On the limiting behaviour of Lévy processes at zero

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Abstract We find an analytical condition characterising when the probability that a Lévy Process leaves a symmetric interval upwards goes to one as the size of the interval is shrunk to zero. We show that this is also equivalent to the probability that the process is positive at time t going to one as t goes to zero and prove some related sequential results. For each $\alpha > 0$ we find an analytical condition equivalent to $X_{T_r} T_r^{-1/\alpha} \xrightarrow{P} \infty$ and $X_t t^{-1/\alpha} \xrightarrow{P} \infty$ as $r, t \to 0$ where X is a Lévy Process and T_r the time it first leaves an interval of radius r.

1 Introduction

Let X be a Lévy Process (LP); that is an $\mathbb{R}$-valued stochastic process with stationary, independent increments whose paths are taken to be almost surely right-continuous. (We assume some familiarity with LPs and for an account refer to Bertoin [2]). It can be shown that there is a one-to-one correspondence between LPs and infinitely divisible distributions. We have

$$E(e^{i\lambda X_t}) = e^{-t\psi(\lambda)}$$

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where we will write ψ (the Lévy exponent) as follows:

$$\psi(\lambda) = \frac{1}{2}\sigma^2\lambda^2 - i\varphi\lambda + \int_{-\infty}^{\infty} \left(1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|\leq 1\}}\right)\Pi(dx); \qquad (1)$$

where $\sigma, \varphi \in \mathbb{R}$ and Π (the jump measure) satisfies $\int_{-\infty}^{\infty} 1 \wedge |x|^2 \Pi(dx) < \infty$.

In this paper we are interested in results relating to the small-time behaviour of Lévy Processes, particularly in regard to positivity in probability and exit from small intervals. We give analytical conditions equivalent to $\mathbb{P}(X_{T_r} > 0) \to 1$, $\mathbb{P}(X_t > 0) \to 1$, $X_{T_r}T_r^{-1/\alpha} \xrightarrow{P} \infty$ and $X_t t^{-1/\alpha} \xrightarrow{P} \infty$ respectively. Many of these form analogues to large-time results for Random Walks (RWs). We discuss when these conditions are satisfied and investigate some related conditions in their sequential forms.

Note that since a non-zero Brownian component would dominate at small times we exclude this case. (i.e. We will henceforth assume that $\sigma = 0$.) Indeed, if we let B_t be a standard Brownian Motion and X_t a Lévy Process with zero Brownian component then it is a consequence of the results in Pruitt [13] (see Theorem 6 below or [6]) that

$$\frac{B_t}{X_t} \xrightarrow{P} \infty \quad \text{as } t \to 0,$$

and thus if X had a non-zero Brownian component then we would have $\lim_{t\downarrow 0} \mathbb{P}(X_t > 0) = \lim_{t\downarrow 0} \mathbb{P}(B_t > 0) = 1/2$ etc.

We likewise assume that X is not a compound Poisson process since otherwise $\lim_{t\downarrow 0} \mathbb{P}(X_t = 0) = 1$ while the probability that X leaves a small interval upwards and other quantities of interest are similarly trivial.

For $x, r, t > 0$ let

$$T_r = \{\inf s \geq 0 : |X_s| > r\};$$
$$M_t = \sup_{s\leq t} |X_s|;$$
$$\Delta_t^+ = \max_{s\leq t} \Delta X_s \quad \text{(where } \Delta X_t \text{ is the jump process of } X\text{)};$$
$$\Delta_t^- = -(\min_{s\leq t} \Delta X_s);$$
$$\Delta_t = \Delta_t^+ \vee \Delta_t^-;$$
$$V(x) = \Pi(x, \infty);$$
$$W(x) = \Pi(-\infty, -x);$$
$$L(x) = V(x) + W(x);$$
$$D(x) = V(x) - W(x);$$
$$A(x) = \varphi + xD(x) - \mathbf{1}_{\{x<1\}} \int_{x<|y|\leq 1} y\,\Pi(dy) + \mathbf{1}_{\{x>1\}} \int_{1<|y|\leq x} y\,\Pi(dy)$$

$$= \varphi + D(1) - \int_x^1 D(y)\,dy;$$

$$U(x) = 2\int_0^x y\,L(y)dy;$$

$$k(x) = \frac{|A(x)|}{x} + \frac{U(x)}{x^2}.$$

We define $\check{X}^r$ as X with jumps whose moduli are larger than r reduced to size r. i.e. $\check{X}_t^r = X_t - \sum_{s\leq t}\Delta X_s \mathbf{1}_{\{|\Delta X_s|>r\}} + r\sum_{s\leq t}\mathbf{1}_{\{\Delta X_s>r\}} - r\sum_{s\leq t}\mathbf{1}_{\{\Delta X_s<-r\}}$. By differentiating characteristic functions we can show

$$E\check{X}_1^r = A(r); \quad Var\check{X}_1^r = U(r). \tag{2}$$

Compare the following: for $r < 1$

$$E\left(X_1 - \sum_{s\leq 1}\mathbf{1}_{\{|\Delta X_s|>r\}}\Delta X_s\right) = \varphi - \int_{r<|x|\leq 1} x\,\Pi(dx) = A(r) - rD(r). \tag{3}$$

2 Main results

Theorem 1 *The following are equivalent as $r, t \to 0$:*

$$\mathbb{P}(X_{T_r} > 0) \to 1; \tag{4}$$

$$\mathbb{P}(X_t > 0) \to 1; \tag{5}$$

$$\frac{X_{T_r}}{\Delta_{T_r}^-} \xrightarrow{p} \infty; \tag{6}$$

$$\frac{X_t}{\Delta_t^-} \xrightarrow{p} \infty; \tag{7}$$

$$\frac{A(r)}{rW(r)} \to \infty. \tag{8}$$

This result has a large time LP analogue [5], derived from a similar result for RWs [10], which states:

$$\mathbb{P}(X_t > 0) \to 1 \Leftrightarrow \mathbb{P}(X_{T_r} > 0) \to 1 \Leftrightarrow \frac{A(r)}{\sqrt{W(r)U(r)}} \to \infty \ (\text{as } r \to \infty). \tag{9}$$

Note that in Theorem 1 $\frac{A(r)}{rW(r)} \to \infty$ may be replaced by $\frac{A(r)}{\sqrt{W(r)U(r)}} \to \infty$ (or vice versa in (9).)

Moreover, Doney [4] has already proved the equivalence of (5), (7) and (8) using a direct proof that (5) is equivalent to (8) similar to the one used by Kesten and Maller [10] to prove the RW version. We proceed here, however, by first showing that (4) is equivalent to (8) using a proof similar to Griffin and McConnell [9] for the RW case, before going on to show that (4) is also equivalent to the other probabilistic conditions.

If X has bounded variation (b.v.) (that is if $\sum_{t \leq 1} |\Delta X_t| \stackrel{\text{a.s.}}{<} \infty$ or equivalently $\int_0^1 L(x)dx < \infty$) then the analytical condition (8) can be somewhat simplified. In this case the process can be reduced to the sum of the difference of two (pure-jump) subordinators with a drift d (where $d = \lim_{r \downarrow 0} A(r)$). The $d \neq 0$ case is not of interest since we then have $X_t t^{-1} \stackrel{a.s.}{\longrightarrow} d$ (see [2] p. 84). If, on the other hand, $d = 0$, then we write

$$X_t = Y_t - Z_t$$

where Y_t and Z_t are (pure-jump) subordinators wih jump measures $\mathbf{1}_{\{x>0\}}\Pi(dx)$ and $\mathbf{1}_{\{x>0\}}\Pi(dx)$ respectively. Rearranging (8) then yields the following equivalent:

$$\frac{\int_0^r (V(x) - W(x))\, dx}{rW(r)} \to \infty. \tag{10}$$

Moreover, it is easily deduced from (10) that for $\lambda > 0$

$$\mathbb{P}(Y_t > \lambda Z_t) \to 1 \Leftrightarrow \frac{\int_0^r (V(\lambda x) - W(x))\, dx}{rW(r)} \to \infty. \tag{11}$$

Thus as t,r go to 0 (or ∞) we have

$$\frac{Y_t}{Z_t} \stackrel{p}{\longrightarrow} \infty \Leftrightarrow \frac{\int_0^r (V(\lambda x) - W(x))\, dx}{rW(r)} \to \infty \quad \forall \lambda > 0. \tag{12}$$

It is interesting to compare (10) with the condition for the irregularity of $(-\infty, 0)$. Namely, Bertoin [3] showed that for a b.v. X with zero drift then as $t \downarrow 0$

$$\mathbf{1}_{\{X_t > 0\}} \stackrel{\text{a.s.}}{\longrightarrow} 1 \Leftrightarrow \frac{Y_t}{Z_t} \stackrel{\text{a.s.}}{\longrightarrow} \infty \Leftrightarrow \int_0^1 \frac{x|W(dx)|}{\int_0^x V(y)dy} < \infty.$$

In view of Bertoin's result it might be wondered if there are any ostensibly stronger probabilistic conditions equivalent to $\mathbb{P}(Y_t > Z_t) \to 1$. Note however that (12) is stronger than (10). In fact, the following example shows that it is possible for $\mathbb{P}(Y_t > Z_t)$ to go to one while $\mathbb{P}((1-\varepsilon)Y_t > Z_t)$ goes to zero for all $\varepsilon > 0$.

