leq k/2 + 1$, ω_g for each $g \in D$ is $(d + 1)$ -compact for Γ with center Y_g for some Y_g with $|Y_g| \leq k/2 + 1$, by Lemma 3.1. We apply Lemma 3.2 to ω_g for each $g \in D$, and obtain a branch-decomposition T_g of $\Gamma|_{\text{rev}(\omega_g)}$ with width at most $d + k$. Applying Corollary 2.1 to $\Gamma|_{\text{rev}(\omega_g)}$, $g \in D$, and Γ_t , we combine the branch-decompositions T_g , $g \in D$, and T_t into a branch-decomposition of Γ of width at most $d + k$. This completes the proof of Theorem 3.1.

4 Upper Bound

The goal of this section is to prove Theorem 1.4, which is restated below.

Theorem 1.4 *Let G be a planar graph and k, h be integers with $k \geq 3$ and $h \geq 1$. Then G has either a minor isomorphic to a $k \times h$ cylinder or a branch-decomposition of width at most $k + 2h - 3$.*

We first sketch the proof ideas. The proof is by induction and at each induction step, we work on a plane hypergraph Γ , which is almost a graph, in the sense that all but one of its edges, say e_0 , are of order 2. The order of e_0 is smaller than k . We construct *contours* each of which consists of vertices of Γ that are equidistant from $F_\Gamma(e_0)$ in the radial graph of Γ . If the nest of contours is shallow then we may apply Theorem 3.1 and obtain a branch-decomposition of small width. If the nest is deep, then we look for promising separations of small order that separate those deep contours from e_0 and commit to those separations, hoping that they will lead to a branch-decomposition of small width. We may fail to find a separation of small order between e_0 and some deep contour, but then there must be a large enough number of vertex-disjoint paths from $V_\Gamma(e_0)$ to the vertices on the contour, which will form a large cylinder minor together with the nested contours.

We need some definitions and lemmas to formalize these ideas.

Let Γ be a plane hypergraph that is a graph, that is, every edge of Γ has order two, and r_0 a face of Γ . (We could alternatively say that Γ is a biconnected plane graph, but we would then need to repeat many definitions already done for plane hypergraphs to adapt them to plane graphs.) Let $\mathcal{R}_\Gamma(r_0, i)$ denote the subgraph of the radial graph $\mathcal{R}_\Gamma$ induced by the set of vertices corresponding to the set of vertices and faces of Γ that are at distance $2i + 2$ or greater from r_0 in $\mathcal{R}_\Gamma$. Then, each connected component of $\mathcal{R}_\Gamma(r_0, i)$ is bounded by a cycle of Γ consisting of vertices at distance $2i + 1$ from r_0 . We call such a cycle a *contour at depth i from r_0* . Let c be a contour at depth i from r_0 . We say that a vertex or face x of Γ is *inside of* contour c , if the vertex of $\mathcal{R}_\Gamma$ corresponding to x belongs to the connected component of $\mathcal{R}_\Gamma(r_0, i)$ bounded by c ; x is *outside of* c if it is not inside of c and is not a vertex on c . We say that Γ -noose v *separates* contour c from r_0 , if $R(v)$ contains all edges of c and $R(\text{rev}(v))$ contains r_0 . We say v *crosses* c if some face of Γ that lies inside of c is on v .

We need the following version of the Menger's theorem.

Lemma 4.1 *Let Γ be a plane hypergraph that is a graph, c_1 and c_2 two vertex-disjoint cycles of Γ , and k a positive integer. We say that a Γ -noose v *separates* c_1*

from c_2 if $R(v)$ contains all the edges of c_1 and $R(\text{rev}(v))$ contains all the edges of c_2 . Then, there are k pairwise vertex-disjoint paths from $V(c_1)$ to $V(c_2)$, if and only if there is no Γ -noose v of length smaller than $2k$ that separates c_1 from c_2 .

Proof One direction is straightforward: if v is a Γ -noose that separates c_1 from c_2 , then every path of Γ from $V(c_1)$ to $V(c_2)$ must intersect v . Therefore, k pairwise vertex-disjoint paths between c_1 and c_2 imply $\text{length}_\Gamma(v) \geq 2k$.

To prove the other direction, we suppose that P is a largest set of pairwise vertex-disjoint paths from $V(c_1)$ to $V(c_2)$ and show that there is a Γ -noose of length $2|P|$ that separates c_1 from c_2 . Recall that paths in our definition based on walks are oriented. The proof is an adaptation of the standard augmenting path argument for the maxflow-mincut theorem. Let $V(P) = \bigcup_{p \in P} V(p)$ and $E(P) = \bigcup_{p \in P} E(p)$. Let q be an arbitrary path in Γ . We say that an edge e in $E(q) \cap E(P)$ is a *backward edge* of q , if the orientation of e in q is the opposite from its orientation in a path in P . We say that a path q of Γ is a *partial augmenting path* if it is from a vertex on c_1 , each edge e in $E(q) \cap E(P)$ is a backward edge of q , and each vertex in $V(q) \cap V(P)$ except for the last vertex of q is incident with a backward edge of q . Let S denote the set of vertices of Γ such that there is a partial augmenting path ending at v . Then, S does not contain any vertex on c_2 , since otherwise there is a partial augmenting path that ends in a vertex on c_2 , which would imply that there are $|P| + 1$ vertex-disjoint paths from $V(c_1)$ to $V(c_2)$, a contradiction. Note that $V(c_1) \subseteq S$, since the path of length zero from each vertex on c_1 is a partial augmenting path by definition. For each path $p \in P$, let v_p denote the last vertex on p that belongs to S and e_p the edge on p that immediately follows v_p . By the definition of partial augmenting paths, all vertices of p up to v_p belong to S .

Let u and v be an arbitrary pair of vertices adjacent in Γ such that $u \in S$ and $v \notin S$. Then there is a partial augmenting path ending with u and this partial augmenting path cannot be extended to v , u and v consecutively appear in some path $p \in P$ in this order. This means that $u = v_p$ and the edge between u and v is e_p . Therefore, every path from $V(c_1)$ to $V(c_2)$ contains v_p and e_p for some $p \in P$.

Let $V_P = \{v_p \mid p \in P\}$, $E_P = \{e_p \mid p \in P\}$, and F_P the set of faces of Γ incident with some edge in E_P . Let H be the subgraph of the radial graph $\mathcal{R}_\Gamma$ induced by the set of vertices corresponding to the elements of $V_P \cup F_P$. Then, every path of Γ from $V(c_1)$ to $V(c_2)$ intersects H since it contains v_p and e_p for some $p \in P$. Therefore, H contains a cycle v that separates $V(c_1)$ from $V(c_2)$. Clearly, we have $\text{length}_\Gamma(v) \leq 2|V_P| = 2|P|$. Since every path in P intersects v , the equality must hold. $\square$

Lemma 4.2 *Let Γ be a plane graph that is a graph and $k \geq 3$, $h \geq 2$ integers. Let r_1 and r_2 be two faces of Γ such that the following conditions are satisfied.*

1. *There is no Γ -noose v of length smaller than $2k$ such that $r_1 \subseteq R(v)$ and $r_2 \subseteq R(\text{rev}(v))$.*
2. $\text{dist}_\Gamma(r_1, r_2) \geq 2h$.

Then, Γ contains a $k \times h$ cylinder as a minor.

