

Tile-Packing Tomography Is NP-hard

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Abstract Discrete tomography deals with reconstructing finite spatial objects from their projections. The objects we study in this paper are called tilings or tile-packings, and they consist of a number of disjoint copies of a fixed tile, where a tile is defined as a connected set of grid points. A row projection specifies how many grid points are covered by tiles in a given row; column projections are defined analogously. For a fixed tile, is it possible to reconstruct its tilings from their projections in polynomial time? It is known that the answer to this question is affirmative if the tile is a bar (its width or height is 1), while for some other types of tiles NP-hardness results have been shown in the literature. In this paper we present a complete solution to this question by showing that the problem remains NP-hard for *all* tiles other than bars.

Keywords Tilings · Discrete tomography · NP-hardness · Affine independence

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1 Introduction

Discrete tomography deals with reconstructing finite spatial objects from their low-dimensional projections. Inverse problems of this nature arise naturally in medical computerized tomography, electron tomography, non-destructive quality control, timetable design and a number of other areas. This wide range of applications inspired significant theoretical interest in this topic and led to studies of computational complexity of various discrete tomography problems. For an extensive and detailed coverage of practical and theoretical aspects of this area, we refer readers to the book by Kuba and Herman, see [7, 8].

In this paper we consider the problem of reconstructing a tile packing from its row and column projections. Formally, consider the integer grid of dimension $m \times n$, consisting of all cells $(i, j) \in [0, m) \times [0, n)$. Every cell (i, j) is adjacent to its neighbor cells

$$\begin{array}{ccc} & (i-1, j), & \\ (i, j-1), & & (i, j+1), \\ & (i+1, j), & \end{array}$$

whichever of those are present in the grid. Alternatively, one can think of each (i, j) as a cell in an $m \times n$ matrix. In the paper, we will often use the matrix notation and terminology, using terms “row” and “column”, with rows numbered top-down and columns numbered from left to right, so that the upper-left cell is $(0, 0)$.

We define a *tile* to be any finite connected set T of grid cells. By “connected” we mean that for any two cells of T there is a path inside T between these cells, where any two consecutive cells on this path are adjacent. The *width* and *height* of T are defined in the obvious manner, as the dimensions of the smallest $h \times w$ rectangle containing T . If $w = 1$ or $h = 1$, then T is called a *bar*. By

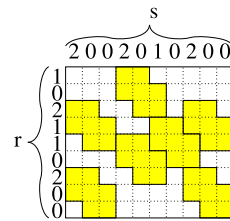
$$T + (i, j) = \{(x + i, y + j) : (x, y) \in T\}$$

we denote the translation of T by the vector (i, j) . $T + (i, j)$ is called a (*translated*) *copy* of T , and (i, j) is the *position* of this copy. Later in the paper, we will sometimes refer to $T + (i, j)$ as a “tile”—which is somewhat ambiguous but more intuitive than a “copy”.

A *tile packing of the $m \times n$ grid using T* —or a *T -packing*, in short, if m and n are understood from context—is a disjoint partial covering of the grid with translated copies of T . Formally, a *T -packing* is defined by a set D of translation vectors such that all translated copies $T + (i, j)$, for all $(i, j) \in D$, are contained in the $m \times n$ grid and are pairwise disjoint. We do not require the tiles to completely cover the grid. We stress here that what we call a *T -packing* here has been sometimes called a *partial tiling* in the literature; see, for example, [5]. An example of a *T -packing* is shown in Fig. 1. Without loss of generality, throughout the paper, we will be assuming that the tile T used in packing is in a *canonical position* in the upper-left corner of the grid, that is $\min \{x : (x, y) \in T\} = \min \{y : (x, y) \in T\} = 0$.