Example Let

$$V(r) = \frac{1}{r\log^2 r}\left(1 + \frac{3}{2\sqrt{|\log r|}}\right); \quad W(r) = \frac{1}{r\log^2 r};$$

such that

$$\int_0^r V(x)dx = \frac{1}{|\log r|}\left(1 + \frac{1}{\sqrt{|\log r|}}\right); \quad \int_0^r W(x)dx = \frac{1}{|\log r|}.$$

Then

$$\frac{\int_0^r (V(x) - W(x))\, dx}{rW(r)} = \sqrt{|\log r|}$$

and thus $\mathbb{P}(Y_t > Z_t)$ goes to one as t goes to 0 by (10).

Whereas for any $\varepsilon > 0$

$$\frac{\int_0^r (W(x(1 - \varepsilon)) - V(x))\, dx}{rV(r)}$$

$$= \frac{(1 - \varepsilon)^{-1} \int_0^{r(1-\varepsilon)} W(x)dx - \int_0^r V(x)dx}{rV(r)}$$

$$\sim (1 - \varepsilon)^{-1}|\log r| - |\log r|\left(1 + \frac{1}{\sqrt{|\log r|}}\right)$$

and hence $\mathbb{P}((1 - \varepsilon)Y_t > Z_t)$ goes to zero as t goes to 0 by (11).

If X has unbounded variation (u.b.v.) (and hence is regular for both half-lines), then first note that (8) implies the prevalence of negative jumps in the sense that

$$\liminf_{r\downarrow 0} A(r) \geq 0. \tag{13}$$

As we are now assuming $\int_0^1 W(x)dx = \infty$ the analytical condition (8) further requires that

$$\lim_{r\downarrow 0} \frac{\int_r^1 W(x)dx}{rW(x)} = \infty.$$

Thus $\int_r^1 W(x)dx$ must be slowly varying at zero (by Lemma 7). Hence by the monotone density theorem $W(r)$ is regularly varying with index -1. So X is of unbounded variation but only just. (i.e. by (13) $\int_r^1 L(x)dx$ goes to infinity 'slowly'.) We may compare this with other results where more variation leads to less extreme (limiting) values of $\mathbb{P}(X_t > 0)$. e.g. For spectrally negative

Lévy processes, $\mathbb{P}(X_t > 0) \to p$ is equivalent to the Laplace exponent (i.e. $-\varphi(-i\cdot)$) regularly varying with index p^{-1} (at 0 or ∞ respectively). Note that in this case $\lim_{t \downarrow 0} \mathbb{P}(X_t > 0) \to 1$ is equivalent to $W(r)$ regularly varying at 0 with index -1. While for a stable process with index α the possible values of $\mathbb{P}(X_1 > 0) = \lim_{t \downarrow 0} \mathbb{P}(X_t > 0)$ range over $[1 - \alpha^{-1}, \alpha^{-1}]$. (See Bertoin (1996) for more details.)

For more on conditions (6) and (7) see the comments preceding Theorem 4 below.

Theorem 2 *For $\alpha > 0$ the following are equivalent as $r, t \to 0$:*

$$\frac{X_{T_r}}{T_r^{1/\alpha}} \xrightarrow{p} \infty; \tag{14}$$

$$\frac{X_t}{t^{1/\alpha}} \xrightarrow{p} \infty; \tag{15}$$

$$\frac{A(r)}{rW(r)} \to \infty, A(r)r^{\alpha-1} \to \infty. \tag{16}$$

Theorem 2 also provides an analogue to results at large times; see Kesten and Maller [11] for the RW case and Doney [5] who shows the equivalence of (14), (15), and (16) for $\alpha = 1$ as $r, t \to \infty$. The sufficiency of (16) for (15) (as $r, t \to 0$) when $\alpha = 1$ was first proved by Doney and Maller [6].

Similarly to above, if X has b.v. and zero drift then the analytical condition may be rewritten. i.e. (16) becomes

$$\frac{\int_0^r V(x) - W(x)dx}{rW(r) + r^{1-\alpha}} \to \infty. \tag{17}$$

Note that for any $\alpha \in (0, 1)$ it is easy to find processes such that (10) holds while (17) fails. Clearly if $\alpha = 1$ then the conditions in Theorem 2 are never satisfied for any b.v. X.

If X has u.b.v. then, since (8) implies that $rW(r)$ is slowly varying as r goes to 0, it also forces $A(r)r^{\alpha-1}$ to go to infinity for all $\alpha \in (0, 1)$ and hence

$$X \text{ has u.b.v.}, \mathbb{P}(X_t > 0) \to 1 \Rightarrow \frac{X_t}{t^{1/\alpha}} \xrightarrow{p} \infty \quad \forall \alpha \in (0, 1).$$

While when $\alpha = 1$ we have, as t and r go to zero,

$$\frac{X_t}{t} \xrightarrow{p} \infty \Leftrightarrow \frac{A(r)}{1 + rW(r)} \to \infty.$$

Whereas, since (8) implies $A(r)$ is bounded above by a slowly varying function, (16) can never hold for $\alpha > 1$.

This is slightly different to what happens at infinity where no matter how large we take α we can find X such that $X_t/t^\alpha \xrightarrow{p} \infty$, while, similarly to what happens

with u.b.v. X at zero, $\lim_{t \to \infty} \mathbb{P}(X_t > 0) = 1$ implies $X_t/t^\alpha \xrightarrow{p} \infty \ \forall \alpha \in (0, 1)$ (see e.g. [6]).

3 Sequential results

Note that the results (and comments) in this section are also valid at infinity. However, as this can be shown using simple adaptations of the arguments used below, and in keeping with the tone of the rest of the paper, we will concentrate on the small-time cases.

Theorem 3 *For any sequence $r_k \downarrow 0$ the following are equivalent*:

$$\frac{W(\lambda r_k)}{k(\lambda r_k)} \to 0 \quad \forall \lambda > 0; \tag{18}$$

$$\frac{\Delta_{T_{r_k}}^-}{r_k} \xrightarrow{p} 0; \tag{19}$$

$$\frac{\Delta X_{T_{r_k}} \wedge 0}{r_k} \xrightarrow{p} 0; \tag{20}$$

$$\frac{\Delta_{T_{r_k}}^-}{X_{T_{r_k}}} \xrightarrow{p} 0. \tag{21}$$

This result is related to various earlier theorems concerning the relative magnitude of overshoots or of biggest jumps at exit times. Indeed, in the full-sequence case it may be seen as a simpler small-time one-sided analogue to a result (Theorem 2.1) from Griffin and Maller [7], which gives an analytical condition equivalent to $|S_{T_r}|r^{-1} \xrightarrow{p} 1$ as $r \to \infty$ for a RW S. The interesting thing about Theorem 3, however, is that it is in strict sequential form, in which case it turns out to be appropriate to study relations relative to the size of the jump at T_{r_k} rather than to $X_{T_{r_k}}$. (i.e. We consider when $\frac{\Delta_{T_{r_k}}}{r_k} \xrightarrow{p} 0$ rather than when $\frac{|X_{T_{r_k}}|}{r_k} \xrightarrow{p} 1$ etc.)