Proof Since the vertex of $\mathcal{R}_\Gamma$ corresponding to r_2 belongs to $\mathcal{R}_\Gamma(r_1, h-1)$, there is a unique contour c_i at depth i from r_1 that contains r_2 in its inside, for each $0 \leq i < h$. By assumption 1, there is no Γ -noose of length smaller than $2k$ that separates c_{h-1} from c_0 . Therefore, by Lemma 4.1, there are k vertex-disjoint paths between $V(c_0)$ and $V(c_{h-1})$. These k vertex-disjoint paths, together with the h contours, constitute a minor of Γ isomorphic to a $k \times h$ cylinder. $\square$

The induction would be straightforward if this lemma held for plane hypergraphs that are not graphs. This unfortunately is not the case and we need a more elaborate structure of induction in which we work on hypergraphs that are almost graphs. An appropriate generalization of Theorem 1.4 for an induction proof turns out to be as follows.

Lemma 4.3 *Let Γ be a plane hypergraph with at least one edge and $k \geq 3$ and $h \geq 2$ be integers. Suppose that every edge of Γ except for one edge e_0 is of order 2 and e_0 is of order smaller than k . Then either Γ has a branch-decomposition of width at most $k + 2h - 3$ or the subhypergraph Γ' of Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \setminus \{e_0\}$, which is a graph, has a $k \times h$ cylinder as a minor.*

To prove this lemma, we need the following technical lemma.

Lemma 4.4 *Let Γ be a plane hypergraph that is a graph, r_0 a face of Γ , and c_1 , and c_2 be two distinct contours of Γ at depth d from r_0 , where d is a positive integer. For $i = 1, 2$, let v_i be a Γ -noose such that*

1. v_i separates c_i from r_0 ,
2. $\text{length}_\Gamma(v_i)$ is the smallest subject to condition 1, and
3. $R(v_i)$ is minimal subject to conditions 1 and 2.

Then, either v_1 separates both c_1 and c_2 from r_0 , v_2 separates both c_1 and c_2 from r_0 , or $\text{int}(R(v_1)) \cap \text{int}(R(v_2)) = \emptyset$.

Proof We first show that v_1 does not cross c_2 . Suppose otherwise. For each maximal subpath μ of v_1 that does not intersect any vertex on or inside of c_2 , let μ' denote the Γ -noose consisting of μ and a path of $\mathcal{R}_\Gamma$ around c_2 such that $R(\text{rev}(\mu'))$ contains all the edges of c_2 . Then, since c_1 and c_2 are edge-disjoint, there is one such μ such that $R(\mu')$ contains all the edges of c_1 . Let μ denote this particular path and let $\mu = v_1[v_1, v_2]$, where $v_1, v_2 \in V(c_2)$.

For each vertex $v \in V(c_1)$, let δ_v be a path in $\mathcal{R}_\Gamma$ of length d from v to r_0 . We assume without loss of generality that, for any pair of distinct vertices $v, w \in V(c_1)$, δ_v and δ_w do not cross each other. For each $v \in V(c_1)$, δ_v intersects v_1 since v_1 separates c_1 from r_0 . Let x_v denote the vertex or face of Γ in which δ_v and v_1 intersect each other; we choose the one closest to v on δ_v if there are multiple intersections. Note that, since every vertex on δ_v except for v is at distance smaller than $2d + 1$ from r_0 , no vertex on c_2 is on δ_v . Therefore, x_v is on the subpath μ of v_1 . For $i = 1, 2$, let $w_i \in V(c_1)$ be the vertex such that x_{w_i} is the closest to v_i on μ . We claim that w_1 and w_2 are adjacent on c_0 . For, otherwise there are two vertices $u_1, u_2 \in V(\Gamma)$ such that

u_1, w_1, u_2, w_2 appear in this order on c_1 while x_{u_1} and x_{u_2} appear between x_{v_1} and x_{w_2} on μ . This contradicts our assumption since δ_{u_i} and δ_{w_j} must cross each other for at least one pair $(i, j) \in \{1, 2\} \times \{1, 2\}$.

Since $\text{dist}_\Gamma(w_i, r_0) = \text{dist}_\Gamma(v_i, r_0) = 2d + 1$ for $i = 1, 2$, we have $\text{length}_\Gamma(\mu[v_1, x_{w_2}]) \geq \text{length}_\Gamma(\delta_{w_1}[w_1, x_{w_1}])$ and $\text{length}_\Gamma(\mu[x_{w_2}, v_2]) \geq \text{length}_\Gamma(\delta_{w_2}[w_2, x_{w_2}])$.

Let ν be a Γ -noose obtained by closing the path $\delta_{w_1}[w_1, x_{w_1}] + \mu[x_{w_1}, x_{w_2}] + \text{rev}(\delta_{w_2}[w_2, x_{w_2}])$ by a path of length two from w_2 to w_1 through a face outside of c_1 . Then $\text{length}_\Gamma(\nu) \leq \text{length}_\Gamma(v_1)$, all the edges of c_1 are contained in $R(\nu)$, and r_0 is contained in $R(\text{rev}(\nu))$. Since $R(\nu)$ is a proper subset of $R(v_1)$, this is a contradiction to the minimality of v_1 . This proves the claim that v_1 does not cross c_2 . Similarly, v_2 does not cross c_1 .

We are now to prove that either v_1 separates c_2 from r_0 , v_2 separates c_1 from r_0 , or $\text{int}(R(v_1)) \cap \text{int}(R(v_2)) = \emptyset$. Suppose that none of these three conditions hold. Since v_{3-i} does not cross c_i , for $i = 1, 2$, the assumption that v_{3-i} does not separate c_i from r_0 implies that no edge of c_i is contained in $R(v_{3-i})$. For the same reason, all the edges of c_i are contained in one connected component of $R(v_i) \setminus R(v_{3-i})$, which we call R_i . Let ν'_i be the Γ -noose such that $R(\nu'_i) = R_i$, for $i = 1, 2$. Then, ν'_i consists of subpaths of v_i and subpaths of $\text{rev}(v_{3-i})$. But if an oriented edge of v_i appears in ν'_i , then it is not contained in $R(v_{3-i})$ and hence its reversal does not appear in ν'_{3-i} . Therefore we have $\text{length}_\Gamma(\nu'_1) + \text{length}_\Gamma(\nu'_2) \leq \text{length}_\Gamma(v_1) + \text{length}_\Gamma(v_2)$ and hence $\text{length}_\Gamma(\nu'_i) \leq \text{length}_\Gamma(v_i)$ for $i = 1$ or 2 . Since we are assuming $\text{int}(R(v_1)) \cap \text{int}(R(v_2)) \neq \emptyset$ and hence $R(\nu'_i)$ is a proper subset of $R(v_i)$, this contradicts the minimality conditions for v_i . $\square$

We are now ready to prove Lemma 4.3. The proof is by induction on the number of edges of Γ . The base case $|E(\Gamma)| \leq 2$ is trivial. So, suppose $|E(\Gamma)| > 2$.

Suppose first that for every face r of Γ , we have $\text{dist}_\Gamma(r, F_\Gamma(e_0)) \leq 2h - 2$. Then, we apply the “dual” version of Theorem 3.1, in which the roles of vertices and faces are interchanged, with d replaced by $2h - 2$ and k replaced by $k - 1$, and obtain a branch-decomposition of Γ of width at most $(k - 1) + 2h - 2 = k + 2h - 3$.

Therefore, suppose that there is some face r of Γ such that $\text{dist}_\Gamma(r, F_\Gamma(e_0)) \geq 2h$. Let Γ' be a plane hypergraph with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \setminus \{e_0\}$. Then $F(\Gamma') = (F(\Gamma) \setminus F_\Gamma(e_0)) \cup \{r_0\}$, where $r_0 = e_0 \cup \bigcup_{r \in F_\Gamma(e_0)} r$. Since every edge of Γ except for e_0 is of order two, Γ' is a graph. Because there is a face r of Γ' with $\text{dist}_{\Gamma'}(r, r_0) \geq 2h$, Γ' has some contours at depth $h - 1$ from r_0 . Let $m > 0$ be the number of those contours and $c_1, \dots, c_m$ the list of those contours.