To simplify notation, instead of counting how many grid points are covered by tiles in a given row (or column), we count how many tiles *start* in a given row (column),

Fig. 1 A tile packing of the 9×10 grid and its projections. By convention the *leftmost upper cell* is $(0, 0)$



which is equivalent up to some base-change. So the *row* and *column projections* of a packing D are defined as a pair $r \in \mathbb{N}^m$ and $s \in \mathbb{N}^n$ of vectors such that

$$r_i = |\{j : (i, j) \in D\}| \quad \text{and} \quad s_j = |\{i : (i, j) \in D\}|.$$

For example, consider tile $T = \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1), (2, 2)\}$. Figure 1 shows an example of a T -packing. This packing is

$$D = \{(0, 3), (2, 0), (2, 7), (3, 5), (4, 3), (6, 0), (6, 7)\}.$$

We study the problem of reconstructing tile packings from its horizontal and vertical projections. More formally, for any fixed tile T , the problem is defined as follows:

Tile Packing Tomography Problem (TPTP(T)) The instance of TPTP(T) consists of vectors $r \in \mathbb{N}^m$, $s \in \mathbb{N}^n$. The objective is to decide if there is a T -packing D whose projections are r and s .

This problem has been introduced in [2] and shown to be NP-hard for some particular tiles. In [3], the proof technique has been adapted to show NP-hardness for any rectangular tile, i.e. a tile that consists of all cells $(i, j) \in [0, h) \times [0, w)$ for some dimensions $w, h \geq 2$.

On the positive side, the classical work of Ryser [11] on projections of 0-1 matrices provides a characterization of vectors that correspond to projections of T -packings for the special case when T is a single cell, and provides a simple polynomial-time algorithm for that case. The ideas from [11] were extended in [4, 9] to the case when T is a bar. In [1], polynomial-time algorithms were given for restricted special cases. The complexity status was unknown for all other tiles, and the current paper completes the picture by proving the following theorem.

Theorem 1 *Problem TPTP(T) is NP-complete for any tile T that is not a bar.*

The general structure of our proof resembles those introduced in [2] and [3], although the reductions we present are substantially more difficult, since the generality of our result means that we cannot take advantage of a specific shape of the tile, and that we need to base the construction on some generic properties shared by infinitely many tiles. Our techniques take advantage of Ryser’s structure results for 0-1 matrices, in combination with some arguments based on linear algebra.

After reviewing some background information in Sect. 2, we introduce the main idea of the reduction in Sect. 3, by formulating the overall framework of the reduction

and conditions on T required for this reduction to be correct. Then, in Sect. 4 we show that all non-bar tiles satisfy these conditions.

2 Main Tools

In this section we briefly review two concepts that will play a crucial role in our proofs: affine independence and Ryser's theorem.

Affine Independence Vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are called *affinely independent* if the unique solution of equations

$$\sum_{i=1}^k \alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^k \alpha_i v_i = 0$$

is $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$. It is easy to show that the following three conditions are equivalent:

- (ai1) v_1, v_2, \dots, v_k are affinely independent,
- (ai2) $v_2 - v_1, v_3 - v_1, \dots, v_k - v_1$ are linearly independent,
- (ai3) $(v_1, 1), (v_2, 1), \dots, (v_k, 1)$ are linearly independent.

We will refer to vectors $v_i - v_1, i = 2, 3, \dots, k$, in (ai2), as *difference vectors*. Condition (ai2) is useful in verifying affine independence. For example, $(1, 1), (3, 4), (5, 5)$ are affinely independent because the difference vectors $(3, 4) - (1, 1) = (2, 3)$ and $(5, 5) - (1, 1) = (4, 4)$ are linearly independent.

Condition (ai3) implies that if v_1, v_2, \dots, v_k are affinely independent then for any vector v and constant β , the equations

$$\sum_{i=1}^k \alpha_i v_i = v \quad \text{and} \quad \sum_{i=1}^k \alpha_i = \beta$$

have a unique solution $\alpha_1, \alpha_2, \dots, \alpha_k$.