Indeed, it is possible to have $X_{T_{r_k}}/r_k \xrightarrow{p} 1$ but not $\Delta X_{T_{r_k}}/r_k \xrightarrow{p} 0$. For example, we may construct a pure-jump subordinator X and a sequence $r_k \downarrow 0$ such that (with high probability) X jumps over the small r_k with a jump of size $(k+1)r_k/k$ before the sum of the smaller jumps has reached r_k/k. Moreover, this example is typical in the sense that it can be shown by reasonings very similar to those used in the proof of (step 2 of) Theorem 3 below that if we have $X_{T_{r_l}}/r_l \xrightarrow{p} 1$ but there exists $\varepsilon > 0$ such that $\mathbb{P}(\Delta X_{T_{r_l}}/r_l > \varepsilon) > 0$ then X is 'mesh-jumping' relative to r_l: that is, for a subsequence r_l of r_k, with probability approaching one as l gets large, X is confined to a set of small (relative to r_l) intervals before T_{r_l}, neighbouring intervals being separated by a distance close to a divisor $d_l(> 1/\varepsilon)$ of r_l (imagine the rungs of a ladder). Consider the subordinator X and sequence r_k in the example above, and note that if we take

any bounded sequence of integers n_k then we similarly have $X_{T_{n_k r_k}}/n_k r_k \xrightarrow{p} 1$ while $\Delta X_{T_{n_k r_k}}/n_k r_k \xrightarrow{p} 0$.

This in turn explains why X cannot be weakly stable ($X_{T_r} r^{-1} \xrightarrow{p} 1$) and have big jumps (in the sense that $\Delta^-_{T_r} r^{-1} \xrightarrow{p} 0$).

It is an immediate consequence of Theorem 3 that $\mathbb{P}(X_{T_{r_k}} > 0) \to 1$ implies $X_{T_{r_k}}/\Delta^-_{T_{r_k}} \xrightarrow{p} \infty$. This may be loosely justified as follows: if $\mathbb{P}(X_{T_{r_k}} < N\Delta^-_{T_{r_k}})$ (or $\mathbb{P}(X_{t_k} < N\Delta^-_{t_k})$) is bounded away from 0 then we could reason that, since X has stationary, independent increments, it could have N such jumps before T_{r_k} (or t_k) with probability bounded away from 0.

An argument of this kind can be formalised for X_{t_k}, whence we have the following Theorem.

Theorem 4

(i) *For any sequence* $r_k \downarrow 0$, $\mathbb{P}(X_{T_{r_k}} > 0) \to 1 \Leftrightarrow \dfrac{X_{T_{r_k}}}{\Delta^-_{T_{r_k}}} \xrightarrow{p} \infty$.

(ii) *For any sequence* $t_k \downarrow 0$, $\mathbb{P}(X_{t_k} > 0) \to 1 \Leftrightarrow \dfrac{X_{t_k}}{\Delta^-_{t_k}} \xrightarrow{p} \infty$.

We give an analytical condition that the probability a LP X leaves a sequence of intervals $[-r_k, r_k]$ approaches one. The proof is similar to that for Griffin and McConnell's [9] analogous result for the RW case.

Theorem 5 *For any sequence* $r_k \downarrow 0$,

$$\mathbb{P}(X_{T_{r_k}} > 0) \to 1 \Leftrightarrow \liminf_{r_k \downarrow 0} \frac{A(\lambda r_k)}{\lambda r_k k(\lambda r_k)} > 0, \quad \frac{W(\lambda r_k)}{k(\lambda r_k)} \to 0 \quad \forall \lambda > 0.$$

4 Preliminaries and proofs

Our approach will be to first prove the sequential results of Sect. 3 before going on to deduce the full-sequence results.

We will repeatedly appeal to the following theorem from Pruitt [13] (see also [6].) It is similarly crucial in the proof of the large-time results.

Theorem 6 *There exists* $C > 0$ *such that for any LP X and for all* $a, t > 0$:

(i) $\mathbb{P}(M_t \geq a) \leq Ctk(a)$;

(ii) $\mathbb{P}(M_t \leq a) \leq \dfrac{C}{tk(a)}$;

(iii) $\dfrac{1}{Ck(a)} \leq ET_a \leq \dfrac{C}{k(a)}$.

Furthermore for all $x > 0$, $\lambda > 1$

$$\lambda^{-3} \leq \frac{k(\lambda x)}{k(x)} \leq 3.$$

We also require the following lemma (from e.g. [6]).

Lemma 7 *Let f be any positive differentiable function such that*

$$\lim_{x \downarrow 0} \frac{xf'(x)}{f(x)} = 0.$$

Then f is slowly varying at 0.

Proof of Theorem 3 To avoid an overabundance of minus signs later in the proof we will prove Theorem 3 for $-X$. Namely, we show that the following are equivalent (for any sequence r_i):

$$\frac{V(\lambda r_i)}{k(\lambda r_i)} \to 0 \qquad \forall \lambda > 0; \tag{22}$$

$$\frac{\Delta^+_{T_{r_i}}}{r_i} \xrightarrow{P} 0; \tag{23}$$

$$\frac{\Delta X_{T_{r_i}} \vee 0}{r_i} \xrightarrow{P} 0; \tag{24}$$

$$\frac{\Delta^+_{T_{r_i}}}{X_{T_{r_i}}} \xrightarrow{P} 0; \tag{25}$$

$$\frac{\Delta^+_{T_{\lambda r_i}}}{\lambda r_i} \xrightarrow{P} 0 \qquad \forall \lambda > 0; \tag{26}$$

$$\frac{\Delta X_{T_{\lambda r_i}} \vee 0}{\lambda r_i} \xrightarrow{P} 0 \qquad \forall \lambda > 0. \tag{27}$$

We let

$$Y_t^r = \sum_{0 \leq s \leq t} \mathbf{1}_{\{\Delta X_t > r\}}.$$

((22)$\Rightarrow$ (26)) Given $\varepsilon > 0$ we have

$$\begin{aligned}
\mathbb{P}(\Delta^+_{T_{\lambda r_i}} > \lambda r_i \varepsilon) &\leq E(Y^{\lambda r_i \varepsilon}_{T_{\lambda r_i}}) \\
&\leq EY^{\lambda r_i \varepsilon}_1 ET_{\lambda r_i} \\
&\leq c \frac{EY^{\lambda r_i \varepsilon}_1}{k(\lambda r_i \varepsilon)} \\
&\leq c \frac{V(\lambda r_i \varepsilon)}{k(\lambda r_i \varepsilon)},
\end{aligned}$$

where the second inequality follows by Optional Stopping and the third from Theorem 6. ((26)$\Rightarrow$ (27))

$$\Delta^+_{T_{\lambda r_i}} \geq \Delta X_{T_{\lambda r_i}} \vee 0.$$

((27)$\Rightarrow$ (22)) Since $\{\Delta X_{T_{\lambda r_i/2}} \vee 0 > 3\lambda r_i/2\}$ implies $\{\Delta X_{T_{\lambda r_i}} \vee 0 > 3\lambda r_i/2\}$ we have

$$\mathbb{P}(\Delta X_{T_{\lambda r_i}} \vee 0 > \lambda r_i) \geq \mathbb{P}(\Delta X_{T_{\lambda r_i/2}} \vee 0 > 3\lambda r_i/2)$$
$$= E(Y_{T_{\lambda r_i/2}}^{3\lambda r_i/2})$$
$$= E Y_1^{3\lambda r_i/2} E T_{\lambda r_i/2}$$
$$\geq c \frac{V(3\lambda r_i/2)}{k(3\lambda r_i/2)},$$

where we have again used Optional Stopping and Theorem 6. Hence we have shown that (22), (26) and (27) are equivalent.

We have trivially that

$$\frac{\Delta_{\lambda T_{r_i}}^+}{\lambda r_i} \xrightarrow{p} 0 \ \forall \lambda > 0 \Rightarrow \frac{\Delta_{T_{r_i}}^+}{r_i} \xrightarrow{p} 0 \Rightarrow \frac{\Delta X_{T_{r_i}} \vee 0}{r_i} \xrightarrow{p} 0.$$

i.e. (26) $\Rightarrow$ (23) $\Rightarrow$ (24). Since, given $\lambda > 0$, $\frac{\Delta_{T_{\lambda r_i}}^+}{\lambda r_i} \leq \frac{1}{\lambda} \frac{\Delta_{T_{2^n r_i}}^+}{r_i}$ for n large enough, '(24) $\Rightarrow$ (26)' (and hence the equivalence of (26), (23), (24)) would follow if we could show that

$$(24) \Rightarrow \frac{\Delta_{T_{2r_i}}^+}{r_i} \xrightarrow{p} 0. \tag{28}$$

We will prove (28) in the following two steps:-

Step 1:   RHS (of (28)) fails $\Rightarrow \exists n \in \mathbb{N}$ s.t. $\dfrac{\Delta X_{T_{r_i/2^n}} \vee 0}{r_i} \xrightarrow{p} \!\!\!\!\!\!/ \ 0$;

Step 2: $\dfrac{\Delta X_{T_{r_i}} \vee 0}{r_i} \xrightarrow{p} \!\!\!\!\!\!/ \ 0 \Rightarrow \dfrac{\Delta X_{T_{2r_i}} \vee 0}{r_i} \xrightarrow{p} \!\!\!\!\!\!/ \ 0.$

Thus, the failure of the RHS of (28) will imply $\frac{\Delta X_{T_{r_i/2^n}} \vee 0}{r_i} \xrightarrow{p} \!\!\!\!\!\!/ \ 0$ (by Step 1), whence we deduce (by Step 2) the failure of (24).