For each $1 \leq i \leq m$, let ν_i be a shortest Γ' -noose that separates c_i from r_0 . Suppose first that $\text{length}_\Gamma(\nu_i) \geq 2k$ for some $1 \leq i \leq m$. Let Γ'_i be the plane hypergraph obtained from Γ' by removing all vertices and edges that lie inside of c_i and thus making the inside of c_i into a single face. Applying Lemma 4.2 to Γ'_i with faces r_0 and r_i , we obtain a minor of Γ'_i isomorphic to a $k \times h$ cylinder and are done.

So, suppose $\text{length}_\Gamma(\nu_i) < 2k$ for every $1 \leq i \leq m$. Let $\mathcal{N} = \{\nu_i \mid 1 \leq i \leq m\}$. By Lemma 4.4, we may choose these Γ' -nooses in $\mathcal{N}$ so that, for each pair $1 \leq i < j \leq m$, $\text{int}(R(\nu_i)) \cap \text{int}(R(\nu_j)) = \emptyset$ unless either ν_i separates c_j from r_0 or ν_j separates c_i from r_0 . If ν_i separates c_j from r_0 then we remove ν_j from the collection; if ν_j separates c_i from r_0 then we remove ν_i from the collection. In this manner, we obtain a subcollection $\mathcal{N}'$ of Γ' -nooses such that

1. for each $v \in \mathcal{N}'$, $\text{length}_\Gamma(v) < 2k$,
2. for each $1 \leq i \leq m$, there is some $v \in \mathcal{N}'$ that separates c_i from r_0 , and
3. for each pair of distinct elements $v, v' \in \mathcal{N}'$, we have $\text{int}(R(v)) \cap \text{int}(R(v')) = \emptyset$.

Let $\mathcal{N}' = \{v'_1, \dots, v'_{m'}\}$. Noting that these Γ' -nooses are also Γ -nooses, let $\Gamma_0 = \Gamma$ and, for $1 \leq i \leq m'$, $\Gamma_i = \Gamma_{i-1} \cup v'_i$.

Since the faces in $F_\Gamma(e_0)$ are contained in face r_0 of Γ' , none of them is contained in $R(v)$ for any $v \in \mathcal{N}'$. Therefore, each of these faces are also a face of $\Gamma_{m'}$. By construction, each face r of $\Gamma_{m'}$ has $\text{dist}_{\Gamma_{m'}}(r, F_\Gamma(e_0)) \leq 2(h-1)$. Therefore, the $\Gamma_{m'}$ satisfies the dual version of Theorem 3.1, with d replaced by $2h-2$ and k by $k-1$, and hence has a branch-decomposition of width $\leq k+2h-3$.

Since $R(v'_i)$ does not contain edges of Γ incident with e_0 , the number of edges in $\Gamma|_{\text{rev}(v'_i)}$ is strictly smaller than the number of edges in Γ , for $1 \leq i \leq m'$. Therefore, we may apply the induction hypothesis to $\Gamma|_{\text{rev}(v'_i)}$ and obtain either a $k \times h$ cylinder as a minor or a branch-decomposition of width at most $k+2h-3$. If all of $\Gamma|_{\text{rev}(v'_i)}$, $1 \leq i \leq m'$, have such a branch-decomposition, we may apply Corollary 2.1 and combine those branch-decompositions and the branch-decomposition of $\Gamma_{m'}$ into a branch-decomposition of Γ of width at most $k+2h-3$. This completes the proof of Lemma 4.3.

In proving Theorem 1.4, we may assume that the given planar graph G is biconnected, since otherwise we may work separately on biconnected components. Then, the statement of the theorem immediately follows from the application of Lemma 4.3 to a plane hypergraph Γ that is a graph and is isomorphic to G . This completes the proof of Theorem 1.4.

5 Tightness of Theorem 3.1

In this section, we show that the bound of Theorem 3.1 is tight. That is, for every integer $k \geq 2$ and every even integer $d \geq 2$, there is a plane hypergraph satisfying the conditions of the theorem and has branchwidth exactly $d+k$.

A (k, h, w) -capped cylinder, denoted by $CC_{k,h,w}$ where $k \geq 2$, $h \geq 1$, $w \geq 2$ are integers, is a hypergraph that consists of the following:

1. a $kw \times h$ cylinder $C_{kw,h}$,
2. an edge e_0 , where the set $V(e_0)$ of vertices incident with e_0 is $\{v_j \mid 0 \leq j < k\}$ and is disjoint from $V(C_{kw,h})$,
3. an edge between vertex v_j of e_0 and vertex $(wj+l, 0)$ of $C_{kw,h}$, for $0 \leq j < k$ and $0 \leq l < w$,
4. an edge e_1 that is incident with vertices $(wj, h-1)$ for $0 \leq j < k$.

It is not difficult to verify that $CC_{k,h,w}$ has an essentially unique plane embedding: draw $C_{kw,h}$ on the side of a geometric cylinder in the standard manner, draw e_0 in the top of this cylinder, and draw e_1 in the bottom. In the following, we will refer to this plane hypergraph as $CC_{k,h,w}$.

The maximum edge order of $CC_{k,h,w}$ is k and for every vertex v of $CC_{k,h,w}$ we have $\text{dist}_{CC_{k,h,w}}(v, V(e_0)) \leq 2h$. Therefore by Theorem 3.1, we have a branch-decomposition of $CC_{k,h,w}$ with width $k+2h$. The following theorem shows that Theorem 3.1 is tight.

Theorem 5.1 *Let $k \geq 2$, $h \geq 1$, $w \geq 3h$ be integers. Then the branchwidth of $CC_{k,h,w}$ is at least $k + 2h$.*

To prove this theorem, we need the following notion and lemma. Let Γ be a plane hypergraph. A *noose-tangle* for Γ of order k , where $k > 0$ is an integer, is a collection $\mathcal{T}$ of separations of Γ of order strictly smaller than k that are induced by Γ -nooses, such that the following three axioms are satisfied.

1. For each separation (A, B) of order strictly smaller than k induced by some Γ -noose, exactly one of (A, B) and (B, A) is in $\mathcal{T}$.
2. If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq E(\Gamma)$.
3. If $(A, B) \in \mathcal{T}$, then $V_\Gamma(A) \neq V(\Gamma)$.

If we change the first axiom and require that exactly one of (A, B) and (B, A) to belong to $\mathcal{T}$ for every separation (A, B) of order smaller than k , then we get the definition of standard tangles [18]. Robertson and Seymour [18] show that a hypergraph Γ has a branch-decomposition of width k if and only if there is no tangle of Γ of order k .

For plane hypergraphs, we have the following parallel.

Lemma 5.1 *Let Γ be a plane hypergraph and let $k > 0$ be an integer. Then, Γ has a branch-decomposition of width k if and only if there is no noose-tangle for Γ of order $k + 1$.*

Proof Since a tangle of order $k + 1$ implies a noose-tangle of order $k + 1$, “if” part follows immediately from the corresponding direction of the tangle characterization of branchwidth. For “only if” part, suppose Γ has a branch-decomposition of width k . The result of Seymour and Thomas on bond carvings (5.1 in [21]) implies that Γ has a sphere-cut decomposition of width k . With an argument almost identical to the one showing that a tangle of order $k + 1$ precludes a branch-decomposition of width k [18], we can show that a noose-tangle of order $k + 1$ precludes a sphere-cut decomposition of width k . Therefore, Γ cannot have such a noose-tangle. $\square$

We are now ready to prove Theorem 5.1. Let $\Gamma = CC_{k,h,w}$, $k \geq 2$, $h \geq 1$, and $w \geq 3h$. Let ν be an arbitrary Γ -noose of length strictly smaller than $2(k + 2h)$. We claim that if ν separates e_0 from e_1 then

1. ν intersects either some vertex of e_0 or some vertex of e_1 , and
2. if ν intersects some vertex of e_0 and $e_0 \subseteq R(\nu)$ then $R(\nu)$ does not contain any edge in row $h - 1$ of the cylinder in Γ .