For any vector $v \in \mathbb{R}^n$ and any set of indices $i_1, i_2, \dots, i_b \in [0, n]$, define the (i_1, i_2, \dots, i_b) -restriction of v to be the vector $v' \in \mathbb{R}^b$ that consists only of the coordinates $i_t, t = 1, \dots, b$, of v . For example, the $(0, 3, 4)$ -restriction of $v = (4, 3, 1, 0, 7, 9, 5)$ is $v' = (4, 0, 7)$. For any set of vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$, to show that they are affinely independent it is sufficient to show that their (i_1, i_2, \dots, i_b) -restrictions are affinely independent, for some set of indices i_1, i_2, \dots, i_b .

Ryser's Theorem Ryser [10] studied the structure of 0-1 matrices with given projections. We adapt his characterization of these matrices and express it in terms of tile packings.

Fix a tile T and let $I \subseteq [0, m]$ be a set of rows and $J \subseteq [0, n]$ a set of columns. We say that a tile copy $T + (i, j)$ belongs to $I \times J$ if $i \in I, j \in J$. Note that here we do not require inclusion of $T + (i, j)$ in $I \times J$.

Define $\xi_{I,J} = \max_D |D \cap (I \times J)|$, where the maximum is taken over all T -packings D of the $m \times n$ grid. Thus $\xi_{I,J}$ is the maximum number of copies of T that can belong to $I \times J$ in a T -packing without overlapping (and without any restriction on their projections).

For a set I of rows, denote $r(I) = \sum_{i \in I} r_i$. Analogously, let $s(J) = \sum_{j \in J} s_j$, for a set J of columns. By $\bar{I} = [0, m) - I$ and $\bar{J} = [0, n) - J$ we denote the complements of sets I and set J , respectively.

Consider a T -packing D with projections r, s . Then we have

$$r(I) - s(\bar{J}) = |D \cap (I \times J)| - |D \cap (\bar{I} \times \bar{J})|.$$

By definition, $|D \cap (I \times J)| \leq \xi_{I,J}$. Therefore we obtain the following lemma (inspired by [10]).

Lemma 1 *Let I be a set of rows and J a set of columns. If $r(I) - s(\bar{J}) = \xi_{I,J}$ then every T -packing D with projections r, s satisfies $|D \cap (I \times J)| = \xi_{I,J}$ and $|D \cap (\bar{I} \times \bar{J})| = 0$.*

3 General Proof Structure

For each non-bar tile T , we show a polynomial-time reduction from the *3-Color Tomography Problem* introduced in [6] and shown to be NP-hard in [3]. In that problem, an object to be reconstructed is a set of “atoms” (in our terminology, single cells) colored red (R), green (G) or blue (B). The instance contains separate projections for each color. The formal definition is this:

3-Color Tomography Problem (3CTP) The instance consists of six vectors $r^R, r^G, r^B \in \mathbb{N}^m, s^R, s^G, s^B \in \mathbb{N}^n$. The objective is to decide whether there is an $m \times n$ matrix M with values from $\{R, G, B\}$ such that, for each color $c \in \{R, G, B\}$, $r_x^c = |\{y : M_{xy} = c\}|$ for each x and $s_y^c = |\{x : M_{xy} = c\}|$ for each y .

From now on, assume that T is some non-bar fixed tile of width w and height h . Let \mathcal{I} be an instance of 3CTP for some $m \times n$ matrix specified by six projections $r^R, r^G, r^B, s^R, s^G, s^B$. We will map \mathcal{I} into an instance \mathcal{J} of TPTP(T) for an $m' \times n'$ grid with projections r, s , such that \mathcal{I} has a matrix M with projections $r^R, r^G, r^B, s^R, s^G, s^B$ if and only if \mathcal{J} has a T -packing with projections r, s .

Without loss of generality we assume that for every color c we have $\sum_x r_x^c = \sum_y s_y^c$, for every row x we have $\sum_c r_x^c = m$, and for every column y we have $\sum_c s_y^c = n$. Otherwise, \mathcal{I} is of course unfeasible, so we could take \mathcal{J} to be any fixed unfeasible instance of TPTP(T).