Proof of Step 1 Assume $\exists \varepsilon > 0, n \in \mathbb{N}$ and $\{r_s\} \subseteq \{r_i\}$ such that

$$\mathbb{P}\left(\Delta_{T_{2r_s}}^+ > \frac{r_s}{2^n}\right) > \varepsilon$$

for all r_s.

For all s, let

$$\tau_0^s = 0; \ \tau_j^s = \inf\left(t > \tau_{j-1}^s : \left|X_t - X_{\tau_{j-1}^s}\right| > r_s/2^{n+1}\right) \text{ for } j \in \mathbb{N}.$$

Then, for any s, any j,

$$\mathbb{P}\left(\tau^s_{2^{n+4}j} < T_{2r_s} \,\Big|\, \tau^s_{2^{n+4}(j-1)} < T_{2r_s}\right)$$

$$\leq 1 - \mathbb{P}\left(X_{T_{r_s/2^{n+1}}} > 0\right)^{2^{n+4}} - \mathbb{P}\left(X_{T_{r_s/2^{n+1}}} < 0\right)^{2^{n+4}}$$

$$\leq 1 - (1/2)^{2^{n+4}}.$$

Therefore there exists an integer N such that $\mathbb{P}(\tau^s_{2^{n+4}N} \leq T_{2r_s}) \leq \varepsilon/2$ for all s, and hence

$$\frac{\varepsilon}{2} \leq \mathbb{P}\left(\Delta^+_{T_{2r_s}} > \frac{r_s}{2^n}\right) - \mathbb{P}\left(\tau^s_{2^{n+4}N} \leq T_{2r_s}\right)$$

$$\leq \mathbb{P}\left(\Delta^+_{T_{2r_s}} > \frac{r_s}{2^n}, \tau^s_{2^{n+4}N} > T_{2r_s}\right)$$

$$\leq \mathbb{P}\left(\Delta^+_{\tau^s_{2^{n+4}N}} > \frac{r_s}{2^n}\right)$$

$$\leq 2^{n+4}N \times \mathbb{P}\left(\Delta^+_{T_{r_s/2^{n+1}}} > \frac{r_s}{2^n}\right).$$

Thus

$$\mathbb{P}\left(\Delta X_{T_{r_s/2^{n+1}}} > r_s/2^n\right) \geq \varepsilon/(2^{n+5}N)$$

completing the proof of Step 1.

Proof of Step 2 We assume for contradiction that

$$\frac{\Delta X_{T_{r_i}} \vee 0}{r_i} \xrightarrow{p} 0 \tag{29}$$

and

$$\frac{\Delta X_{T_{2r_i}} \vee 0}{r_i} \xrightarrow{p} 0. \tag{30}$$

Thus we must have either $\{r_a\} \subseteq \{r_i\}$, $\lambda' > 0$, $\alpha' > 0$ such that

$$\mathbb{P}\left(X_{T_{r_a}} > (1 + \lambda')r_a\right) > \alpha' \quad \forall r_a, \tag{31}$$

or $\{r_b\} \subseteq \{r_i\}$, $\lambda'' > 0$, $\alpha'' > 0$ such that

$$\mathbb{P}\left(X_{T_{r_b}} > 0, X_{T_{r_b}-} < (1 - \lambda'')r_b\right) > \alpha'' \quad \forall r_b. \tag{32}$$

If (31) holds then by the Markov property applied on entry into $[r_a(1-\lambda'/2), r_a]$ we have

$$\mathbb{P}\left(\Delta X_{T_{2r_a}} > \lambda' r_a/2\right) \geq \mathbb{P}\left(X \text{ visits } \left[r_a(1-\lambda'/2), r_a\right] \text{ before } T_{2r_a}\right)$$
$$\times \mathbb{P}\left(X_{T_{r_a}} > (1+\lambda')r_a\right),$$

and thus by (30)

$$\mathbb{P}\left(X \text{ visits } \left[r_a(1-\lambda'/2), r_a\right] \text{ before } T_{2r_a}\right) \to 0$$

as $r_a \downarrow 0$. If on the other hand (32) holds then by stopping X when it enters $\left[r_b, r_b\left(1+\lambda''/2\right)\right]$ we similarly have

$$\mathbb{P}\left(X \text{ visits } \left[r_b, r_b\left(1+\lambda''/2\right)\right] \text{ before } T_{2r_b}\right) \to 0,$$

which in turn (by stopping X in $[r_b\left(1-\lambda''/4\right), r_b]$ and spotting $\liminf_{r_a} \mathbb{P}(X_{T_{2a}} > 0) > 0$) implies that

$$\mathbb{P}\left(X \text{ visits } \left[r_b\left(1-\lambda''/4\right), r_b\right] \text{ before } T_{2r_b}\right) \to 0.$$

Hence in all cases (when (29) and (30) hold) we may define a positive d such that

$$d = \sup_{\{r_s\}\subseteq\{r_i\}} \sup\left\{\delta : \mathbb{P}\left(X \text{ visits } \left[r_s(1-\delta), r_s\right] \text{ before } T_{2r_s}\right) \to 0\right\}.$$

Now choose $\varepsilon < 1/50$, $\{r_s\} \subseteq \{r_i\}$ and $\xi > 0$ such that

$$\mathbb{P}\left(X \text{ visits } \left[r_s(1-d(1-\varepsilon)), r_s\right] \text{ before } T_{r_s}\right) \to 0 \tag{33}$$

as $r_s \downarrow 0$ and

$$\mathbb{P}\left(X \text{ visits } \left[0 \vee \left[r_s(1-d(1+\varepsilon))\right], r_s(1-d(1-\varepsilon))\right] \text{ before } T_{2r_s}\right) > \xi \tag{34}$$

for all r_s. Then, by stopping in $[2r_s d\varepsilon, r_s d(1-\varepsilon)]$ and considering (34) and (33), we must have

$$\mathbb{P}\left(X \text{ visits } \left[2r_s d\varepsilon, r_s d(1-\varepsilon)\right] \text{ before } T_{r_s}\right) \to 0. \tag{35}$$

Consequently, stopping in $\left[2r_s - r_s d(1-\varepsilon), 2r_s - 2r_s d\varepsilon\right]$, we deduce from (30) that

$$\mathbb{P}\left(X \text{ visits } \left[2r_s - r_s d(1-\varepsilon), 2r_s - 2r_s d\varepsilon\right] \text{ before } T_{2r_s}\right) \to 0. \tag{36}$$

Then, noting that,

$$\mathbb{P}\left(X \text{ visits}[2r_s - 2r_s d\varepsilon, 2r_s] \text{ before } T_{2r_s} \,\middle|\, X_{T_{2r_s}} > 0\right) \to 1$$

we must have

$$\mathbb{P}\left(X \text{ visits } \left[-r_s d(1-\varepsilon), -2r_s d\varepsilon\right] \text{ before } T_{2r_s}\right) \to 0, \tag{37}$$

since otherwise (36) would fail. From (35) and (37) we see that the probability X leaves $[-2r_s d\varepsilon, 2r_s d\varepsilon]$ by a jump with modulus bigger than $dr_s/2$ goes to one as r_s goes to 0. For each s we may consider X as the sum of two independent processes Y^s and Z^s such that Z^s consists of the jumps of X with modulus bigger than $dr_s/2$. By (29) we may choose $\alpha > 0$ such that

$$\liminf_{r_s} \mathbb{P}(X_{T_{r_s}} > 0) > \alpha,$$

and hence we must also have a positve p such that

$$p = \liminf_{s \to \infty} \mathbb{P}\left(\text{the first jump of } Z^s \text{ is positive}\right) > 0.$$

Let

$$E_s = [2r_s - r_s d\varepsilon, 2r_s]$$

such that

$$\liminf_{r_s} \mathbb{P}\left(X \text{ visits } E_s \text{ before } T_{2r_s}\right) \geq \liminf_{r_s} \mathbb{P}\left(X_{T_{2r_s}} > 0\right) > \alpha^2.$$

Hence (by (30)), stopping in E_s,

$$\liminf_{s \to \infty} \mathbb{P}\left(Z^s \text{ jumps before } Y^s \text{ visits } (0, \infty)\right) \to 0. \tag{38}$$

Thus, if we define

$$A_{c_s}^s = \mathbb{P}\left(Y_s \text{ visits } (c_s r_s, \infty) \text{ before } Z_s \text{ jumps}\right),$$

then for each large s, there must exist $c_s > 0$ such that

$$\mathbb{P}\left(A_{c_s}^s\right) \in (1/3, 2/3);$$

to see this spot that $\mathbb{P}(A_y^s)$ decreases as y increases, that $\mathbb{P}(A_{2dr_s\varepsilon}^s)$ is close to 0 (by (35)), while by (38) $\exists \chi_s > 0$ s.t. $\mathbb{P}(A_{\chi_s}^s)$ is close to one; whereas we can exclude a discontinuity bridging $(1/3, 2/3)$ since for $x > 0$, $\mathbb{P}(A_{2x}^s) > \mathbb{P}(A_x^s)^2$.