The first claim holds, since if ν separates e_0 from e_1 and intersects only the vertices of the cylinder then it must intersect at least one vertex in each of the $kw \geq 3kh \geq k + 2h$ columns. For the second claim, suppose ν intersects some vertex of e_0 and $e_0 \subseteq R(\nu)$. If ν does not intersect some vertex v_j , $0 \leq j < k$, then it must intersect at least one vertex in each of the w columns indexed by wj through $wj + w - 1$. This is impossible since $w \geq 3h$ and therefore ν intersects all the k vertices of e_0 . To have an edge of in row $h - 1$, ν must intersect at least 2 vertices in each of the h rows of the cylinder, which is impossible.

Our noose-tangle $\mathcal{T}$ for Γ of order $k + 2h$ is defined as follows. Let (A, B) be an arbitrary separation of Γ induced by some Γ -noose, say ν , whose order is strictly smaller than $k + 2h$. If $e_0, e_1 \in B$ then we put (A, B) in $\mathcal{T}$; if $e_0, e_1 \in A$ then we put (B, A) in $\mathcal{T}$; if $e_0 \in A, e_1 \in B$ then we put (A, B) in $\mathcal{T}$ if ν intersects vertices of e_0 and put (B, A) in $\mathcal{T}$ otherwise.

$\mathcal{T}$ satisfies the first axiom for noose-tangles due to its construction.

To verify the second axiom, suppose for contradiction that $\bigcup_{1 \leq i \leq 3} A_i = E(\Gamma)$ where $(A_i, B_i) \in \mathcal{T}$ for $1 \leq i \leq 3$. Since it is impossible for any A_i to have both e_0 and e_1 , at least one of them, say A_1 contains e_0 and at least one of the others, say A_2 contains e_1 . By the second claim above, A_1 does not contain any edge of row $h - 1$ of the cylinder. Let F be the set of edges in row $h - 1$ contained in A_2 . Then the simple Γ -noose ν_2 that induces separation (A_2, B_2) must intersect at least one vertex in each column of the cylinder that contains a vertex incident to some edge in F . Therefore, we have $|F| \leq k + 2h - 1$. Similarly, the number of edges of row $h - 1$ belonging to A_3 is at most $k + 2h - 1$. Therefore, $A_1 \cup A_2 \cup A_3$ contains at most $2k + 4h - 2 < kw$ vertices, a contradiction.

For the last axiom, let (A, B) be an arbitrary member of $\mathcal{T}$. If this separation separates e_0 from e_1 then $V_\Gamma(A)$ cannot contain both vertices of e_0 and e_1 , by our second claim above. If $e_0, e_1 \in B$, on the other hand, $V_\Gamma(A)$ may contain at most $k + 2h$ vertices of any single row of the cylinder. In either case, we have $V_\Gamma(A) \neq V(\Gamma)$.

Therefore, $\mathcal{T}$ is indeed a noose-tangle for Γ of order $k + 2h$ and by Lemma 5.1, we have the result. This completes the proof of Theorem 5.1.

Remark 5.1 The proof would work, with finer calculations, for somewhat smaller values of w .

6 The Branchwidth and Grid-Minor Size of a Cylinder

In this section, we study the branchwidth and the grid-minor size of the cylinder.

Theorem 6.1 *For arbitrary integers $k \geq 3$ and $h \geq 1$, we have $\text{bw}(C_{k,h}) = \min\{k, 2h\}$.*

Proof The upper bound can be exhibited by simple “linear” branch-decompositions as follows.

Consider first a linear branch-decomposition of $C_{k,h}$ induced by the “row-major” total order $<_{\text{row}}$ defined as follows. We first define the row major order $<_{\text{row}}$ on $V(C_{k,h})$ to be a lexicographic total order: $(i_1, j_1) <_{\text{row}} (i_2, j_2)$ if and only if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$. We extend this order to a total order on $E(C_{k,h})$ as follows. For $e_1, e_2 \in E(C_{k,h})$, let $e_1 = \{u_1, v_1\}$ with $u_1 <_{\text{row}} v_1$ and let $e_2 = \{u_2, v_2\}$ with $u_2 <_{\text{row}} v_2$. Then $e_1 <_{\text{row}} e_2$ if and only if $u_1 <_{\text{row}} u_2$ or $u_1 = u_2$ and $v_1 <_{\text{row}} v_2$.

For each separation (A, B) in this branch-decomposition, vertices in $V(A) \cap V(B)$ are contained in one row and therefore the width of this branch-decomposition is k .

Next consider another linear branch-decomposition of $C_{k,h}$ induced by the “column-major” order defined similarly but with a slight modification so that the edges between column 0 and column $k - 1$ are the largest in this ordering.

For each separation (A, B) in this branch-decomposition, vertices in $V(A) \cap V(B)$ are contained in two columns (column 0 and another) and therefore the width of this branch-decomposition is $2h$. These two types of branch-decompositions prove that $\text{bw}(C_{k,h}) \leq \min\{k, 2h\}$.

To prove the inequality in the other direction, we use noose-tangles of Sect. 5.

We assume an arbitrary plane hypergraph Γ that is a graph and is isomorphic to $C_{k,h}$ as a graph and construct a noose-tangle $\mathcal{T}$ of order k with respect to this embedding. Let D_1 be the set of edges in row 0 of $C_{k,h}$ and D_2 the set of edges in row $h - 1$. Let (A, B) be an arbitrary noose-induced separation of Γ of order smaller than $\min\{k, 2h\}$, and let ν denote the noose that induces this separation. Since there are k vertex-disjoint paths between row 0 and row $h - 1$ of $C_{k,h}$, by Menger’s theorem ν cannot separate D_1 and D_2 . Moreover, ν cannot separate D_1 into more than one non-empty subsets and at the same time separate D_2 into more than one non-empty subset, since if it did then its length would be at least $2h$. To summarize, exactly one of A and B contains D_1 or D_2 . If A does then we put (B, A) in $\mathcal{T}$ and if B does then we put (A, B) in $\mathcal{T}$. As a result, for each $(A, B) \in \mathcal{T}$, we have either $A \cap D_1 = \emptyset$ and $D_2 \setminus A \neq \emptyset$ or $A \cap D_2 = \emptyset$ and $D_1 \setminus A \neq \emptyset$.

The first axiom of noose-tangles holds because of the construction.

The second axiom holds, since if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then it is impossible for $A_1 \cup A_2 \cup A_3$ to contain all the edges in D_1 and D_2 ; at least two of A_1, A_2, A_3 must be dedicated to cover D_1 and the same holds for D_2 .

The third axiom holds, since for each $(A, B) \in \mathcal{T}$, $V_\Gamma(A)$ does not contain either the vertices in row 0 or the vertices in row $h - 1$ and hence $V_\Gamma(A) \neq V(\Gamma)$.

Therefore, G has a noose-tangle of order k and the branchwidth of G is at least k . $\square$

Next we prove Theorem 1.3 that states $\text{gm}(C_{2h,h}) = h$. One direction, $\text{gm}(C_{2h,h}) \geq h$, is trivial as a $h \times h$ grid is clearly a subgraph of $C_{2h,h}$. The other direction, $\text{gm}(C_{2h,h}) \leq h$, may be somewhat surprisingly the most difficult result to prove in this paper.

We first review a few basic facts about the branch-decomposition of grids.