We now describe \mathcal{J} . We will choose a grid of size $m' \times n'$ for $m' = mk$ and $n' = n\ell$, where k and ℓ are positive integer constants to be specified later. We will use the term *block* for a $k \times \ell$ grid. We can partition our $m' \times n'$ grid into mn rectangles of dimension $k \times \ell$, and we can think of each such rectangle as a translated block. The rectangle $[xk, (x + 1)k) \times [y\ell, (y + 1)\ell)$ will be referred to as block (x, y) .

Next, we need to specify the projections r and s . We will describe these projections in a somewhat unusual way, by fixing three packings of a block denoted D^R , D^G , and D^B (obviously, corresponding to the three colors), and then expressing r and s as linear combinations of these packings. More specifically, denoting by \bar{r}^c and \bar{s}^c the horizontal and vertical projections of packing D^c , for each $c \in \{R, G, B\}$, we define

$$r_{xk+i} = \sum_c r_x^c \cdot \bar{r}_i^c \quad \text{and} \quad s_{y\ell+j} = \sum_c s_y^c \cdot \bar{s}_j^c \tag{1}$$

for every $i \in [0, k)$, $j \in [0, \ell)$, $x \in [0, m)$, and $y \in [0, n)$. The idea is that replacing each cell in a solution to the 3CTP instance r^c, s^c , for $c = R, G, B$, by its color-corresponding block gives a solution to the TPTP(T) instance r, s .

To complete the description of the reduction, it still remains to define the three packings D^R , D^G , and D^B . This will be done in the next section. In the remainder of this section we establish conditions that will guarantee correctness of our reduction.

Our three packings will be designed to satisfy the following two requirements:

Requirement 1: Vectors $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are affinely independent and vectors $\bar{s}^R, \bar{s}^G, \bar{s}^B$ are affinely independent. Note that, by property (ai3), this implies that for any vector v there is at most one possible way to represent it in a form $v = n_R \bar{r}^R + n_G \bar{r}^G + n_B \bar{r}^B$, where $n_R + n_G + n_B = n$. Naturally, an analogous statement holds for column projections.

Requirement 2: In any packing D of \mathcal{J} with projections r, s , the restriction of D to each block of the grid has projections equal to \bar{r}^c, \bar{s}^c , for some $c \in \{R, G, B\}$.

Lemma 2 *Assume that the three packings D^R, D^G, D^B satisfy Requirements 1 and 2. Then \mathcal{I} has a solution if and only if \mathcal{J} has a solution.*

Proof (\Rightarrow) Let $M \in \{R, G, B\}^{m \times n}$ be a solution to \mathcal{I} . We transform M into the following packing D for the $m' \times n'$ grid:

$$D = \bigcup_{x \in [0, m)} \bigcup_{y \in [0, n)} (D^{M_{xy}} + (xk, y\ell)).$$

In other words, if $M_{xy} = c$ then block (x, y) of the $m' \times n'$ grid contains a copy of D^c . By simple inspection, the projections of D are indeed equal to the vectors r and s in (1).

(\Leftarrow) For the converse, suppose that there is a packing D with projections r, s . By Requirement 2, every block of the $m' \times n'$ grid has projections \bar{r}^c and \bar{s}^c , for some $c \in \{R, G, B\}$. We then *associate* this block with color c . We can thus define a matrix $M \in \{R, G, B\}^{m \times n}$ such that $M_{xy} = c$ if block (x, y) of D is associated with color c .