As noted above we have

$$\mathbb{P}\left(X \text{ enters } E_s \text{ before } T_{2r_s} \,\middle|\, X_{T_{2r_s}} > 0\right) \to 1,$$

and hence

$$\liminf_{s\to\infty} \mathbb{P}\left(X \text{ enters } \{E_s - c_s r_s\} \text{ before } T_{2r_s}\right)$$

$$\geq \liminf_{s\to\infty} \mathbb{P}\left(X \text{ enters } E_s \text{ before } T_{2r_s} \,\middle|\, A_{c_s}, X_{T_{r_s}} > 0, X_{T_{2r_s}>0}\right)$$

$$\geq \liminf_{s\to\infty} \mathbb{P}\left(A_{c_s}, X_{T_{r_s}} > 0, X_{T_{2r_s}} > 0\right)$$

$$> \alpha^2/3.$$

Thus, finally, we have

$$\liminf_{s\to\infty} \mathbb{P}(\Delta X_{T_{2r_s}} > dr_s/3) > \liminf_{s\to\infty} \mathbb{P}\left(X \text{ enters } \{E_s - c_s r_s\} \text{ before } T_{2r_s}\right)$$
$$\times \mathbb{P}(\text{the first } Z_s\text{-jump is +ve and occurs before } Y_s \text{ leaves } [-2r_s d\varepsilon, c_s r_s])$$
$$> \alpha^2 p/9,$$

which contradicts (30) and completes the proof of Step 2.

We complete the proof of Theorem 3 by showing the the equivalence of (23) and (25).

(23) $\Rightarrow$ (25) is immediate. Now assume (23) fails. Then, by above (24) fails too and hence there exists $\{r_j\} \subseteq \{r_i\}$, $\varepsilon > 0$ and $c > 0$ such that $\mathbb{P}\left(\Delta X_{T_{r_j}}/r_j > c\right) > \varepsilon \ \forall r_j$. Thus for each r_j we have that with probability greater than ε

$$\frac{\Delta_{T_{r_j}}^+}{X_{T_{r_j}}} \geq \frac{\Delta X_{T_{r_j}}}{\Delta X_{T_{r_j}} + r_j} > \frac{1}{1 + c^{-1}}$$

and hence (25) fails.   □

Informally, Theorem 3 shows that if X sometimes has large negative jumps before T_{r_k} then these jumps will on occasions carry it out of the interval $[-r_k, r_k]$. Thus when we come to the question of determining when the probability that X leaves an interval upwards goes to 1 as the interval is shrunk to 0 we may exclude all processes where $\Delta_{T_{r_k}}^- \overset{p}{\not\to} 0$.

Furthermore, it can be deduced from the well known fact that a simple asymmetric random walk diverges that for a LP with no negative jumps

$$\liminf_{r\downarrow 0} \mathbb{P}(X_{T_r} > 0) = p > 1/2 \Rightarrow \mathbb{P}(X_{T_r} > 0) \to 1.$$

Indeed, define a simple RW such that $\mathbb{P}(S_1 = 1) = p = 1 - \mathbb{P}(S_1 = -1) > 1/2$. We may then find $N > 0$ such that the probability S leaves an interval $[-N, N]$ upwards is arbitrarily close to 1. Since X has no negative overshoots

$$\liminf_{r\downarrow 0} \mathbb{P}(X_{T_r} > 0) = \liminf_{r\downarrow 0} \mathbb{P}(X_{T_{Nr}} > 0) \geq \mathbb{P}(S_{T_N} > 0).$$

It turns out that a similar reasoning can be applied in this instance, and is used in the following proof, which is similar to that used in Griffin and McConnell [9] for the RW case.

Proof of Theorem 5 First note that

$$\mathbb{P}(X_{T_{r_k}} > 0) \to 1 \Rightarrow \mathbb{P}(X_{T_{\delta r_k}} > 0) \to 1 \quad \forall \delta > 0, \tag{39}$$

since if $\delta < 1$ then we can let N be the smallest integer $\geq 1/\delta$ and reason via:

$$\mathbb{P}(X_{T_{\delta r_k}} < 0)^N \leq \mathbb{P}(X_{T_{r_k}} < 0);$$

whereas if $\delta \geq 1$ then for any integer $N \geq \delta$ we have

$$\mathbb{P}(X_{T_{\delta r_k}} > 0) \geq \mathbb{P}(X_{T_{r_k}} > 0)^N.$$

Recall that we defined $\{\check{X}_t^{\lambda r_k}, t \geq 0\}$ as X with all jumps whose modulus is bigger than λr_k being reduced to a size of λr_k. In other words we adjust the jump measure as follows:

$$\check{W}(x) = W(x), \quad \check{V}(x) = V(x) \ \text{ for } \ x < \lambda r_k;$$
$$\check{W}(x) = 0, \quad \check{V}(x) = 0 \ \text{ for } \ x \geq \lambda r_k.$$

Let

$$Y_t = \check{X}_t^{\lambda r_k} - E\check{X}_1^{\lambda r_k} t.$$

As Y is a (well-behaved) martingale and $\check{T}_{r_k}$ a stopping time (the first time the process $\check{X}$ leaves the interval $[-r_k, r_k]$), we have from Optional Stopping that

$$E\check{X}_{\check{T}_{r_k}}^{\lambda r_k} = E\check{X}_1^{\lambda r_k} E\check{T}_{r_k}.$$

Moreover

$$E\check{X}_{\check{T}_{r_k}}^{\lambda r_k} \geq r_k \mathbb{P}(\check{X}_{\check{T}_{r_k}}^{\lambda r_k} > 0) - (\lambda + 1) r_k \mathbb{P}(\check{X}_{\check{T}_{r_k}}^{\lambda r_k} < 0)$$
$$\geq (\lambda + 2) r_k \mathbb{P}(\check{X}_{\check{T}_{r_k}}^{\lambda r_k} > 0) - (\lambda + 1) r_k;$$

while by similar reasoning

$$E\check{X}_{\check{T}_{r_k}}^{\lambda r_k} \leq (\lambda + 2) r_k \mathbb{P}(\check{X}_{\check{T}_{r_k}}^{\lambda r_k} > 0) - r_k.$$

Hence

$$\frac{E\check{X}_1^{\lambda r_k} E\check{T}_{r_k} r_k^{-1} + \lambda + 1}{\lambda + 2} \geq \mathbb{P}(\check{X}_{\check{T}_{r_k}}^{\lambda r_k} > 0) \geq \frac{E\check{X}_1^{\lambda r_k} E\check{T}_{r_k} r_k^{-1} + 1}{\lambda + 2}. \tag{40}$$

If we set $\lambda = 2$ then (since any jump with modulus greater than $2r_k$ occurring before T_{r_k} will carry X out of $[-r_k, r_k]$) $\check{T}_{r_k} = T_{r_k}$ and $\{\check{X}_{\check{T}_{r_k}} > 0\} = \{X_{T_{r_k}} > 0\}$. Therefore, by (40), Theorem 6 and (2)

$$\mathbb{P}(X_{T_{r_k}} > 0) \leq \frac{1}{4} E\check{X}_1^{2r_k} ET_{r_k} r_k^{-1} + \frac{3}{4} \leq c\frac{A(2r_k)}{r_k k(2r_k)} + \frac{3}{4}. \tag{41}$$