It is well-known that $\text{bw}(G_{k,h}) = \min\{k, h\}$. This can be shown similarly to the proof of Theorem 6.1 which determines the branchwidth of cylinders. We need a slightly stronger observation. We call a vertex set $W \subseteq V(G_{k,h})$ *semi-vertical*, if $W = \{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\}$ with $j_1 < j_2 < \dots < j_m$ and $|i_{a+1} - i_a| \leq j_{a+1} - j_a$ for $1 \leq a < m$.

Lemma 6.1 *Let $k \geq h \geq 1$ be integers. Let $W \subseteq V(G_{k,h})$ be an arbitrary non-empty semi-vertical set of vertices and Λ a hypergraph with $V(\Lambda) = V(G_{k,h})$ and $E(\Lambda) = E(G_{k,h}) \cup \{e_0\}$, where $V_\Lambda(e_0) = W$. Then $\text{bw}(\Lambda) = h$.*

Proof Since $\text{bw}(\Lambda) \geq \text{bw}(G_{k,h}) \geq \min\{k, h\} = h$, it suffices to exhibit a branch-decomposition of Λ of width h . First observe that each semi-vertical set W can be

extended into a semi-vertical set $W' \supseteq W$ with $|W'| = h$. Moreover, the width of a branch-decomposition of a hypergraph does not increase when we replace an edge e of that hypergraph with some edge e' with $V(e') \subseteq V(e)$. Therefore, it suffices to give a construction for the case $|W| = h$. So let $W = \{(i_0, 0), (i_1, 1), \dots, (i_{h-1}, h-1)\}$. Let $W_1 = \{(i, j) \in V(G_{k,h}) \mid (i < i_j)\}$ and $W_2 = \{(i, j) \in V(G_{k,h}) \mid (i > i_j)\}$. Note that $W \cup W_1 \cup W_2 = V(G_{k,h})$ and that W , W_1 , and W_2 are pairwise disjoint. Let $E_1 = \{e \in E(G_{k,h}) \mid V_{G_{k,h}}(e) \cap W_1 \neq \emptyset\}$ be the set of edges incident with a vertex in W_1 , $E_2 = \{e \in E(G_{k,h}) \mid V_{G_{k,h}}(e) \cap W_2 \neq \emptyset\}$ the set of edges incident with a vertex in W_2 , and $E_3 = E(G_{k,h}) \setminus (E_1 \cup E_2)$. Note that $E_1 \cap E_2 = \emptyset$ and therefore E_1 , E_2 , and E_3 partitions $E(G_{k,h})$.

Let $<_{\text{col}}$ be the column major total order on $V(G_{k,h}) (= V(C_{k,h}))$ as defined in the proof of Theorem 6.1. We define a total order $<$ on $|E(\Lambda)|$ as follows.

1. For each $e_1 \in E_1$, each $e_2 \in E_2$ and each $e_3 \in E_3$, $e_1 < e_3 < e_0 < e_2$.
2. For each pair $e_1, e_2 \in E(G_{k,h})$ for which the order is not specified by the above, let $e_1 = (u_1, v_1)$ with $u_1 <_{\text{col}} v_1$ and $e_2 = (u_2, v_2)$ with $u_2 <_{\text{col}} v_2$. Then, $e_1 < e_2$ if and only if $u_1 <_{\text{col}} u_2$ or $u_1 = u_2$ and $v_1 <_{\text{col}} v_2$.

Let (A, B) be an arbitrary separation of $E(\Lambda)$ such that $e_1 < e_2$ for each $e_1 \in A$ and each $e_2 \in B$. We claim that $V(A) \cap V(B)$ contains at most one vertex from each row. To verify this claim, suppose for a contradiction that $(i, j_1), (i, j_2) \in V(A) \cap V(B)$ for some row i , where $j_1 < j_2$. Let $e_1 \in B$ be an edge incident to (i, j_1) and $e_2 \in A$ an edge incident to (i, j_2) . From the assumption on the separation (A, B) , we must have $e_2 < e_1$. Since $(i, j_1) \in e_1$ and $(i, j_2) \in e_2$ with $j_1 < j_2$ implies either $e_1 = e_2$ or $e_1 <_{\text{col}} e_2$, this order $e_2 < e_1$ must be determined by the first case of the definition of $<$ and therefore we have either $e_2 \in E_1$ and $e_1 \in E_3 \cup \{e_0\} \cup E_2$, $e_2 \in E_3$ and $e_1 \in \{e_0\} \cup E_2$, or $e_2 = e_0$ and $e_1 \in E_2$. In each of these cases, we must have $j_2 \leq j_1$, a contradiction. This verifies the claim.

Therefore, the width of the branch-decomposition of Λ induced by this total order $<$ is h . $\square$

Since $C_{2h,h}$ contains $G_{h,h}$ as a subgraph and hence as a minor, that $\text{gm}(C_{2h,h}) \geq h$ clearly holds. Therefore, it suffices to show that $C_{2h,h}$ does not contain $G_{h+1,h+1}$ as a minor in order to prove Theorem 1.3.

We fix $h \geq 2$ and, for a contradiction, assume that $C_{2h,h}$ contains a minor isomorphic to $G_{h+1,h+1}$. To deal with $C_{2h,h}$, we often need arithmetic on column indices, which are naturally modulo $2h$. We use arithmetic symbols $\oplus$ and $\ominus$ for this purpose: $a \oplus b$ is integer c with $0 \leq c < 2h$ such that $a + b \equiv c \pmod{2h}$ and $a \ominus b$ is integer c with $0 \leq c < 2h$ such that $a - b \equiv c \pmod{2h}$.

Let M be a minor of $C_{2h,h}$ isomorphic to $G_{h+1,h+1}$ and let H be a minimal subgraph of $C_{2h,h}$ that is contracted into M . Let $\psi : V(H) \rightarrow V(M)$ denote the mapping such that $\psi(v)$ is the vertex of M into which v is contracted. By the minimality of H , the following holds.

1. For each $v \in V(M)$, the subgraph of H induced by the vertex set $\psi^{-1}(v)$ is a tree. We denote this tree by T_v .
2. For each pair of vertices u, v adjacent in M , there is exactly one edge between $\psi^{-1}(u)$ and $\psi^{-1}(v)$.

We embed $C_{2h,h}$ as a plane hypergraph Γ , H as a plane hypergraph Δ such that $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$. The embedding of M is a plane hypergraph Ω obtained from Δ as follows. For each $v \in V(M)$, we replace the tree in Δ representing T_v in H by a single vertex that is the union of all the vertices and edges in the tree. For each $0 \leq i < 2h$ and $0 \leq j < h$, we denote by c_{ij} the vertex of Γ representing vertex (i, j) of $C_{2h,h}$.

Proposition 6.1 *We have $E(\Omega) \subseteq E(\Gamma)$ and, for every $e \in E(\Gamma) \setminus E(\Omega)$, $e \subseteq x$ for some $x \in V(\Omega) \cup F(\Omega)$. For every $v \in V(\Gamma)$, we have $v \subseteq x$ for some $x \in V(\Omega) \cup F(\Omega)$. For every $r \in F(\Gamma)$, we have $r \subseteq r'$ for some $r' \in F(\Omega)$.*

From now on, we identify the graphs H and M with their embeddings Δ and Ω , respectively. In particular, we view the mapping $\psi : V(H) \rightarrow V(M)$ as a mapping from $V(\Delta)$ to $V(\Omega)$. Thus, we have $v \subseteq \psi(v)$ for each $v \in V(\Delta)$.

We call the face of Γ bounded by the cycle in row 0 the *top face* and the one bounded by the cycle in row $h - 1$ the *bottom face* of Γ .