We now need to show that M is a solution for \mathcal{I} . To this end, fix some arbitrary $0 \leq x < m$ and consider vector

$$v = (r_{xk}, r_{xk+1}, \dots, r_{(x+1)k-1}),$$

which is the projection of the “row” of all blocks (x, y) , for all y . By the construction (1), v can be written as

$$v = n_R \bar{r}^R + n_G \bar{r}^G + n_B \bar{r}^B,$$

where $n_R = r_x^R$, $n_G = r_x^G$, and $n_B = r_x^B$. Now, using Requirement 1, we obtain that this representation is unique under the assumption that $n_R + n_G + n_B = n$. We can thus conclude that the projection of row x of M is correct, that is $|\{y : M_{xy} = c\}| = r_x^c$ for all c . By the same argument, column projections of M are correct as well, completing the proof. \square

In summary, to complete the proof for the given tile T , we need to do this: (i) define a rectangular $k \times \ell$ block with three packings D^R, D^G, D^B , (ii) show that the row projections of D^R, D^G, D^B and the corresponding column projections are affinely independent (Requirement 1), and (iii) show that in any solution to \mathcal{J} , each block (x, y) has projections equal to those of one of D^c , for some c (Requirement 2). We show the construction of such block packings in the next section.

4 Construction of Block Packings

As in the previous section, T is a fixed (but arbitrary) non-bar tile. We call (i, j) a *conflicting* vector if T and $T + (i, j)$ overlap, that is $T \cap (T + (i, j)) \neq \emptyset$. Obviously, the vectors (i, j) and $(-i, -j)$ are either both conflicting or both non-conflicting. Since T is not a bar, it has a conflicting vector (i, j) with $i, j \neq 0$. To see this, observe that since T is connected and not a bar, it contains two cells (i, j) and (i', j') with $|i - i'| = 1$ and $|j - j'| = 1$, so one of $(1, 1)$ or $(-1, 1)$ is a conflicting vector.

For the construction of the proof, fix a conflicting vector of T that maximizes the L_1 norm under the constraint that none of the coordinates is 0. We denote this vector by $(-p, q)$ and assume without loss of generality $p, q > 0$, for otherwise we can flip T horizontally or vertically and give the proof for the resulting tile.

So any vector (i, j) with $i, j \neq 0$ and $|i| + |j| > |p| + |q|$ is not conflicting. Let a be the smallest positive integer such that $(ap, 0)$ is not a conflicting vector. Similarly let b be the smallest positive integer such that $(0, bq)$ is not a conflicting vector. Without loss of generality we assume that $a \leq b$, since otherwise we can exchange the roles of columns and rows in the proof.

We now divide the proof into four cases, and for each of them we show that Requirements 1 and 2 are satisfied.

4.1 Case $a = 1$ and $b = 1$

In this case, we use the following three packings:

$$\begin{aligned} D^R &= \{(p, 0), (p, q)\}, \\ D^G &= \{(0, q), (p, q)\}, \\ D^B &= \{(p, q)\}. \end{aligned}$$

The values of k and ℓ are chosen to be the smallest integers for which these three packings are contained in the $k \times \ell$ grid.

The packings are depicted on Fig. 2. The squares represent possible positions for tiles. Two positions are connected with a solid line if the difference of the positions is