And so then by (39) and Theorem 3

$$\text{LHS(of theorem)} \Rightarrow \left(\Delta X_{T_{r_k}} \wedge 0\right) r_k^{-1} \xrightarrow{p} 0, \ \mathbb{P}(X_{T_{\beta r_k}} > 0) \to 1 \quad \forall \beta > 0$$
$$\Rightarrow \text{RHS}.$$

We now assume RHS and will proceed from (40) to show that

$$\lim_{\lambda \downarrow 0} \liminf_{r_k \downarrow 0} \mathbb{P}(\check{X}_{\check{T}_{r_k}}^{\lambda r_k} > 0) = 1 \tag{42}$$

from whence we will immediately have LHS since then by Theorem 3

$$\lim_{\lambda \downarrow 0} \liminf_{r_k \downarrow 0} \mathbb{P}(\check{X}_{\check{T}_{r_k}}^{\lambda r_k} > 0) \leq \lim_{\lambda \downarrow 0} \liminf_{r_k \downarrow 0} \left[\mathbb{P}(X_{T_{r_k}} > 0) + \mathbb{P}(\Delta_{T_{r_k}}^- \geq \lambda r_k)\right]$$
$$\leq \liminf_{r_k \downarrow 0} \mathbb{P}(X_{T_{r_k}} > 0).$$

Given $\lambda \leq 1/2$, define for all r_k:

$$\tau_0^{\lambda r_k} = 0; \quad \tau_{j+1}^{\lambda r_k} = \inf\{t > \tau_j^{\lambda r_k} : |X_t - X_{\tau_j}| > \lambda r_k\} \ \text{ for } j = 0, 1, 2, \ldots$$

Then, for any integer n, $|\check{X}_{\tau_n^{\lambda r_k}}^{\lambda r_k}| \leq 2n\lambda r_k$ and hence if we let $s(\lambda)$ be the largest integer less than $1/2\lambda$, $\check{T}_{r_k} \geq \tau_{s(\lambda)}^{\lambda r_k}$. Thus

$$E\check{T}_{r_k} \geq E\tau_{s(\lambda)}^{\lambda r_k} = s(\lambda)ET_{\lambda r_k}$$

and so

$$3\lambda E\check{T}_{r_k} \geq ET_{\lambda r_k}.$$

Then by (40), Theorem 6 and (2) we have

$$\mathbb{P}(\check{X}^{\lambda r_k}_{\check{T}_{r_k}} > 0) \geq \frac{cA(\lambda r_k)/\lambda r_k k(\lambda r_k) + 1}{\lambda + 2}. \tag{43}$$

Replacing r_k with δr_k and λ with $\lambda \delta^{-1}$ we have for $\delta > 0$, $\lambda \leq \delta/2$

$$\mathbb{P}\left(\check{X}^{\lambda r_k}_{\check{T}_{\delta r_k}} > 0\right) \geq \frac{c_1 A(\lambda r_k)/\lambda r_k k(\lambda r_k) + 1}{\lambda \delta^{-1} + 2}.$$

Thus $\exists p_1 > 1/2$ such that $\forall \delta > 0$

$$\lim_{\lambda \downarrow 0} \liminf_{r_k \downarrow 0} \mathbb{P}(\check{X}^{\lambda r_k}_{\check{T}_{\delta r_k}} > 0) \geq p_1 \tag{44}$$

As alluded to above, if we consider the Simple Random Walk $\{S_n, n \in \mathbb{N}\}$ with parameter p_1 (i.e. $\mathbb{P}(S_1 = 1) = p_1$, $\mathbb{P}(S_1 = -1) = 1 - p_1$) then $\mathbb{P}(S_{T_n} > 0) \to 1$ as $n \to \infty$. (We are taking T_N for a RW as the first time it leaves $[-N, N]$.) Hence for $\varepsilon > 0$ we may choose $N, H \in \mathbb{N}$ such that

$$\mathbb{P}(S_{T_N} > 0, \; T_N \leq H) > 1 - \varepsilon.$$

For given λ, r_k let R_n be a simple RW such that R_1 has the same distribution as $\check{X}^{\lambda r_k}_{\check{T}_{r_k/2N}} |\check{X}^{\lambda r_k}_{\check{T}_{r_k/2N}}|^{-1}$. Then

$$\mathbb{P}\left(\check{X}^{\lambda r_k} \text{ leaves } [-r_k/2 - \lambda r_k H, \; r_k/2 - \lambda r_k H] \text{ upwards }\right)$$
$$\geq \mathbb{P}(R_{T_N^R} > 0, \; T_N^R \leq H)$$
$$\geq \mathbb{P}(S_{T_N^S} > 0, \; T_N^S \leq H) \geq 1 - \varepsilon.$$

Therefore for any $\xi > 0$

$$\lim_{\lambda \downarrow 0} \liminf_{r_k \downarrow 0} \mathbb{P}(\check{X}^{\lambda r_k} \text{ leaves } [-r_k(1/2 + \xi), \; r_k(1/2 - \xi)] \text{ upwards}) \geq 1 - \varepsilon,$$

which implies $\lim_{\lambda \downarrow 0} \liminf_{r_k \downarrow 0} \mathbb{P}(\check{X}^{\lambda r_k}_{\check{T}_{r_k}} > 0) \geq (1 - \varepsilon)^2$ and as epsilon is arbitrary (42) follows and the proof is complete. $\qquad \square$

Proof of Theorem 4 (i) First note that

$$\text{RHS} \Rightarrow \mathbb{P}(X_{T_{r_k}} > \Delta^-_{T_{r_k}}) \to 1 \Rightarrow \text{LHS},$$

while the converse follows from Theorem 3 as

$$\mathbb{P}(X_{T_{r_k}} > 0) \to 1 \Rightarrow (-\Delta X_{T_{r_k}} \wedge 0)/r_k \xrightarrow{p} 0 \Rightarrow \Delta^-_{T_{r_k}}/r_k \xrightarrow{p} 0.$$

We then have $\left|X_{T_{r_k}}\right|/\Delta_{T_{r_k}}^- \xrightarrow{p} \infty$ and hence $X_{T_{r_k}}/\Delta_{T_{r_k}}^- \xrightarrow{p} \infty$ from LHS.

(ii) RHS$\Rightarrow$LHS is again immediate. To prove the converse we assume for contradiction that $\exists\, t_j \downarrow 0$, $k \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\lim_{t_j} \mathbb{P}(X_{t_j} > 0) = 1 \quad \text{and} \quad \mathbb{P}(X_{t_j} < 2k\Delta_{t_j}^-) > 8\varepsilon \quad \text{for all } t_j. \tag{45}$$

Let

$$E_j = \{X_{t_j} < 2k\Delta_{t_j}^-\};$$

and for each t_j choose c_j such that

$$\mathbb{P}(E_j, \Delta_{t_j}^- \le c_j) \ge 2\varepsilon \tag{46}$$

and

$$\mathbb{P}(E_j, \Delta_{t_j}^- \ge c_j) \ge 6\varepsilon. \tag{47}$$

Then (by (47)) we must have either a subsequence $\{t_m, c_m, E_m\} \in \{t_j, c_j, E_j\}$ such that

$$\mathbb{P}(E_m, \Delta_{t_m}^- > 2c_m) \ge 2\varepsilon \quad \forall m \tag{48}$$

or a subsequence $\{t_n, c_n, E_n\} \in \{t_j, c_j, E_j\}$ such that

$$\mathbb{P}(E_n, c_n \le \Delta_{t_n}^- \le 2c_n) \ge 4\varepsilon \quad \forall n. \tag{49}$$

First assume that (48) holds and for each m let

$$X_t = Y_t^m + Z_t^m$$

where Z_t^m consists of all the jumps smaller than $-2c_m$ by time t. (i.e. $Z_t^m = \sum_{0 \le s \le t} \Delta X_s \mathbf{1}_{\{\Delta X_s < -2c_m\}}$.) Further let

$$N_t^m = \sum_{0 \le s \le t} \mathbf{1}_{\{\Delta X_s < -2c_m\}},$$

the number of jumps of Z_t^m by time t. Then since we have $\mathbb{P}\left(N_{t_m}^m \ge 1\right) \ge 2\varepsilon$ (from (48)) and $\mathbb{P}\left(N_{t_m}^m = 0\right) \ge 2\varepsilon$ (from (46)) the parameters, say p_m, of the Poisson distributions $N_{t_m}^m$ must be bounded uniformly away from 0 and ∞ for all m. Therefore $\exists\, \xi > 0$ such that for all large m

$$\mathbb{P}\left(N_{t_m}^m \ge k\right) > e^{-p_m} p_m^k/k! > \xi.$$

From (45) and (46)

$$\mathbb{P}\left(Y_{t_m}^m \in (0, 2kc_m), Z_{t_m}^m = 0\right) > \varepsilon$$

for all large m and as $Y^m_{t_m}$ and $Z^m_{t_m}$ are independent we have (for all large m)

$$\mathbb{P}\left(X_{t_m} < 0\right) \geq \mathbb{P}\left(Y^m_{t_m} \in (0, 2kc_m), N^m_{t_m} \geq k\right)$$
$$= \mathbb{P}\left(Y^m_{t_m} \in (0, 2kc_m)\right)\mathbb{P}\left(N^m_{t_m} \geq k\right)$$
$$> \varepsilon\xi,$$

which gives the required contradiction (when (48) holds).