Lemma 6.2 *Suppose Ω has a face that contains both the top and bottom faces of Γ . Then, a $k \times h$ grid $G_{k,h}$ for sufficiently large k contains a minor isomorphic to Ω .*

Proof Let r be a face of Ω that contains both the top and bottom faces of Γ . Choose an arbitrary path μ of Γ contained in r that connects the top and the bottom faces of Γ . Let Γ' be the subhypergraph of Γ with $E(\Gamma') = \{e \in E(\Gamma) \mid e \cap \mu = \emptyset\}$ and $V(\Gamma') = V(\Gamma)$. Then, $k \times h$ grid $G_{k,h}$ for sufficiently large k contains a subgraph G isomorphic to Γ' : the isomorphism from Γ' to G is obtained by viewing the sphere as a geometrical cylinder whose top/bottom faces are the top/bottom faces of Γ and cutting the side of this cylinder along μ into a strip. Since Γ' , as a graph, has Ω as a minor, $G_{k,h}$ has a minor isomorphic to Ω . $\square$

We call the unique face of Ω bounded by $4h$ edges the *large face* of Ω . We call the four vertices of degree 2 on the large face of Ω the *corner vertices* of Ω . Let S_0 , S_1 , S_2 , and S_3 be oriented paths between adjacent corner vertices such that their cyclic concatenation in this order is a directed cycle S that bounds the large face of Ω .

We define some more symbols B , B_0 , B_1 , B_2 , B_3 , R_{out} and R_{in} to be used throughout the proof as follows.

B is an oriented cycle of Δ , hence of Γ , such that $\psi(B) = S$. For each i , $0 \leq i < 4$, B_i is the maximal subpath of B such that $\psi(B_i) = S_i$. Of the two discs of Σ bounded by B , R_{out} is the one that contains the large face of Ω and R_{in} is the other.

We assume the following throughout the proof.

A1 In the embedding Γ of the cylinder $C_{2h,h}$, the columns are arranged clockwise around the bottom face under the ascending order of the column indices.

A2 When we proceed along the oriented cycle B of Γ , we see R_{in} to the right.

Let ω be an arbitrary walk on $\mathcal{R}_\Gamma$. By Proposition 6.1, ω does not intersect any edge of Ω and defines a walk on $V(\Omega) \cup F(\Omega)$. Let ω' be the walk on $\mathcal{R}_\Omega$ that defines the same walk on $V(\Omega) \cup F(\Omega)$ as ω . We call ω' the walk in $\mathcal{R}_\Omega$ corresponding to ω . Clearly, $\text{length}_\Omega(\omega') \leq \text{length}_\Gamma(\omega)$.

We say that vertices $u, v \in V(B)$ are *antipodal* to each other, if $u \in V(B_i)$ and $v \in (B_{(i+2) \bmod 4})$ for some $0 \leq i < 4$. We call a path μ of $\mathcal{R}_\Gamma$ an *antipodal chord* if μ is between an antipodal pair of vertices and avoids R_{out} .

Lemma 6.3 *Every antipodal chord has length $2h$ or greater in Γ .*

Proof Let u and v be arbitrary vertices of B that are antipodal and μ an antipodal chord between u and v . Let μ' be the path in $\mathcal{R}_\Omega$ corresponding to μ . Then, since μ' is between $\psi(u)$ and $\psi(v)$ which belong to two opposite sides of the $(h+1) \times (h+1)$ grid, we have $\text{length}_\Gamma(\mu) \geq \text{length}_\Omega(\mu') \geq 2h$. $\square$

A *slit* is a path in $\mathcal{R}_\Gamma$ that connects distinct vertices $u, v \in V(B)$, intersects each row of Γ at most once, and stays in R_{in} . Let σ be a slit connecting $u, v \in V(B)$. Since $\text{length}_\Gamma(\sigma) < 2h$, the pair u, v is not antipodal by Lemma 6.3. It follows that, of the two parts of R_{in} separated by σ , one is incident with B_i for three or more indices i and the other is incident with B_i for two or fewer indices i . We call the former the *major side* and the latter the *minor side* of slit σ .

Lemma 6.4 *Let σ be a slit. Then the major side of σ contains either the top face or the bottom face of Γ .*

Proof Let σ be a slit and assume σ connects vertices b_1 and b_2 in $V(B)$. Let σ' be the path in $\mathcal{R}_\Omega$ corresponding to σ and let W be a set of vertices of Γ on σ such that, for each vertex v on σ' , exactly one vertex in $\psi^{-1}(v)$ is contained in W .

Let R_1 and R_2 be the major and minor sides of this slit respectively. Let Ω_i , $i = 1, 2$, be the subhypergraph of Ω such that $E(\Omega_i)$ consists of the edges of Ω contained in R_i and $V(\Omega_i)$ consists of vertices of Ω that are intersected by R_i . Note that $V(\Omega_1) \cap V(\Omega_2) = \psi(W)$ and $(E(\Omega_1), E(\Omega_2))$ is a separation of Ω with order $|\psi(W)|$.

Suppose R_1 contains neither the top nor the bottom face of Γ . We argue similarly to the proof of Lemma 6.2. Choose an arbitrary curve ν that connects the top and the bottom faces of Γ avoiding R_1 . Let Γ' be a subhypergraph of Γ induced by the edge set $\{e \in E(\Gamma) \mid e \cap \nu = \emptyset\}$. Let ϕ be a natural isomorphism, obtained similarly to the one in the proof of Lemma 6.2, from Γ' to a subgraph of $k \times h$ grid $G_{k,h}$, where k is large enough. Let M_2 be a graph isomorphic to Ω_2 with an isomorphism $\chi : V(\Omega_2) \rightarrow V(M_2)$. Let J be a graph obtained from $G_{k,h}$ by attaching M_2 by identifying, for each $w \in W$, the vertex $\phi(w)$ of $G_{k,h}$ and the vertex $\chi(\psi(w))$ of M_2 . In this construction of J we disregard the embeddings of $G_{k,h}$ and M_2 , dealing with them simply as graphs.

We claim that $\text{bw}(J) = h$. This gives us the desired contradiction, because J contains $G_{h+1,h+1}$ as a minor and hence $\text{bw}(J) \geq h+1$. Since σ is a slit, $\phi(W)$ is semi-vertical. Therefore, the hypergraph Λ with $V(\Lambda) = G_{k,h}$ and $E(\Lambda) = E(G_{k,h}) \cup \{e_0\}$, where e_0 is incident with each vertex in $\phi(W)$, has branchwidth $\text{bw}(\Lambda) = h$, by Lemma 6.1. Moreover, the hypergraph M'_2 with $V(M'_2) = V(M_2)$ and $E(M'_2) = E(M_2) \cup \{e_0\}$ has branchwidth $\text{bw}(M'_2) \leq h$ since M_2 is a subgraph of the $(h+1) \times (h+1)$ grid M drawn on the minor side of the slit. Therefore, by

Lemma 2.1, we have $\text{bw}(J) = h$. This verifies the claim and completes the proof of the lemma. $\square$

Lemma 6.5 *There does not exist a pair of slits σ_1 and σ_2 that satisfies the following conditions.*

1. σ_1 and σ_2 do not intersect each other,
2. the major side of σ_1 contains the minor side of σ_2 (and hence vice versa), and
3. the minor side of σ_1 contains the bottom face of Γ , and the minor side of σ_2 contains the top face of Γ .

Proof Since the proof is similar to the proof of Lemma 6.4, we only give a sketch. Suppose there is such a pair of slits σ_1 and σ_2 satisfying the above conditions. Let Ω_i , for $i = 1, 2$, be the subhypergraph of Ω consisting of the vertices and faces of Ω contained in or intersecting the minor side of σ_i . We attach a graph isomorphic to Ω_1 and a graph isomorphic to Ω_2 to a grid $G_{k,h}$ for sufficiently large k in an appropriate manner to construct a graph of branchwidth h that has a minor isomorphic to $G_{h+1,h+1}$, deriving a contradiction. $\square$

Lemma 6.6 *Region R_{in} must contain at least one of the top and bottom faces of Γ .*

Proof Suppose that R_{in} does not contain either the top or bottom face of Γ . Then, the large face of Ω contains both the top and bottom face of Γ . It follows from Lemma 6.2 that $G_{k,h}$ for sufficiently large k contains $G_{h+1,h+1}$ as a minor, a contradiction since $\text{bw}(G_{k,h}) = h$ for $k \geq h$ and $\text{bw}(G_{h+1,h+1}) = h + 1$. $\square$

From now on, we add the following to our list of assumptions.