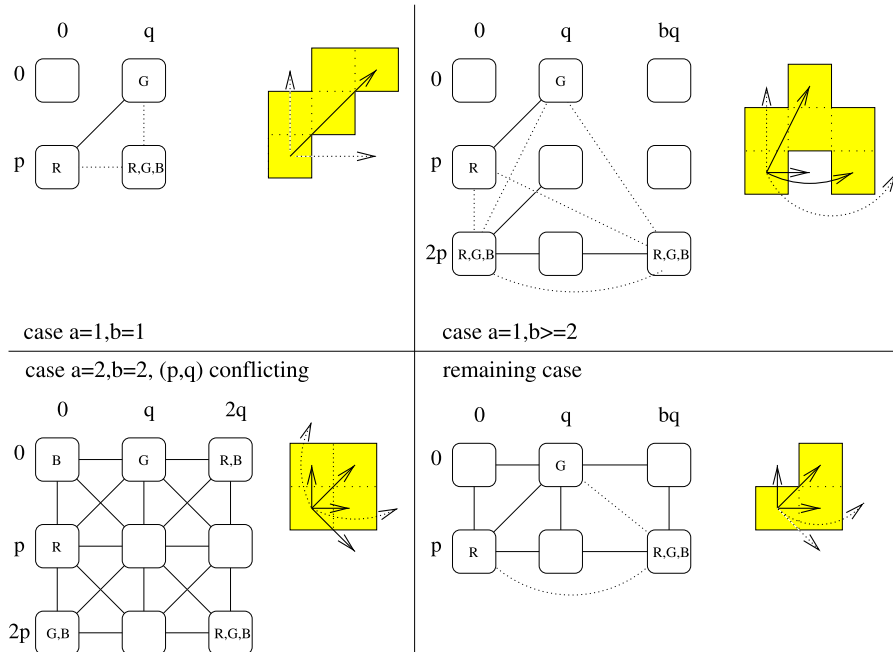


Fig. 2 For each case of the proof, the figure shows three packings (*left-hand side*) and an example of a tile satisfying this case (*right-hand side*). Dotted vectors are non-conflicting, either by maximality of $(-p, q)$ or by the case condition. In the third case, all vectors that are not shown (to avoid clutter) are non-conflicting

a conflicting vector. That means that no packing can contain simultaneously a tile in both positions. Dotted lines indicate non-conflicting vectors, i.e. they connect pairs of compatible positions. We show lines only between position pairs relevant to the proof. In the figure we mark with letter c the tile positions of D^c , for $c \in \{R, G, B\}$. For illustration, for each case, on the right-hand side of the figure we show a tile satisfying the case conditions. Again, solid vectors are conflicting and dotted vectors are non-conflicting.

We first verify Requirement 1. The $(0, p)$ -restrictions of $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are, respectively, $(0, 2), (1, 1)$ and $(0, 1)$, and the $(0, q)$ -restrictions of $\bar{s}^R, \bar{s}^G, \bar{s}^B$ are, respectively, $(1, 1), (0, 2)$ and $(0, 1)$. For both the row and column projections, routine calculations show that their restrictions are affinely independent.

We now focus on Requirement 2. Let r, s be the projections obtained by the reduction, and consider a packing D with these projections. We use Lemma 1, with I being the set of all row indices that are p modulo k , and J being the set of all column indices that are q modulo ℓ . By inspecting the definition of the projections we have

$$r(I) = \sum_{x \in \{0, m\}} r_x^R + mn = \sum_{y \in \{0, n\}} s_y^R + mn = s(\bar{J}) + mn,$$

so $r(I) - s(\bar{J}) = mn$, which is $|I \times J|$. In consequence Lemma 1 applies. Therefore every block in D contains a tile at position (p, q) and none at position $(0, 0)$. The

remaining possible positions for tiles are $(p, 0)$, $(0, q)$, but both cannot be occupied in the same block. This shows that every block of D is one of the packings D^R, D^G, D^B .

4.2 Case $a = 1$ and $b \geq 2$

In this case, we use the following three packings:

$$D^R = \{(2p, 0), (2p, bq), (p, 0)\},$$

$$D^G = \{(2p, 0), (2p, bq), (0, q)\},$$

$$D^B = \{(2p, 0), (2p, bq)\}.$$

The values of k and ℓ are chosen to be the smallest integers for which these three packings are contained in the $k \times \ell$ grid.

The $(0, p, 2p)$ -restrictions of $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are linearly independent vectors $(0, 1, 2)$, $(1, 0, 2)$, $(0, 0, 2)$; therefore $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are affinely independent. By a similar argument, we obtain that the corresponding projection column vectors $\bar{s}^R, \bar{s}^G, \bar{s}^B$ are affinely independent as well. Thus, Requirement 1 holds.