If on the other hand (49) holds then we now let $X_t = Y^n_t + Z^n_t$ where

$$Z^n_t = \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s \leq -c_n\}}$$

and define

$$N^n_t = \mathbf{1}_{\{\Delta X_s \leq -c_n\}}.$$

By (45) and (49) we then have for all large n

$$3\varepsilon < \mathbb{P}\left(X_{t_n} \in (0, 4ck_n), \Delta^-_{t_n} \geq c_n\right)$$
$$= \sum_{\alpha=1}^{\infty} \mathbb{P}\left(0 < Y^n_{t_n} + Z^n_{t_n} < 4kc_n, N^n_{t_n} = \alpha\right)$$
$$= \sum_{\alpha=1}^{\infty} \mathbb{P}\left(-Z^n_{t_n} < Y^n_{t_n} < 4kc_n - Z^n_{t_n} \mid N^n_{t_n} = \alpha\right)\mathbb{P}\left(N^n_{t_n} = \alpha\right).$$

Similarly to above, each $N^n_{t_n}$ has a Poisson distribution with parameter say p_n (where $p_n = t_n\Pi(-\infty, -c_n]$). Since for all n $\mathbb{P}\left(N^n_{t_n} \geq 1\right) \geq 4\varepsilon$ the p_n must be uniformly bounded away from 0. (i.e. $\liminf_n p_n > 0$.) For the moment we will assume that they are also uniformly bounded away from infinity. Thus we may choose $C \in \mathbb{N}$ and $\varsigma > 0$ such that for all n

$$\mathbb{P}\left(N^n_{t_n} > C\right) < \varepsilon \quad \text{and} \quad \frac{\mathbb{P}\left(N^n_{t_n} = \alpha + 4k\right)}{\mathbb{P}\left(N^n_{t_n} = \alpha\right)} = \frac{p_n^{4k}\alpha!}{(\alpha + 4k)!} > \varsigma \;\; \forall \alpha \leq C.$$

Thus, as $Y^n_{t_n}$ and $Z^n_{t_n}$ are independent, we have for all large n

$$\mathbb{P}\left(X_{t_n} < 0\right) > \sum_{\alpha=1}^{C} \mathbb{P}\left(-Z^n_{t_n} < Y^n_{t_n} < 4kc_n - Z^n_{t_n} \mid N^n_{t_n} = \alpha\right)\mathbb{P}\left(N^n_{t_n} = \alpha + 4k\right)$$
$$> \sum_{\alpha=1}^{C} \mathbb{P}(-Z^n_{t_n} < Y^n_{t_n} < 4kc_n - Z^n_{t_n} \mid N^n_{t_n} = \alpha)\,\mathbb{P}\left(N^n_{t_n} = \alpha\right)\frac{\mathbb{P}(N^n_{t_n}=\alpha+4k)}{\mathbb{P}(N^n_{t_n}=\alpha)}$$
$$> 2\varepsilon\varsigma,$$

which gives a contradiction (under the assumption that $\limsup_n p_n < \infty$.) Thus we may now suppose without loss of generality that $p_n \to \infty$. But then

$$\sup_{m \in \mathbb{N}} \mathbb{P}\left(N^n_{t_n} \in [m, m + 4k]\right) \to 0 \quad \text{as } n \to \infty.$$

Therefore (by (45) and (49)) for all large n there exists $H_n \in \mathbb{N}$ such that

$$\mathbb{P}\left(X_{t_n} \in (0, 4kc_n), N^n_{t_n} \in [1, H_n]\right) > \varepsilon, \quad \mathbb{P}\left(N^n_{t_n} > H_n + 4k\right) > \varepsilon,$$

and hence

$$\mathbb{P}\left(X_{t_n} < 0\right) \geq \mathbb{P}\left(X_{t_n} \in (0, 4kc_n) \,\middle|\, N^n_{t_n} \leq H_n\right) \mathbb{P}\left(N^n_{t_n} > H_n + 4k\right) > \varepsilon^2$$

for all large n, which completes the proof. $\qquad\square$

We now move on to the relation between the probability X is positive at small times and the probability X leaves small intervals upwards. We use a proof similar to that used by Kesten and Maller [11] to prove the analogous result for RW.

Proposition 8 (i) $\limsup_{t \downarrow 0} \mathbb{P}(X_t > 0) = 1 \Leftrightarrow \limsup_{r \downarrow 0} \mathbb{P}(X_{T_r} > 0) = 1$.
(ii) $\lim_{t \downarrow 0} \mathbb{P}(X_t > 0) = 1 \Leftrightarrow \lim_{r \downarrow 0} \mathbb{P}(X_{T_r} > 0) = 1$.
(iii) *For* $\alpha > 0$, $\frac{X_t}{t^\alpha} \xrightarrow{P} \infty$ *as* $t \downarrow 0 \Leftrightarrow \frac{X_{T_r}}{T_r^\alpha} \xrightarrow{P} \infty$ *as* $r \downarrow 0$.

Proof Recall that we have assumed that X is not a compound Poisson process and so

$$\mathbb{P}(X_t = 0) = 0 \quad \forall t > 0; \quad \lim_{r \downarrow 0} k(r) = \infty.$$

Choose large $l > 0$, and then for each $r > 0$ define

$$t(r) = \frac{l}{k(r)}.$$

Note that $t(r) \to 0$ (continuously) as $r \downarrow 0$ and so

$$\liminf_{s \downarrow 0} \mathbb{P}(X_s > 0) = \liminf_{r \downarrow 0} \mathbb{P}(X_{t(r)} > 0).$$

By Theorem 6 we have $C > 0$ such that

$$\mathbb{P}\left(\frac{t(r)}{l^2} \leq T_r \leq t(r)\right) \geq 1 - \mathbb{P}(M_{1/lk(r)} \geq r) - \mathbb{P}(M_{l/k(r)} \leq r)$$

$$\geq 1 - 2C/l. \tag{50}$$

For $r > 0$ we let

$$\tau_0^r = 0; \ \tau_j^r = \inf\left\{s > \tau_{j-1}^r : \left|X_s - X_{\tau_{j-1}^r}\right| > r\right\} \quad \text{for } j \in \mathbb{N}.$$

Proof of (i) *and* (ii) For $\forall r > 0$, applying the Markov property

$$\mathbb{P}(X_{t(r)} \leq 0) \geq \mathbb{P}\left(X_{\tau_{j+1}^r} - X_{\tau_j^r} \leq -r, \quad \forall j \leq l^2 - 1, \ \tau_1^r \leq t(r) \leq \tau_{l^2}^r\right)$$

$$\geq \left(\mathbb{P}\left(\frac{t(r)}{l^2} \leq \tau_1^r \leq t(r), X_{\tau_1^r} \leq -r\right)\right)^{l^2}$$

$$\geq \left([\mathbb{P}(X_{T_r} < 0) - 2C/l] \vee 0\right)^{l^2} \quad \text{by (50).}$$

Hence

$$\liminf_{r \downarrow 0} \mathbb{P}(X_{t(r)} \leq 0) = 0 \Rightarrow \liminf_{r \downarrow 0} \mathbb{P}(X_{T_r} < 0) \leq 2C/l,$$

$$\lim_{r \downarrow 0} \mathbb{P}(X_{t(r)} \leq 0) = 0 \Rightarrow \lim_{r \downarrow 0} \mathbb{P}(X_{T_r} < 0) \leq 2C/l$$

and since l is arbitrary we have '$\Rightarrow$' for (i) and (ii).
Similarly to above we have

$$\mathbb{P}(X_{t(r)} \geq 0) \geq \mathbb{P}\left(X_{\tau_{j+1}} - X_{\tau_j} \geq r \text{ for all } j \leq 2l^2, \ \tau_1^r \leq t(r) \leq \tau_{2l^2}^r\right)$$

$$\geq \mathbb{P}(X_{T_r} > 0)^{2l^2} - \frac{C}{l} - \mathbb{P}\left(\tau_{2l^2} < t(r)\right).$$

Now, letting $Z(n, p)$ be a random variable with binomial distribution $B(n, p)$,

$$\mathbb{P}\left(\tau_{2l^2} < t(r)\right) \leq \mathbb{P}\left(\sum_{j=1}^{2l^2} \mathbf{1}\left\{\tau_{j+1} - \tau_j \geq t(r)l^{-2}\right\} \leq l^2\right)$$

$$= \mathbb{P}\left(Z(2l^2, \mathbb{P}(T_r < t(r)l^{-2})) \geq l^2\right)$$

$$\leq \mathbb{P}(Z(2l^2, Cl^{-1}) \geq l^2).$$

And thus by Chebyshev's inequality

$$\mathbb{P}\left(\tau_{2l^2} < t(r)\right) \leq \frac{2C}{l^3} + \frac{4C^2}{l^2} \leq \frac{C_1}{l},$$

and '$\Leftarrow$' follows trivially.