A3 R_{in} contains the bottom face of Γ .

We also need some more global symbols: L , L_0 , L_1 , L_2 , and L_3 . We say that vertex $v = c_{ij} \in V(B)$ of the cylinder Γ is *extremal* if $j \geq j'$ for every j' such that $c_{ij'} \in V(B)$. Then, L is the set of all extremal vertices in $V(B)$ and $L_i = L \cap V(B_i)$ for $0 \leq i < 4$.

The following lemma is straightforward.

Lemma 6.7 *When we arrange the elements of L cyclically according to the ascending order of column indices, the elements of L_i for each i appear consecutively, and as a whole, they appear in the cyclic order of L_0, L_1, L_2, L_3 , skipping empty sets if any.*

We fix a point b_0 in the bottom face. For each vertex $v = c_{ij}$ of Γ contained in the closure of R_{in} , let β_v^+ and β_v^- be fixed paths in $\mathcal{R}_\Gamma$ chosen as follows. Path β_v^+ starts from $v = c_{ij}$, threads vertices $c_{(i \oplus a)(j+a)}$, for $a = 1, \dots, h - j - 1$, in this order and ends in b_0 . Similarly path β_v^- starts from $v = c_{ij}$, threads vertices $c_{(i \ominus a)(j+a)}$, for $a = 1, \dots, h - j - 1$, in this order and ends in b_0 .

Let $v = c_{ij} \in V(B)$. We say that vertex $u = c_{i'j'} \in V(B)$ *obstructs* β_v^+ if $i' = i \oplus a$ and $j' > j + a$ for some $0 \leq a < h - j$; that u obstructs β_v^- if $i' = i \ominus a$ and $j' > j + a$ for some $0 \leq a < h - j$. We also say that u obstructs v if it obstructs either β_v^+ or β_v^- .

We say that $v \in V(B)$ is *exposed* if both β_v^+ and β_v^- stays in R_{in} . Note that v is exposed if and only if no $u \in V(B)$ obstructs it. It is also true that v is exposed if and only if no exposed vertex $u \in V(B)$ obstructs it.

Let X be the set of exposed vertices of $V(B)$ and $X_i = V(B_i) \cap X$ for $0 \leq i < 4$. Since $X_i \subseteq L_i$ for each $0 \leq i \leq 4$, the following lemma follows from Lemma 6.7.

Lemma 6.8 *When we arrange the elements of X cyclically according to the ascending order of column indices, the elements of X_i for each i appear consecutively, and as a whole, they appear in the cyclic order of X_0, X_1, X_2, X_3 , skipping empty sets if any.*

Lemma 6.9 *At least one of X_0 and X_2 is empty and at least one of X_1 and X_3 is empty.*

Proof We prove the first part of the statement; the proof of the second part is similar. For contradiction, suppose X_0 and X_2 are both non-empty. If there are $v_0 = c_{i_0 j_0} \in X_0$ and $v_2 = c_{i_2 j_2} \in X_2$ with $i_2 \ominus i_0 < h$, let μ be the concatenation of $\beta_{v_0}^+$ and $\beta_{v_2}^-$. Then, μ is an antipodal chord of length strictly smaller than $2h$, a contradiction to Lemma 6.3. If there are $v_0 = c_{i_0 j_0} \in X_0$ and $v_2 = c_{i_2 j_2} \in X_2$ with $i_0 \ominus i_2 < h$, we have a similar contradiction.

Therefore, the only possibility is that X_0 has only one element $v_0 = c_{i_0 j_0}$, X_2 has only one element $v_2 = c_{i_2 j_2}$, and $i_2 \ominus i_0 = h$. If $j_0 + j_2 \geq h$, a path in $\mathcal{R}_\Gamma$ between v_0 and v_2 going through the bottom face is an antipodal chord of length strictly smaller than $2h$, a contradiction to Lemma 6.3. So suppose $j_0 + j_2 < h$. Then, vertices in column $i_0 \oplus a$ for $j_0 \leq a \leq h - j_2$ are not obstructed by v_0 or v_2 . Therefore, there must be some vertex $v_1 = c_{i_1 j_1} \in X$ where $0 < i_1 \ominus i_0 < h$. Since $X_0 = \{v_0\}$ and $X_2 = \{v_2\}$, by Lemma 6.8, v_1 must belong to X_1 . Similarly, there is a vertex $v_3 = c_{i_3 j_3} \in X_3$, where $0 < i_3 \ominus i_2 < h$. By an argument similar to the one in the beginning of this proof, $X_1 = \{v_1\}$, $X_3 = \{v_3\}$, and $i_3 \ominus i_1 = h$. Assume without loss of generality that j_0 is the largest among j_q , $0 \leq q < 4$. It is clear that $j_0 > 0$, since otherwise none of v_0, v_1, v_2, v_3 obstructs any other vertices of B and hence there must be other vertices in X . Since v_0 is the vertex of B with the largest row index in column i_0 , either $c_{(i_0 \oplus 1) j_0}$ or $c_{(i_0 \ominus 1) j_0}$ must be in $V(B)$. We assume the former: the other case is similar. Then, $c_{(i_0 \oplus 1) j_0}$ is not obstructed by any vertex of X and hence must be exposed. This means that $i_1 = i_0 \oplus 1$ and $i_3 = i_2 \oplus 1$. Let $i = i_0 \oplus 1 \oplus j_0$. Then, vertices in column i are not obstructed by v_0 or v_1 since $j_0 = i \ominus (i_0 + 1)$; they are not obstructed by v_2 since $j_0 + j_2 < h$ and hence

$$\begin{aligned} j_2 &< h - j_0 \\ &= (i_2 \ominus i_0) - j_0 - 1 \\ &= i_2 \ominus i. \end{aligned}$$

It is impossible for v_3 to obstruct vertices in column i , since it must obstruct either v_0 or v_2 in order to do so, but v_0 and v_2 are exposed. Therefore, there must be some exposed vertex in column i , a contradiction since only columns i_q for $0 \leq q < 4$ have exposed vertices. $\square$

Lemma 6.10 *Either L_0 or L_2 is empty and either L_1 or L_3 is empty.*

Proof We first assume that both L_0 and L_2 are non-empty and show that both L_1 and L_3 are non-empty. To show that L_1 is non-empty, we suppose L_1 is empty and derive a contradiction. Let i_0 and i_2 be column indices such that column i_0 contains an element of L_0 , column i_2 contains an element of L_2 , and column $i_0 \oplus a$ for $0 < a < i_2 \ominus i_0$ does not contain any element of L_0 or of L_2 . If $i_2 = i_1 \oplus 1$, there is trivially an antipodal chord of length strictly smaller than $2h$ between v_0 and v_2 , a contradiction to Lemma 6.3. So, it must be that $i_2 \ominus i_0 > 1$, that is, there are columns after i_0 and before i_2 and they are free of vertices of B . Along the directed subpath of B from v_0 to v_2 , the next vertex of v_0 must be $c_{i_0(j_0-1)}$, which we call v'_0 . Since $L_1 = \emptyset$, $v_0 \notin V(B_1)$ and therefore $v'_0 \in V(B_0)$ (v'_0 may also be in $V(B_1)$ but that is irrelevant). Then, there is an antipodal chord between v'_0 and v_2 of length $j_0 + j_2$ through the top face and one of length $2h - j_0 - j_2 - 1$ through the bottom face. At least one of them has length smaller than $2h$, a contradiction. Therefore, L_1 is non-empty. Similarly, L_3 is non-empty under the assumption that both L_0 and L_2 are non-empty.