Now we verify Requirement 2. Row $2p$ of a block can contain at most 2 tiles. Let I be the set of rows i with $i \bmod k = 2p$. By inspecting the definition of the projections we have $r(I) = 2mn$, so every block in a solution D must contain exactly 2 tiles in row $2p$, and they are at positions $(2p, 0), (2p, bq)$.

Now let J be the set of all columns j with $j \bmod \ell = bq$. We only have $s(J) = mn$, so in every block of D the positions $(0, bq), (p, bq)$ are empty. The tile at $(2p, 0)$ forces position (p, q) to be empty. By the case assumption that $a = 1$, there is no conflict between $(2p, 0)$ and $(p, 0)$. By maximality of $(-p, q)$ there is no conflict between positions $(2p, 0)$ and $(0, q)$, or between $(2p, bq)$ and $(p, 0)$, or $(2p, bq)$ and $(0, q)$.

That leaves only 3 positions where the block packings can differ, namely $(0, 0)$, $(p, 0)$ and $(0, q)$. Let d be the number of blocks in D with a tile in $(0, 0)$. Similarly let e be the number of blocks in D with a tile in $(0, q)$. Now we use the fact that in the original instance we had $\sum r_x^G = \sum s_y^G$; let λ_G denote this quantity. This time, let I' be the set of rows i with $i \bmod k = 0$, and J' the set of columns j with $j \bmod \ell = q$. Then, by the definition of d and e , we have $r(I') = d + e$, and by the definition of the chosen three packings, we have $r(I') = \lambda_G$. Similarly, we have $r(J') = e$ and $r(J') = \lambda_G$, which shows $d = 0$. Therefore every block packing in D is one of D^R, D^G or D^B .

4.3 Case $a = 2, b = 2$ and Vector (p, q) Conflicting

In this case we assume $a = 2, b = 2$ and that the vector (p, q) is conflicting. Since $(-p, q)$ is conflicting as well, this makes the construction very symmetric. The three packings used in this case are:

$$D^R = \{(0, 2q), (p, 0), (2p, 2q)\},$$

$$D^G = \{(0, q), (2p, 0), (2p, 2q)\},$$

$$D^B = \{(0, 0), (0, 2q), (2p, 0), (2p, 2q)\}.$$

Again, the values of k and ℓ are chosen to be the smallest integers for which these three packings are contained in the $k \times \ell$ grid. The construction is illustrated in Fig. 2. The idea behind this construction is similar to the reduction used in [2] to show \mathbb{NP} -hardness of the packing problem for the 2×2 square tile.

The $(0, p, 2p)$ -restrictions of $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are $(1, 1, 1), (1, 0, 2), (2, 0, 2)$; therefore $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are affinely independent. By symmetry, the same holds for $\bar{s}^R, \bar{s}^G, \bar{s}^B$.

Now we verify Requirement 2. Let D be a packing with projections r, s . Due to conflicts, there can be at most 2 tiles in rows $p, 2p$ of a block. Let $I = \{i : i \bmod k \in \{p, 2p\}\}$. Since $r(I) = 2mn$ and there are mn blocks in D , every block of D must contain exactly 2 tiles in rows $p, 2p$. By symmetry, the same holds for columns $q, 2q$. There are only 4 packings that satisfy these constraints and avoid conflicts, namely D^R, D^G, D^B and packing $D^A = \{(0, 2q), (2p, 0), (2p, 2q)\}$. Let $\lambda_R, \lambda_G, \lambda_B, \lambda_A$ be the respective numbers of these different block packings in D . Since D has row projection r , by expressing the total number of tiles in two different ways, we have

$$3 \sum r_x^R + 3 \sum r_x^G + 4 \sum r_x^B = 3\lambda_R + 3\lambda_G + 4\lambda_B + 3\lambda_A. \tag{2}$$

From the assumption that $\sum_c \sum_x r_x^c = mn$, we have also

$$\sum r_x^R + \sum r_x^G + \sum r_x^B = mn = \lambda_R + \lambda_G + \lambda_B + \lambda_A. \tag{3}$$

Now let $I' = \{i : i \bmod k = p\}$. Then $\sum_x r_x^R = r(I') = \lambda_R$. Similarly, for $J' = \{j : j \bmod \ell = q\}$ we obtain $\sum_x r_x^G = \sum_y s_y^G = s(J') = \lambda_G$. After subtracting these two equations from (2) and (3), we are left with two equations

$$\sum r_x^B = \lambda_B + \lambda_A, \quad 4 \sum r_x^B = 4\lambda_B + 3\lambda_A,$$

from which we conclude $\lambda_A = 0$. This verifies Requirement 2.