Proof of (iii) Assume RHS (and hence by (ii) that $\lim_{s\downarrow0}\mathbb{P}(X_s > 0) = 1$). Thus for given $K > 0$, we have by (50) and the strong Markov property

$$\liminf_{s\downarrow0}\mathbb{P}(X_s > Ks^\alpha) \geq \liminf_{r\downarrow0}\mathbb{P}(X_{t(r)} > Kt(r)^\alpha)$$
$$\geq \liminf_{r\downarrow0}\mathbb{P}(X_{T_r} > Kl^{2\alpha}T_r^\alpha, \; t(r)l^{-2}$$
$$\leq T_r \leq t(r), \; X_{t(r)} - X_{T_r} > 0)$$
$$\geq \liminf_{r\downarrow0}\mathbb{P}(X_{T_r} > Kl^{2\alpha}T_r^\alpha) - 2C/l$$
$$\geq 1 - 2C/l,$$

and LHS follows as l is arbitrary.

Now assume LHS. For given $K > 0$, $l \in \mathbb{N}$ and any $t > 0$ we have

$$\mathbb{P}(X_{t(r)} \leq Kl^2t(r)^\alpha) \geq \mathbb{P}(\{E_1\} \cap \{E_2\})$$

where

$$\{E_1\} = \{0 \leq X_{\tau_j} - X_{\tau_{j-1}} \leq K(\tau_j - \tau_{j-1})^\alpha \text{ for } j = 1, 2, \ldots, l^2\};$$
$$\{E_2\} = \{t(r)l^{-2} \leq \tau_j - \tau_{j-1} \leq t(r) \text{ for } j = 1, 2, \ldots, l^2\}.$$

Thus

$$\limsup_{r\downarrow0}\left(\mathbb{P}(0 \leq X_{T_r} \leq KT_r^\alpha) - 2C/l\right)^{l^2} \leq 0,$$

and RHS follows easily.   □

Proof of Theorem 1 The equivalence of (4), (5), (6) and (7) is immediate from Theorem 4 and Proposition 8. From Theorem 5 and (41) we have

$$(4) \Leftrightarrow \liminf_{r\downarrow0}\frac{A(r)}{rk(r)} > 0, \quad \lim_{r\downarrow0}\frac{W(r)}{k(r)} = 0.$$

It thus remains to show that

$$\lim_{r\downarrow0}\frac{A(r)}{rW(r)} = \infty \Leftrightarrow \lim_{r\downarrow0}\frac{W(r)}{k(r)} = 0, \quad \liminf_{r\downarrow0}\frac{A(r)}{rk(r)} > 0. \tag{51}$$

Assume RHS (of (51)). Then LHS follows easily as

$$\lim_{r\downarrow0}\frac{A(r)}{rW(r)} = \lim_{r\downarrow0}\frac{k(r)}{W(r)}\frac{A(r)}{rk(r)} = \infty.$$

Now assume LHS (of (51)). First note that $\frac{k(r)}{W(r)} \geq \frac{A(r)}{rW(r)}$ and hence

$$\lim_{r\downarrow 0} \frac{W(r)}{k(r)} = 0.$$

Then since $\lim_{r\downarrow 0} \frac{A(r)}{rW(r)} = \infty$ implies $A(r) > 0$ for small r we have

$$\liminf_{r\downarrow 0} \frac{A(r)}{rk(r)} > 0 \quad \text{iff} \quad \limsup_{r\downarrow 0} \frac{U(r)}{rA(r)} < \infty. \tag{52}$$

For any $\varepsilon > 0$ we have $\varepsilon A(x) > xW(x)$ for all small x. Thus, given ε, we have (for small r)

$$\int_0^r xW(x)dx \leq \varepsilon \int_0^r A(x)dx$$

$$\leq \varepsilon \left(r\varphi + rD(1) - \int_0^r \int_x^1 D(y)dydx \right)$$

$$\leq \varepsilon \left(r\varphi + rD(1) - \int_0^r \int_0^y D(y)dxdy - \int_r^1 \int_0^r D(y)dxdy \right)$$

$$\leq \varepsilon rA(r) - \varepsilon \int_0^r yV(y)dy + \varepsilon \int_0^r yW(y)dy. \tag{53}$$

Thus

$$\int_0^r xW(x)dx(1-\varepsilon)\varepsilon^{-1} \leq rA(r).$$

Furthermore, setting $\varepsilon = 1$ in (53), we see that

$$\int_0^r xV(x)dx \leq rA(r).$$

Hence we must have $c > 0$ such that for small r

$$U(r)/2 = \int_0^r x(W(x) + V(x))dx \leq crA(r).$$

$\square$

Before proving Theorem 2 we need the following Lemma.

Lemma 9 *For $\alpha > 0$ the following are equivalent as $r_n \downarrow 0$:*

$$\frac{M_{r_n^\alpha}}{r_n} \xrightarrow{p} \infty; \tag{54}$$

$$r_n^\alpha k(r_n) \to \infty; \tag{55}$$

$$\frac{r_n^\alpha}{T_{r_n}} \xrightarrow{p} \infty. \tag{56}$$

Proof ((54) $\Rightarrow$ (55)) For any $K \in \mathbb{N}$ we have

$$\mathbb{P}(M_{r_n^\alpha} \geq K r_n) \to 1,$$

which implies

$$\mathbb{P}\left(M_{r_n^\alpha/K} \geq r_n\right) \to 1,$$

and hence

$$\liminf_{r_n} C K^{-1} r_n^\alpha k(r_n) \geq 1$$

by Theorem 6.

((55) $\Rightarrow$ (56)) Assume (56) fails. Then $\exists p, K > 0$ and $\{r_m\} \in \{r_n\}$ such that for all r_m

$$p \leq \mathbb{P}(r_m^\alpha / T_{r_m} < K)$$
$$\leq \mathbb{P}\left(M_{r_m^\alpha/K} \leq r_m\right)$$
$$\leq \frac{CK}{r_m^\alpha k(r_m)}.$$

Thus

$$\liminf_{r_n} r_n^\alpha k(r_n) \leq CK/p$$

and so (55) also fails.

((56) $\Rightarrow$ (54)) Given $K \in \mathbb{N}$ we have $\mathbb{P}(T_{r_n} < r_n^\alpha/K) \to 1$. Thus $\mathbb{P}(M_{r_n^\alpha/K} > r_n) \to 1$ and hence $\mathbb{P}(M_{r_n^\alpha} > r_n K) \to 1$. $\qquad \square$

Proof of Theorem 2 We have '(14) $\Leftrightarrow$ (15)' from Proposition 8.

Assume that (16) holds. But then as $A(r)r^{\alpha-1} \to \infty$ implies $r^\alpha k(r) \to \infty$ we have $rT_r^{-1/\alpha} \xrightarrow{p} \infty$ from Lemma 9. Thus since $\mathbb{P}(X_{T_r} > 0) \to 1$ by Theorem 1, (14) holds.

Now assume that (15) holds. We have from Lemma 9 that $r^\alpha k(r) \to \infty$ while from (51) $\liminf_{r \downarrow 0} \frac{A(r)}{rk(r)} > 0$. Hence as $r \downarrow 0$

$$A(r)r^{\alpha-1} = \frac{A(r)}{rk(r)} r^\alpha k(r) \to \infty.$$

$\square$

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