If we assume that L_1 and L_3 are both non-empty, then we have both L_2 and L_4 non-empty by an argument similar to the above. Therefore, if the conclusion of the Lemma does not hold, then all of L_0 , L_1 , L_2 , and L_3 are non-empty; from this we derive a contradiction below.

Owing to Lemma 6.9 and using symmetry, we have to consider only two cases.

(Case 1) X_0 and X_1 are non-empty but X_2 and X_3 are empty. Recall Lemma 6.8 and let $x_0 = c_{i_0 j_0} \in X_0$ and $x_1 = c_{i_1 j_1} \in X_1$ such that column $i_1 \oplus a$ for $0 < a < i_0 \ominus i_1$ does not contain any element of X . Then, each element of $L_2 \cup L_3$ must be obstructed by either x_0 or x_1 ; otherwise it would be exposed and contradicts the assumption $X_2 = X_3 = \emptyset$. Recall Lemma 6.7 and let i^* be the index of the last column, under the standard cyclic order, whose extremal vertex $v = c_{i^* j^*}$ is in L_2 . Either x_0 or x_1 obstructs v and, because of the relative column positions of x_0 , x_1 and v , x_0 obstructs β_v^+ in the first case and x_1 obstructs β_v^- in the second case. Suppose first that x_0 obstructs β_v^+ . Let $v' = c_{i^* j'}$ be the vertex in column i^* such that $\beta_{v'}^+$ goes through x_0 . Since x_0 obstructs β_v^+ , we have $j' > j^*$. Let μ be a path in $\mathcal{R}_\Gamma$ from v to x_0 , that first threads vertices in column i^* between v and v' and then follow $\beta_{v'}^+$ up to x_0 . Then μ is an antipodal chord of length $< 2h$, a contradiction. Suppose next that x_0 does not obstruct β_v^+ and therefore x_1 obstructs β_v^- . Let u be the extremal vertex in column $i^* \oplus 1$ and let j'' be the row index of u . By the choice of i^* , $u \in L_3$. If β_u^- is obstructed by x_1 , we have a contradiction similar to the above. So, suppose that β_u^- is not obstructed by x_1 and therefore β_u^+ is obstructed by x_0 . To summarize current assumptions, x_1 obstructs β_v^- but not β_u^- and x_0 obstructs β_u^+ but not β_v^+ . This is possible only if $j^* = j''$, β_v^+ goes through x_0 , and β_u^- goes through x_1 . The part of β_v^+ connecting v and x_0 provides an antipodal chord of length smaller than $2h$, a contradiction.

(Case 2) X_0 is non-empty but X_1 , X_2 , X_3 are all empty. Since some element of L_2 is obstructed by some element of X_0 , we have a contradiction similarly to Case 1. $\square$

An ordered pair of columns of Γ , with indices i_1 and i_2 , is a *band* if it satisfies the following conditions.

1. Both column i_1 and column i_2 of Γ have a vertex of B .
2. For each $i = i_1 \oplus a$, $0 < a < i_2 \ominus i_1$, each edge e of Γ between a vertex in column $i \oplus a$ and a vertex in column $i \oplus a \oplus 1$ is drawn in R_{in} , i.e., $e \subseteq R_{\text{in}}$.

Lemma 6.11 *There is a unique band.*

Proof Since B is a directed cycle of Γ , R_{out} is connected and hence there is at most one band. Suppose, for contradiction, that there is no band. By Lemma 6.10, we may assume without loss of generality that $L_0 \neq \emptyset$ and $L_2 = L_3 = \emptyset$. Since there is no band, for some column i , the extremal vertex $v_1 = c_{ij_1}$ of column i is in $L_1 \cup L_0$, the extremal vertex $v_2 = c_{(i \oplus 1)j_2}$ of column $i \oplus 1$ is in L_0 and the directed subpath from v_1 to v_2 of B contains B_2 and B_3 . Then there is a slit σ connecting v_1 and v_2 : σ goes through the vertices $c_{i(j_1+1)}, \dots, c_{i(j_2-1)}$ if $(j_1 + 1) < j_2$ and other cases are similar. Since the minor side of σ contains the bottom face, by Lemma 6.4, the major side of σ must contain the top face.

Now, we view the cylinder Γ upside down and use the same argument to find another slit σ' that contains the top face in its minor side. This pair σ, σ' of slits satisfies the conditions of Lemma 6.5 and therefore we have a contradiction. $\square$

We are ready to finish the proof of Theorem 1.3. Let columns i_1 and i_2 form a band. Let v_2 (v_3 , resp.) be the vertex of $V(B)$ in column i_1 (i_2 , resp.) with the smallest row index and v_1 (v_0 , resp.) be the vertex of $V(B)$ in column i_1 (i_2 , resp.) with the largest row index. Note that v_0, v_1, v_2, v_3 appear in the directed cycle B in this cyclic order. Let P_i , $0 \leq i \leq 3$, denote the subpath of B from v_i to $v_{i+1 \bmod 4}$.

Assume without loss of generality that $v_0 \in V(B_0)$ and suppose P_0 intersects B_i for $0 \leq i \leq k$ (and only for $0 \leq i \leq k$), where $0 \leq k \leq 3$. If $k = 3$, then similarly to the proof of Lemma 6.11, we find a slit whose minor side contains the bottom face. But, because of the band, the minor side then contains the top face as well, contradicting Lemma 6.4. Since $v_0 \in V(B_0)$ is extremal and hence $L_0 \neq \emptyset$, we must have $L_2 = \emptyset$ by Lemma 6.10 and hence $v_1 \notin V(B_2)$, or $k \neq 2$. Therefore, we have $k \leq 1$. Similarly, P_2 intersects B_i for at most two values of i .

We also observe that P_3 intersects B_i for at most two values of i , since otherwise there is a slit between v_3 and v_0 whose minor side contains both the top and the bottom faces. Similarly, P_1 intersects B_i for at most two values of i .

We have shown that for every j , $0 \leq j \leq 3$, P_j intersects B_i for at most two values of i . If any P_j intersects B_i for only one value of i then this cannot hold. Therefore, we conclude that for every j , $0 \leq j \leq 3$, P_j intersects B_i for exactly two values of i . From this conclusion and the assumption $v_0 \in V(B_0)$, we have $v_i \in V(B_i)$ for $0 \leq i \leq 3$. Let v'_1 be the vertex of B immediately after v_1 and v'_2 the vertex of B immediately before v_2 . Because of the band assumption, v_1, v'_1, v'_2, v_2 are all in column i_1 . If v_1 and v_2 are not adjacent, then either $v'_1 \in V(B_1)$ or $v'_2 \in V(B_2)$. If v_1 and v_2 are adjacent, then either v_1 or v_2 belongs to $V(B_1) \cap V(B_2)$ and therefore we also have either $v'_1 \in V(B_1)$ or $v'_2 \in V(B_2)$. Suppose $v'_1 \in V(B_1)$. The case $v'_2 \in V(B_2)$ is similar. Let μ_1 be an antipodal chord from v'_1 to v_3 first going through vertices in column i_1 to the top face and then through vertices in column i_2 . Let μ_2 be a similar path in $\mathcal{R}_\Gamma$ from v_1 to v_3 through the bottom face. Since the sum of

the lengths of these two antipodal chords is $4h - 2$, at least one of them has length smaller than $2h$, a contradiction. This completes the proof of Theorem 1.3.

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