4.4 The Remaining Case

Assume now that none of the previous cases holds. Since $a \leq b$, this means that either

- (a) $a \geq 2$ and $b \geq 3$, or
- (b) $a = 2, b = 2$ and vector (p, q) is not conflicting.

We claim that the vector $(p, (b - 1)q)$ is not conflicting. Indeed, in case (b) above, it follows by case assumption and in case (a), it follows from the maximality of $(-p, q)$. Therefore, in any block of a T -packing both positions $(0, q)$ and (p, bq) could contain a tile.

We use the following three packings (see Fig. 2):

$$\begin{aligned} D^R &= \{(p, 0), (p, bq)\}, \\ D^G &= \{(0, q), (p, bq)\}, \\ D^B &= \{(p, bq)\}. \end{aligned}$$

Again, the values of k and ℓ are chosen to be the smallest integers for which these three packings are contained in the $k \times \ell$ grid.

The $(0, p)$ -restrictions of $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are $(0, 2), (1, 1), (0, 1)$, and their difference vectors $(1, -1), (0, -1)$ are linearly independent. Therefore $\bar{r}^R, \bar{r}^G, \bar{r}^B$ are affinely independent. The $(0, q, bq)$ -restrictions of $\bar{s}^R, \bar{s}^G, \bar{s}^B$ are linearly independent vectors $(1, 0, 1), (0, 1, 1), (0, 0, 1)$; therefore $\bar{s}^R, \bar{s}^G, \bar{s}^B$ are affinely independent. Thus, Requirement 1 holds.

Now we verify Requirement 2. The proof is similar to those of previous cases, except that now we have more candidate packings to consider. Fix some T -packing D with projections r, s . First, using the same arguments as in the previous case, we observe that every block of D must contain exactly one tile in column bq , that is either in location $(0, bq)$ or (p, bq) . Taking conflicts into accounts, straightforward case analysis produces nine possible packings, including two pairs of “equivalent” packings with identical projections. We now introduce notation for the numbers of these packings:

- $\lambda_R =$ the number of packings $D^R = \{(p, 0), (p, bq)\}$,
- $\lambda_G =$ the number of packings $D^G = \{(0, q), (p, bq)\}$ or $\{(p, q), (0, bq)\}$,
- $\lambda_B =$ the number of packings $D^B = \{(p, bq)\}$,
- $\lambda_A =$ the number of packings $\{(0, 0), (p, bq)\}$ or $\{(p, 0), (0, bq)\}$,
- $\lambda_C =$ the number of packings $\{(0, 0), (p, q), (0, bq)\}$,
- $\lambda_D =$ the number of packings $\{(0, bq)\}$,
- $\lambda_E =$ the number of packings $\{(0, 0), (0, bq)\}$.

Let I be the set of all rows i with $i \bmod k = 0$ and J be the set of all columns j with $j \bmod \ell = q$. Then, by inspecting the projection definitions, we have $r(I) = \sum_x r_x^G = \sum_y s_y^G = s(J)$. Since r and s are the projections of D , we also have $r(I) = \lambda_G + \lambda_A + 2\lambda_C + \lambda_D + 2\lambda_E$ and $s(J) = \lambda_G + \lambda_C$. This shows $\lambda_A = \lambda_C = \lambda_D = \lambda_E = 0$, completing the analysis of this case and the proof of NP-hardness.

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